Construction of stiffly accurate Two–Step Runge–Kutta Methods of order three and their continuous extensions using Nordsieck representation

Z. Bartoszewski [b,1] and H. Podhaisky [a] and R. Weiner [a]

[a] *FB Mathematik und Informatik, Martin-Luther-Universität Halle-Wittenberg, Postfach, 06099 Halle, Germany*

[b] *Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, Narutowicza 11/12, 80-952 Gdańsk, Poland*

Abstract

We describe a construction of implicit two–step Runge–Kutta methods for ordinary differential equations in Nordsieck form and their continuous extensions. This representation allows accurate and reliable estimation of the local discretization errors and the application to differential equations with delays. Two stiffly accurate methods of order three with quadratic interpolants are derived, one of it is shown to be L-stable.

Key words: Two–step Runge–Kutta methods, stiffly accurate methods, continuous interpolants, delay differential equations, Nordsieck representation

1 Introduction

Two step Runge-Kutta methods (TSRK in short) for constant stepsize were first introduced a decade ago by Jackiewicz and Tracogna [12] and further investigated by many authors (see [2] for historical notes related to TSRK methods). A new approach to their efficient implementation was described

Email addresses: `zbart@mif.pg.gda.pl.` (Z. Bartoszewski), `podhaisky@mathematik.uni-halle.de` (H. Podhaisky), `weiner@mathematik.uni-halle.de` (R. Weiner).

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in [2], where examples of explicit TSRK methods of order three were constructed. To demonstrate their potential their implementations in a variable environment were used to carry out numerical experiments on a number of examples and the results of the tests were compared with the results obtained by applying to the same problems the state-of-the-art `ode23` code from Matlab.

Quadratic interpolations to the above-mentioned explicit methods were constructed in [3], and numerical test carried out on a number of delay differential equations showed that they may compete with other methods for some problems. Construction of parallel continuous TSRK methods was described in [4].

In this paper, we will deal with construction of stiffly accurate TSRK methods of order and stage order three and their continuous extensions with a view to applying them to delay differential equations.

The organization of the paper is as follows. In Section 2 we give a short description of Nordsieck representation of TSRK methods and all the necessary data that makes the paper self-contained and makes it possible to implement the continuous TSRK (CTSRK in short) methods without referring to paper [2]. In Section 3 we describe the construction of stiffly accurate TSRK methods and derive two methods of order three: one L(84.6°)-stable and the second L-stable. In Section 4 we derive continuous extensions to these two TSRK methods of order 3.

2 A short description of Nordsieck representation of TSRK methods

To solve the initial–value problem for the system of ODEs

$$y'(x) = f(y(x)), \quad x \in [x_0, X], \quad y(x_0) = y_0, \tag{1}$$

$f : \mathbb{R}^m \to \mathbb{R}^m$, on the uniform grid given by

$$x_i = x_0 + ih, \quad i = 0, 1, \ldots, N, \quad h = \frac{X - x_0}{N},$$

we consider the class of implicit TSRK methods defined by:

$$\begin{cases} Y_i^{[n]} = u_i y_{n-1} + (1 - u_i) y_n + h_n \sum_{j=1}^{s} \left(a_{ij} f(Y_j^{[n-1]}) + b_{ij} f(Y_j^{[n]}) \right), \\ y_{n+1} = y_n + h_n \sum_{j=1}^{s} \left(v_j f(Y_j^{[n-1]}) + w_j f(Y_j^{[n]}) \right), \end{cases} \tag{2}$$

$n = 1, 2, \ldots, N-1$. Here: $y_n \approx y(x_n)$, $Y_i^{[n]} \approx y(x_n + c_i h_n)$, $i = 1, 2, \ldots, s$, $c = [c_1, \ldots, c_s]^T$ – a given vector and $u = [u_1, \ldots, u_s]^T$, $A = [a_{ij}]$, $B = [b_{ij}]$, $v = [v_1, \ldots, v_s]^T$, $w = [w_1, \ldots, w_s]^T$ – coefficients of the method.

An alternative vector form of (2) is

$$\begin{cases} Y^{[n]} &= (u \otimes I_m)\, y_{n-1} + \Big((e-u) \otimes I_m\Big)\, y_n \\ & \quad + h_n \Big((A \otimes I_m) f(Y^{[n-1]}) + (B \otimes I_m) f(Y^{[n]})\Big), \\ y_{n+1} &= y_n + h_n \Big((v^T \otimes I_m) f(Y^{[n-1]}) + (w^T \otimes I_m) f(Y^{[n]})\Big), \end{cases} \tag{3}$$

for $n = 1, 2, \ldots, N-1$, where $e = [1, \ldots, 1]^T \in \mathbb{R}^s$, I_m is the identity matrix of dimension m, and

$$Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ \vdots \\ Y_s^{[n]} \end{bmatrix}, \quad f(Y^{[n]}) := \begin{bmatrix} f(Y_1^{[n]}) \\ \vdots \\ f(Y_s^{[n]}) \end{bmatrix}.$$

The order conditions for the method (2) can be derived by using Taylor series expansions and are summarized in the following theorem (cf. [2, Theorem 1]):

Theorem 1 *The method (2) has order p and stage order $q = p$ if and only if*

$$\begin{cases} A(c-e)^{j-1} + Bc^{j-1} = \dfrac{c^j - (-1)^j u}{j}, \\ v^T(c-e)^{j-1} + w^T c^{j-1} = \dfrac{1}{j}, \quad j = 1, 2, \ldots, p, \end{cases} \tag{4}$$

where $e = [1 \ \cdots \ 1]^T \in \mathbb{R}^s$ and c^k denotes componentwise operation. Moreover, the error constant E is given by

$$E = \frac{1}{(p+1)!} - \frac{v^T(c-e)^p + w^T c^p}{p!}, \tag{5}$$

The formula for the local discretization error $le(x_{n+1})$ of the method (2) of order p and stage order $q = p$ at x_{n+1} is given by the formula

$$le(x_{n+1}) := y(x_{n+1}) - y_{n+1} = E h^{p+1} y^{(p+1)}(x_{n+1}) + O(h^{p+2}).$$

2.1 Computation of approximations to the Nordsieck vector $z(x_{n+1}, h)$ and $h^{p+1}y^{(p+1)}(x_{n+1})$

To implement the method (2) in a variable stepsize environment we need an approximation $h_{n+1}f(\bar{Y}^{[n]})$ to $h_{n+1}y'\Big(x_{n+1}+(c-e)h_n\Big)$ and $\bar{y}_n$ to $y(x_{n+1}-h_{n+1})$ after the step from x_n to x_{n+1} is completed. They are expressed in terms of approximation $\tilde{z}(x_{n+1}, h_n)$ to the Nordsieck vector

$$z(x_{n+1}, h_n) = [y(x_{n+1}), h_n y'(x_{n+1}), \ldots, h_n^p y^{(p)}(x_{n+1})]^T.$$

The vector $\tilde{z}(x_{n+1}, h_n)$ has the form

$$\tilde{z}(x_{n+1}, h_n) = (\alpha \otimes I_m)y_n + (\beta \otimes I_m)y_{n+1} + h_n(\Gamma \otimes I_m)f(Y^{[n]}), \tag{6}$$

where $\quad \alpha = [\alpha_i]^T_{i=1,\ldots,p}, \quad \beta = [\beta_i]^T_{i=1,\ldots,p}, \quad \Gamma = [\gamma_{ij}]_{\substack{i=0,\ldots,p \\ j=1,\ldots,s}}.$

If some additional conditions are fulfilled then (cf. [2, Theorem 2]) $\tilde{z}$ satisfies

$$\tilde{z}(x_{n+1}, h_n) = z(x_{n+1}, h_n) + O(h_n^{p+2}), \tag{7}$$

if and only if α, β, and Γ satisfy the system of equations

$$\begin{cases} \alpha\,\tau^T + \beta\,e_1^T + \Gamma\,C = I_{p+1}, \\ \dfrac{(-1)^{p+1}\alpha}{(p+1)!} - E\beta + \dfrac{\Gamma(c-e)^p}{p!} = 0. \end{cases} \tag{8}$$

Here, $e_1 = [1, 0, \ldots, 0]^T \in \mathbb{R}^{p+1}$, $\tau = [1, -1, \frac{1}{2!}, \ldots, \frac{(-1)^p}{p!}]^T$, I_{p+1} is the identity matrix of dimension $p+1$ and

$$C = \left[\begin{array}{ccccc} 0 & e & c-e & \cdots & \dfrac{(c-e)^{p-1}}{(p-1)!} \end{array}\right].$$

If some additional conditions are fulfilled then (cf. [2, Theorem 3]) we also have the estimate $h_n^{p+1}y^{(p+1)}(x_{n+1})$ in the form

$$h_n^{p+1}y^{(p+1)}(x_{n+1}) = \alpha_{p+1}y_n + \beta_{p+1}y_{n+1} + (\gamma_{p+1} \otimes I_m)h_n f(Y^{[n]}) + O(h_n^{p+2}), \tag{9}$$

if and only if $\alpha_{p+1}, \beta_{p+1} \in \mathbb{R}$ and $\gamma_{p+1} = [\gamma_{p+1,1}, \ldots, \gamma_{p+1,s}] \in \mathbb{R}^s$ satisfy the system of equations

$$\begin{cases} \alpha_{p+1}\,\tau^T + \beta_{p+1}\,e_1^T + \gamma_{p+1}\,C = 0, \\ \dfrac{(-1)^{p+1}}{(p+1)!}\,\alpha_{p+1} - E\,\beta_{p+1} + \gamma_{p+1}\,\dfrac{(c-e)^p}{p!} = 1. \end{cases} \tag{10}$$

It means that the local discretization error of the method can be estimated by the formula

$$est(x_{n+1}) = E\,\delta(x_{n+1}, h_n)$$

where E is the error constant given by (5) and $\delta(x_{n+1}, h_n)$ is defined by

$$\delta(x_{n+1}, h_n) = \alpha_{p+1}\, y_n + \beta_{p+1}\, y_{n+1} + (\gamma_{p+1} \otimes I_m)\, h_n f(Y^{[n]}). \tag{11}$$

2.2 Computation of $\bar{y}_n$ and $h_{n+1} f(\bar{Y}^{[n]})$

Put $\quad r_{n+1} = h_{n+1}/h_n, \quad D(r_n) = \mathrm{diag}\Big(1, r_n, \ldots, r_n^p\Big)$. Then (see [2]) we have

$$\begin{aligned}\bar{y}_n = &\Big(r^T D(r_{n+1}) \otimes I_m\Big)\tilde{z}(x_{n+1}, h_n) \\ &+ \left(\frac{(-1)^{p+1}}{(p+1)!} r_{n+1}^{p+1} - E\Big(1 - r_{n+1}\Big)^{p+1}\right)\delta(x_{n+1}, h_n).\end{aligned} \tag{12}$$

Then for $h_{n+1} f(\bar{Y}^{[n]})$ we have the formula

$$\begin{aligned}h_{n+1} f(\bar{Y}^{[n]}) = &\Big(CD(r_{n+1}) \otimes I_m\Big)\tilde{z}(x_{n+1}, h_n) \\ &+ \left(\frac{(c-e)^p}{p!} \otimes I_m\right) r_{n+1}^{p+1}\delta(x_{n+1}, h_n).\end{aligned} \tag{13}$$

Observe that (12) and (13) can be computed without any extra evaluations of the right hand side of the equation (1).

3 Construction of stiffly accurate TSRK methods

In this section we will describe the construction of stiffly accurate L(α)-stable and L-stable TSRK methods. The basis for our discussion is the linear scalar test equation $y' = \xi y$ for which the TSRK method reduces to a matrix-vector recurrence

$$y^{[n+1]} = M(z) y^{[n]}, \quad z = \xi h$$

where the vector $y^{[n]} := \Big[Y^{[n-1]}, y_{n-1}, y_n\Big]^T$ contains all quantities which are passed between two steps. The amplification or stability matrix $M(z)$ has the form

$$M(z) = \begin{bmatrix} zGA & Gu & G(e-u) \\ 0 & 0 & 1 \\ zv^T + z^2 w^T GA & zw^T Gu & 1 + zw^T G(e-u) \end{bmatrix}, \tag{14}$$

with $G = (I - zB)^{-1}$. The stability domain can be defined as the set $\mathcal{S} := \{z \in \mathbb{C} : M(z) \text{ is power bounded}\}$ and A-stability becomes $\mathbb{C}^- \subseteq \overline{\mathcal{S}}$. The last row of $M(z)$ describes the propagation of the primary approximation y_n to the solution. We call a method stiffly accurate if for fixed y_{n-1}, y_n it holds

$$y_{n+1} \to 0 \text{ for } \mathrm{Re} z \to -\infty. \tag{15}$$

An A-stable (A(α)-stable) method which is stiffly accurate is called L-stable (L(α)-stable, respectively). Advantages of the requirement (15) compared with $\varrho(M(\infty)) = 0$ are faster damping of errors and an easy characterization of appropriate methods.

Stiffly accurate two-step methods were introduced in [14] for sequential and in [15] for parallel methods, where for the case $u_i = 0$ necessary and sufficient conditions for stiff accuracy have been derived. Numerical tests there have shown the advantage of L-stable methods for very stiff problems.

Stiff accuracy requires

$$e_{s+2}^T M(\infty) = 0, \tag{16}$$

where $e_{s+2} = (0, \ldots, 0, 1)^T$ For nonsingular B we have

$$(I - zB)^{-1} = -\frac{1}{z} B^{-1} - \frac{1}{z^2} B^{-2} + \mathcal{O}(\frac{1}{z^3}).$$

Then (16) ist equivalent to

$$v^T - w^T B^{-1} A = 0 \tag{17}$$

$$w^T B^{-2} A = 0 \tag{18}$$

$$w^T B^{-1} u = 0 \tag{19}$$

$$1 - w^T B^{-1}(e - u) = 0. \tag{20}$$

Following the investigations in [14] we set

$$w^T = e_s^T B. \tag{21}$$

This leads immediately to

$$u_s = 0, \quad v^T = e_s^T A, \tag{22}$$

satisfying conditions (17), (19) and (20). We arrive at

$$M(\infty) = \begin{bmatrix} -B^{-1}A & 0_1 & 0_2 \\ 0 & 0 & 1 \\ -e_s^T B^{-1} A & 0 & 0 \end{bmatrix}, \tag{23}$$

where 0_1 and 0_2 are zero matrices of proper dimensions.

Thus, the required L-stability property leads to the conditions

$$\rho(B^{-1}A) \leq 1, \quad e_s^T B^{-1} A = 0. \tag{24}$$

We will illustrate our approach in the case of methods with $p = q = s = 3$. Assume that $c = [1/3, 2/3, 1]^T$. Then we have to determine the entries of A and B and the components of u, v and w, 21 parameters in all. The stage order and order conditions (4), conditions for stiff accuracy (21),(22), L-stability condition (24) and a given in advance error constant provide 19 conditions leaving 2 degree of freedom (2 free parameters). We choose u_1 and λ as these 2 free parameters and carry out a computer search so that they minimize the absolute values of eigenvalues of $B^{-1}A$ and secure A-stability of the TSRK method.

Using this approach we have found two TSRK methods. The first method is L(84.6°)-stable and the second is L-stable. We use the following form of tableau for the coefficients :

$$\begin{array}{c|c|c} u & A & B \\ \hline & v^T & w^T \end{array} = \begin{array}{c|cccc|cccc} u_1 & a_{11} & a_{12} & \ldots & a_{1s} & b_{11} & b_{12} & \ldots & b_{1s} \\ u_2 & a_{21} & a_{22} & \ldots & a_{2s} & b_{21} & b_{22} & \ldots & b_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_s & a_{s1} & a_{s2} & \ldots & a_{ss} & b_{s1} & b_{s2} & \ldots & b_{ss} \\ \hline & v_1 & v_2 & \ldots & v_s & w_1 & w_2 & \ldots & w_s \end{array}.$$

Their coefficients are:

L(84.6°)-*stable* method

$$\begin{array}{c|ccc|ccc} \frac{1}{63} & \frac{-31}{630} & \frac{7}{45} & \frac{3}{70} & \frac{1}{5} & 0 & 0 \\ \frac{-1}{504} & \frac{-5227}{50400} & \frac{49}{225} & \frac{3559}{50400} & \frac{7}{25} & \frac{1}{5} & 0 \\ 0 & \frac{-159}{1250} & \frac{609}{2500} & \frac{103}{1250} & \frac{783}{2500} & \frac{36}{125} & \frac{1}{5} \\ \hline & \frac{-159}{1250} & \frac{609}{2500} & \frac{103}{1250} & \frac{783}{2500} & \frac{36}{125} & \frac{1}{5} \end{array}$$

L-*stable* method

$$\begin{array}{c|ccc|ccc} \frac{-78}{35} & \frac{-33923}{16380} & \frac{137}{117} & \frac{-25121}{16380} & \frac{7}{13} & 0 & 0 \\ \frac{-8539}{1344} & \frac{-1407199}{232960} & \frac{78313}{23040} & \frac{-8431733}{2096640} & \frac{131143}{299520} & \frac{7}{13} & 0 \\ 0 & \frac{16183}{135200} & \frac{-4269}{135200} & \frac{-123291}{135200} & \frac{335057}{135200} & \frac{-1008}{845} & \frac{7}{13} \\ \hline & \frac{16183}{135200} & \frac{-4269}{135200} & \frac{-123291}{135200} & \frac{335057}{135200} & \frac{-1008}{845} & \frac{7}{13} \end{array}$$

For the L(84.6°)-*stable* method we have

$$
\alpha = \begin{bmatrix} \frac{1701}{1076} \\ 0 \\ \frac{-1250}{269} \\ \frac{-11250}{269} \end{bmatrix}, \quad \beta = \begin{bmatrix} \frac{-625}{1076} \\ 0 \\ \frac{1250}{269} \\ \frac{11250}{269} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \frac{5103}{4304} & 0 & \frac{1701}{4304} \\ 0 & 0 & 1 \\ \frac{-534}{269} & -6 & \frac{898}{269} \\ \frac{-12033}{538} & -18 & \frac{-783}{538} \end{bmatrix}, \quad \begin{bmatrix} \alpha_4 \\ \beta_4 \\ \gamma_4^T \end{bmatrix} = \begin{bmatrix} \frac{-33750}{269} \\ \frac{33750}{269} \\ \frac{-50625}{538} \\ 0 \\ \frac{-16875}{538} \end{bmatrix}
$$

and for the L-*stable* method

$$
\alpha = \begin{bmatrix} \frac{27}{127} \\ 0 \\ \frac{800}{127} \\ \frac{7200}{127} \end{bmatrix}, \quad \beta = \begin{bmatrix} \frac{100}{127} \\ 0 \\ \frac{-800}{127} \\ \frac{-7200}{127} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \frac{81}{508} & 0 & \frac{27}{508} \\ 0 & 0 & 1 \\ \frac{1581}{254} & -6 & \frac{1543}{254} \\ \frac{6543}{127} & -18 & \frac{2943}{127} \end{bmatrix}, \quad \begin{bmatrix} \alpha_4 \\ \beta_4 \\ \gamma_4^T \end{bmatrix} = \begin{bmatrix} \frac{21600}{127} \\ \frac{-21600}{127} \\ \frac{16200}{127} \\ 0 \\ \frac{5400}{127} \end{bmatrix}
$$

Remark 2 *The angle of $L(\alpha)$ stability can be computed numerically by approximating the boundary $p(z, \exp(\varphi i)) := \det(M(z) - \exp(\varphi i)I) = 0$ for $\varphi \in [0, 2\pi]$ of the stability domain. A strict proof of L-stability is obtained by using the Schur criterion recursively on the imaginary axis to show $p(iy, x) = 0 \Rightarrow |x| \leq 1$ for all $y \in \mathbb{R}$.*

The stability domains for the two methods are shown in Fig. 1.

4 Continuous extensions to the stiffly accurate TSRK methods of order 3

We will construct the continuous extension to TSRK method (2) in the form

$$
\begin{cases} Y_i^{[n]} = u_i \bar{y}_{n-1} + (1 - u_i) y_n + h_n \sum_{j=1}^{s} \left(a_{ij} f(\bar{Y}_j^{[n-1]}) + b_{ij} f(Y_j^{[n]}) \right) \\ \qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots, s, \\ y_n(x_n + \theta h_n) = \eta(\theta) y_{n-1} + (1 - \eta(\theta)) y_n + \\ \qquad h_n \sum_{j=1}^{s} \left(v_j(\theta) f(\bar{Y}_j^{[n-1]}) + w_j(\theta) f(Y_j^{[n]}) \right), \ \theta \in [0, 1], \end{cases} \tag{25}
$$

$n = 0, 1, \ldots, N - 1.$

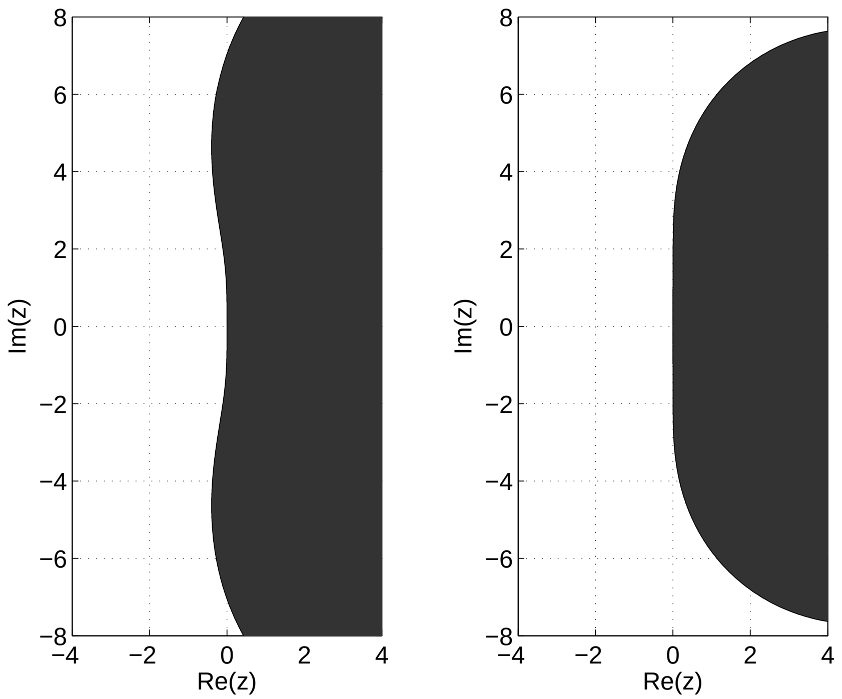

Fig. 1. The stability domain for the two methods. The stable part is unshaded. Left: method 1 with L(84.6°), right: L-stable method

The method (25) is zero stable if and only if (cf. [13, Theorem 1])

$$-1 < \eta(1) \leq 1.$$

4.1 Order conditions

Similarly like in [13], for our continuous extensions of TSRK methods (with constant step) we define for $\mu = 1, 2, \ldots$ the error constants

$$C_\mu = \frac{c^\mu - (-1)^\mu u}{\mu!} - \frac{A(c-e)^{\mu-1}}{(\mu-1)!} - \frac{Bc^{\mu-1}}{(\mu-1)!} \tag{26}$$

$$\widehat{C}_\mu(\theta) = \frac{\theta^\mu}{\mu!} - \frac{(-1)^\mu \eta(\theta)}{\mu!} - \frac{v(\theta)^T(c-e)^{\mu-1}}{(\mu-1)!} - \frac{w(\theta)^T c^{\mu-1}}{(\mu-1)!}, \tag{27}$$

and

$$E(\theta) = \frac{\theta^{p+1}}{(p+1)!} - \frac{v(\theta)^T(c-e)^p}{p!} - \frac{w(\theta)^T c^p}{p!}. \tag{28}$$

Remark 3 *Observe that the equations $C_\mu = 0$, $\mu = 0, \ldots, p$ and $\widehat{C}_\mu(1) = 0$, $\mu = 0, \ldots, p$, are equivalent to (4) for $\eta(1) = 0$ and that for $\theta = 1$ (28) is equivalent to (5).*

These equations and the theorem below are valid for continuous TSRK methods of an arbitrary order p. Following exactly the method that was used in [13] to prove Theorem 3 we can prove the following result.

Theorem 4 *Assume that the continuous TSRK method (25) is zero stable, the errors of the initial approximations y_0 and y_1 are of order $O(h^p)$ and that it has order of consistency $p-1$ and stage order of consistency p, i.e.,*

$$\widehat{C}_\mu(\theta) = 0, \qquad C_\mu = 0, \tag{29}$$

for $\mu = 1, 2, \ldots, p-1$, $\theta \in [0, 1]$ and that

$$\widehat{C}_p(1) = 0. \tag{30}$$

Then the method has order of convergence p and stage order of convergence p.

4.2 Stability properties

We apply (25) with a constant stepsize $h = \frac{\tau}{m}$, (m - a positive integer) to the test equation

$$\begin{cases} y'(t) = ay(t) + by(t-\tau),\ t \geq 0, \\ y(t) = g(t), \qquad\qquad t \in [-\tau, 0], \end{cases} \tag{31}$$

where a and b are complex parameters, $\tau > 0$ is a constant delay and g a given initial function. As a result we obtain a sequence

$$\{y_n(m; \alpha, \beta)\}_{n=0}^{\infty},$$

$\alpha = ha$, $\beta = hb$, where $y_n(m; \alpha, \beta) \approx y(t_n)$. The region of stability $\mathcal{S}$ of (31) is a subset of $C \times C$ given by

$$\mathcal{S} = \{(\alpha, \beta) : \ y_n(m; \alpha, \beta) \to 0 \text{ as } n \to \infty \text{ for every } m\},$$

(cf. [1]). Here, C stands for a set of complex numbers. The method (25) is said to be P–stable if

$$\{(\alpha, \beta) : \ |\beta| < -\Re(\alpha)\} \subset \mathcal{S}.$$

This means that if the method (25) is P–stable then the numerical solution $y_n(m; \alpha, \beta) \to 0$ as $n \to \infty$ for every m if $y(t) \to 0$ as $t \to \infty$, where $y(t)$ is the solution to (1). Put

$$\begin{aligned} &\tilde{u} = [\eta_1, \eta_2, \ldots, \eta_s]^T,\ \eta_i = \eta(c_i), \\ &\Gamma = [\gamma_{ij}]_{i,j=1}^s, \qquad \gamma_{ij} = v_j(c_i), \\ &\Delta = [\delta_{ij}]_{i,j=1}^s, \qquad \delta_{ij} = w_j(c_i). \end{aligned}$$

In [1] it was proved the following theorem which reduces P–stability of TSRK method for DDEs to A–stability of TSRK formulas for ODEs.

Theorem 5 *Assume that the TSRK method for ODEs such that* $\tilde{u} = u$, $\Gamma = A$, *and* $\Delta = B$ *is* A*–stable. Then the corresponding TSRK method (25) for DDEs is* P*–stable.*

So, in order to construct a P-stable continuous extension to the given stiffly accurate L-stable TSRK method, of the form (25), we impose on the continuous weights $\eta(\theta)$, $v_i(\theta)$, and $w_i(\theta)$, $i = 1, 2, \ldots, s$ the following conditions:

$$\eta(0) = 0,\ \eta(1) = 0 = u_s,\ \tilde{u} = u,\ v_i(0) = w_i(0) = 0,\ \Gamma = A,\ \Delta = B. \quad (32)$$

Observe that if conditions (32) are satisfied then conditions $v_i(1) = v_i,\ w_i(1) = wi$ are automatically satisfied as for stiffly accurate TSRK methods $c_s = 1$ and the last row of A is equal to v and the last row of B is equal to w.

We look for continuous weights $\eta(\theta)$ $v_i(\theta)$ and $w_i(\theta)$ in the form

$$\eta(\theta) = \theta\left(\eta_1 + \eta_2\,\theta + \eta_3\,\theta^2 + \cdots + \eta_s\,\theta^{s-1}\right), \quad (33)$$

$$v_i(\theta) = \theta\left(v_{i,1} + v_{i,2}\,\theta + v_{i,3}\,\theta^2 + \cdots + v_{i,s}\,\theta^{s-1}\right), \quad (34)$$

$$w_i(\theta) = \theta\left(w_{i,1} + w_{i,2}\,\theta + w_{i,3}\,\theta^2 + \cdots + w_{i,s}\,\theta^{s-1}\right), \quad (35)$$

$i = 1, 2, \ldots, s$, $\theta \in [0, 1]$, for which, obviously, the conditions $\eta(0) = 0,\ v_i(0) = w_i(0) = 0$ are fulfilled.

To obtain the coefficients of the continuous weights satisfying (32) we have to solve the systems of linear equations

$$\eta(c_j) = u_j, \quad j = 1, 2, \ldots, s, \quad (36)$$

$$v_i(c_j) = a_{ji}, \quad j = 1, 2, \ldots, s,\ i = 1, 2, \ldots, s, \quad (37)$$

and

$$w_i(c_j) = b_{ji}, \quad j = 1, 2, \ldots, s,\ i = 1, 2, \ldots, s. \quad (38)$$

The order of uniform convergence of the method (25), whose continuous weights satisfy (36), (37) and (38) gives the following theorem.

Theorem 6 *Assume that the discrete s-stage stiffly accurate TSRK method (2) with* $c_i \neq 0$, $c_i \neq c_j$ *for* $i \neq j$, *has order* $\tilde{p}$ *and stage order* $q = \tilde{p}$. *Assume also that the continuous weights* $\eta(\theta)$, $v(\theta)$, *and* $w(\theta)$ *of the method (25) are* *polynomials of degree less than or equal to s such that*

$$\eta(0) = 0, \quad \eta(c_i) = u_i,$$
$$v_i(0) = 0, \; v_i(c_j) = a_{ji},$$
$$w_i(0) = 0, \; w_i(c_j) = b_{ji},$$

$i, j = 1, 2, \ldots, s$. Then the continuous TSRK method (25) is convergent with uniform order $p = \tilde{p}$.

PROOF. It follows from the assumptions of the theorem that the error constants $\widehat{C}_\mu(\theta)$ defined by (27) are polynomials of degree less than or equal to s. We have also

$$\widehat{C}_\mu(0) = 0,$$

$$\begin{aligned} \widehat{C}_\mu(c_j) &= \frac{c_j^\mu}{\mu!} - \frac{(-1)^\mu \eta(c_j)}{\mu!} - \frac{v(c_j)(c-e)^{\mu-1}}{(\mu-1)!} - \frac{w(c_j)c^{\mu-1}}{(\mu-1)!} \\ &= \frac{c_j^\mu}{\mu!} - \frac{(-1)^\mu u_j}{\mu!} - \frac{r_j(A)(c-e)^{\mu-1}}{(\mu-1)!} - \frac{r_j(B)c^{\mu-1}}{(\mu-1)!} \\ &= C_{\mu,j} \; = \; 0, \end{aligned}$$

$\mu = 1, 2, \ldots, \tilde{p} - 1$, $j = 1, 2, \ldots, s$, where, we have taken advantage of the fact that the method (2) has stage order $q = \tilde{p}$. Here, $r_j(A)$ stands for the jth row of A and $C_{\mu,j}$ is the jth component of the error vector C_μ defined by (26). Since $\deg(\widehat{C}_\mu(\theta)) \leq s$ it follows that $\widehat{C}_\mu(\theta) \equiv 0$, $k = 1, 2, \ldots, \tilde{p} - 1$, and from Theorem 4 (see also Remark 3) the method (25) is convergent with uniform order $p = \tilde{p}$.

Using the described method we have found the continuous weights for the extension to the 3-stage L(84.6°)-stable method of order 3, constructed in Section 3. Its coefficients are

L(84.6°)-*stable* method

$\bar{\eta}$	$\bar{v}_1$	$\bar{v}_2$	$\bar{v}_3$	$\bar{w}_1$	$\bar{w}_2$	$\bar{w}_3$
$\frac{17}{112}$	$\frac{-28941}{280000}$	$\frac{-13107}{70000}$	$\frac{45753}{280000}$	$\frac{2133}{2500}$	$\frac{-4347}{5000}$	$\frac{1647}{5000}$
$\frac{-11}{28}$	$\frac{1659}{2500}$	$\frac{-3381}{5000}$	$\frac{1281}{5000}$	$\frac{-153}{250}$	$\frac{288}{125}$	$\frac{-351}{250}$
$\frac{27}{112}$	$\frac{42097}{280000}$	$\frac{-4481}{70000}$	$\frac{-1101}{280000}$	$\frac{1}{5}$	$\frac{-9}{10}$	$\frac{9}{10}$

where we used the notation: $\bar{\eta} = [\eta_1, \eta_2, \eta_3]^T$, $\bar{v}_i = [v_{i,1}, v_{i,2}, v_{i,3}]^T$, $\bar{w}_i = [w_{i,1}, w_{i,2}, w_{i,3}]^T$, $i = 1, 2, 3$. The coefficients of the continuous weights for the second L-stable method constructed in Section 3 can be easily found in a similar way and we do not report them.

5 Conclusions

We have derived L- and L(α)-stable two-step Runge-Kutta methods using the property of stiff accuracy. Similar methods with $u_i = 0$ have been tested in [14] and [15] and have shown a good performance for stiff problems. We therefore assume that the new methods with $u_i \neq 0$ combined with the proposed continuous extension will be well suited for stiff delay equations. The numerical test of these methods and the construction of corresponding higher order methods will be the topic of future work.

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