On Lipschitz Functions with Values in Partially Ordered Spaces

Kristin Winkler

Abstract

We investigate two rather technical properties concerning Lipschitz mappings. In the first part, we show that the result of Roberts and Varberg on the Lipschitz conditions for convex functions also holds for mappings with values in partially ordered spaces. In the second part, we prove that the distance functions can be used for exact penalization in vector optimization problems. Our results are close to similar assertions in real-valued convex optimization.

Keywords. vector-valued optimization, Lipschitz condition, abstract constraints, exact penalization

1 Introduction

We consider the vector-valued optimization problem

$$\mathfrak{f}(x) \to \min, \qquad x \in X \subset \mathfrak{X}$$

with $\mathfrak{f} : \mathfrak{X} \to Y$. For this, let $\mathfrak{X}$ be a Banach space and X a nonempty closed subset of $\mathfrak{X}$. $\mathfrak{Y}$ denotes a topological vector space which is partially ordered by the convex cone $C \subset \mathfrak{Y}$. The aim of our note is double:

1) We investigate a Lipschitz condition for mappings with values in partially ordered spaces and show that convex, locally bounded mappings satisfy are Lipschitz.

2) We study how the original problem can be replaced by a global vector optimization problem by including the constraints into the goal function via penalization.

The results are close to similar assertions in real-valued optimization. Our assertions are of rather technical nature and differ sometimes only slightly from results that can be found in literature. Nevertheless, they are crucial for the development of efficiency criteria in vector-valued optimization.

In the real-valued case, penalization via distance function is a fundamental technical method within constraint optimization, compare e.g. Clarke [1, Proposition 2.4.3]. It is the basis for handling abstract constraints, for example if one wishes to characterize optimal points via subdifferentials: The resulting goal functions is again convex if the original function was so, and the subdifferential can be calculated in an easy way (if one can do so for the original goal function).

Our note is organized as follows. Section 3 deals with the notion of (vector-valued) Lipschitz mappings, and we prove that convex, locally bounded mappings are Lipschitz. In section 4 we show that the distance functions can used for exact penalization in vector optimization problems, too.

2 Basic notions and preliminaries

We only repeat some basic notions concerning partially ordered vector spaces and efficiency. For more details, we recommend the monographs of Jahn [2] and Peressini [6].

Let $\mathfrak{Y}$ be a topological vector space which is partially ordered by the convex cone $C \subset \mathfrak{Y}$, $C \neq \mathfrak{Y}$. We assume $C \cap -C = \{0\}$. An element $\bar{y} \in \mathfrak{Y}$ is called supremum of a set $A \subset \mathfrak{Y}$, $\bar{y} = \sup A$, if $\bar{y} \in y + C$ for all $y \in A$ and $z \in \bar{y} + C$ whenever $z \in y + C$ for all $y \in A$. The infimum of a set can be defined dually; if the infimum and the supremum exists for any pair of elements of $\mathfrak{Y}$, the space $\mathfrak{Y}$ is a vector lattice. In that case, by $|y| := \sup\{y, -y\}$ we denote the absolute value of $y \in \mathfrak{Y}$. Obviously, $|y| \in C$, $|y| \in y + C$ and $|y| \in -y + C$.

Let $X \subset \mathfrak{X}$ a nonempty subset, and $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$. An element $\bar{x} \in X$ is called **efficient in X with respect to $\mathfrak{f}$**, if

$$\mathfrak{f}(X) \cap (\mathfrak{f}(\bar{x}) - C)\backslash\{0\}) = \emptyset.$$

If $\operatorname{int} C \neq \emptyset$, $\bar{x} \in X$ is called **weakly efficient in X with respect to $\mathfrak{f}$**, if

$$\mathfrak{f}(X) \cap (\mathfrak{f}(\bar{x}) - \operatorname{int} C) = \emptyset.$$

The set of all (weakly) efficient elements in X with respect to $\mathfrak{f}$ is denoted by $\operatorname{Eff}(\mathfrak{f}(X), C)$ and $\operatorname{Eff}_w(\mathfrak{f}(X), C)$, respective. Obviously $\operatorname{Eff}(\mathfrak{f}(X), C) \subseteq \operatorname{Eff}_w(\mathfrak{f}(X), C)$.

Within this note, let $(\mathfrak{X}, \|.\|)$ be a Banach space and $X \subset \mathfrak{X}$ a (closed) subset. We denote by $d_X : \mathfrak{X} \to \mathbb{R}_+$, $d_X(x) := \inf_{x' \in X} \|x - x'\|$ the **distance function** (with respect to the set X). Remind $d_X(x) = 0$ iff $x \in \operatorname{cl} X$.

3 Lipschitz mappings

Definition 3.1 A mapping $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is called **locally Lipschitz**, if for all $x \in \mathfrak{X}$ there exists a neighborhood N of x and a Lipschitz rank $k \in C$ such that

$$\|x_1 - x_2\| \, k \in |f(x_1) - f(x_2)| + C, \qquad x_1, x_2 \in N$$

holds. $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is called **Lipschitz (of rank $k \in C$)**, if the same Lipschitz rank k can be chosen for all $x \in \mathfrak{X}$.

This definition of Lipschitz mappings with values in partially ordered spaces is near to Kusraev's and to Papageorgiou's notions of Lipschitz mappings, compare [3] and [4], [5], respectively. The hardest difference to their definitions consists in the existence of an unique Lipschitz rank k for all $x \in \mathfrak{X}$. Other concepts of Lipschitz mappings and comparisons of that notions have been investigated by Reiland [7, 8], Staib [10], Thibault [11, 12, 13], for example.

Note that, if $\mathfrak{Y}$ is an order lattice (i.e. a normed vector space such that $|y_1| - |y_2| \in C$ implies $\|y_1\|_{\mathfrak{Y}} \geq \|y_2\|_{\mathfrak{Y}}$), then the above Lipschitz conditions implies that $\mathfrak{f}$ is norm Lipschitz.

From real-valued analysis we know that convex, locally bounded functions are locally Lipschitz (compare Clarke [1, Proposition 2.2.6] and for the original result Roberts and Varberg [9]). We show that these results hold for mappings with values in partially ordered spaces, too. Related results are known from Papageorgiou [4], Reiland [8] and Thibault [11].

For this, let U be an open convex subset of $\mathfrak{X}$. Remind that a mapping $\mathfrak{f} : U \to \mathfrak{Y}$ is said to be **convex** if for all $x_1, x_2 \in U$ and all $\lambda \in [0, 1]$, we have

$$\lambda\mathfrak{f}(x_1) + (1 - \lambda)\,\mathfrak{f}(x_2) \in \mathfrak{f}(\lambda x_1 + (1 - \lambda)x_2) + C.$$

$\mathfrak{f}$ is called **bounded above** on some set $U' \subset U$ if there exists some $y \in \mathfrak{Y}$ such that $y - \mathfrak{f}(x) \in C$ for all $x \in U'$. Finally, $\mathfrak{f}$ is called **bounded** on some set $U' \subset U$ if there exists some $y \in \mathfrak{Y}$ such that $y - |\mathfrak{f}(x)| \in C$ for all $x \in U'$.

Lemma 3.1 *Let $U \subset \mathfrak{X}$ be open and convex, $\mathfrak{f} : U \to \mathfrak{Y}$ convex. Assume $\mathfrak{f}$ to be bounded above on a neighborhood of some point $\bar{x} \in U$. Then, for any $x \in U$, $\mathfrak{f}$ is bounded near x.*

Proof. We only have to verify that the original proof of Roberts and Varberg [9] can be translated in our setting. So, assume that $\mathfrak{f}$ is bounded above by some $y \in \mathfrak{Y}$ on the set $\varepsilon B := \{x \in \mathfrak{X} : \|x - \bar{x}\| < \varepsilon\} \subseteq U$ and fix an arbitrary $\tilde{x} \in U$. We choose $\alpha > 1$ such that $x_1 = \alpha\tilde{x} \in U$. With $\lambda = 1/\alpha$ the set

$$V := \{x \in \mathfrak{X} :\ x = (1-\lambda)x' + \lambda x_1,\ x' \in \varepsilon B\}$$

is a neighborhood of $\tilde{x}$ with radius $(1-\lambda)\varepsilon$. By boundedness above and convexity we get for all $x \in V$

$$y + \lambda\mathfrak{f}(x_1) \in \mathfrak{f}(x') + \lambda\mathfrak{f}(x_1) + C \subseteq (1-\lambda)\mathfrak{f}(x') + \lambda\mathfrak{f}(x_1) + C \subseteq f(x) + C, \tag{1}$$

i.e. $\mathfrak{f}$ is bounded above on a neighborhood of $\tilde{x}$. On the other hand, if $x \in \tilde{x} + (1-\lambda)\varepsilon B$, there is a point x' such that $\tilde{x} = (x + x')/2$. Convexity and boundedness above yield

$$\mathfrak{f}(x) \in 2\mathfrak{f}(\tilde{x}) - \mathfrak{f}(x') + C \subseteq 2\mathfrak{f}(\tilde{x}) - y - \lambda\mathfrak{f}(x_1) + C. \tag{2}$$

With $\tilde{y} := \sup\{y + \lambda\mathfrak{f}(x_1), y + \lambda\mathfrak{f}(x_1) - 2\mathfrak{f}(\tilde{x})\}$, inclusions (1) and (2) can be summarized to $\tilde{y} - \mathfrak{f}(x) \in C$ and $\tilde{y} + \mathfrak{f}(x) \in C$, hence $\tilde{y} - |\mathfrak{f}(x)| \in C$ for all $x \in \tilde{x} + (1-\lambda)\varepsilon B$, i.e. $\mathfrak{f}$ is bounded on this set. ∎

Theorem 3.1 *Let $U \subset \mathfrak{X}$ be open and convex, $\mathfrak{f} : U \to \mathfrak{Y}$ convex. Assume $\mathfrak{f}$ to be bounded above on a neighborhood of some point $\bar{x} \in U$. Then, for any $x \in U$, $\mathfrak{f}$ is Lipschitz near $\tilde{x}$.*

Proof. Once again, we follow the the original proof of Roberts and Varberg [9]. So, by Lemma 3.1 we know that, for any $x \in U$, $\mathfrak{f}$ is bounded near x. To be more precise, assume that $y \in \mathfrak{Y}$ is a bound on $|\mathfrak{f}|$ on the set $x + 2\varepsilon B$ with $\varepsilon > 0$. For $x_1, x_2 \in x + \varepsilon B$, $x_1 \neq x_2$, we define

$$x_3 = x_2 + \frac{\varepsilon}{\alpha}(x_2 - x_1) \qquad \text{with} \quad \alpha := \|x_2 - x_1\|.$$

It is easy to verify $x_3 \in x + 2\varepsilon B$ and

$$x_2 = \frac{\varepsilon}{\alpha + \varepsilon}\, x_1 + \frac{\alpha}{\alpha + \varepsilon}\, x_3,$$

so from convexity of $\mathfrak{f}$ we deduce

$$\frac{\varepsilon}{\alpha + \varepsilon}\,\mathfrak{f}(x_1) + \frac{\alpha}{\alpha + \varepsilon}\,\mathfrak{f}(x_3) \in \mathfrak{f}(x_2) + C.$$

Therefore,

$$\frac{\alpha}{\varepsilon}\,[|\mathfrak{f}(x_3)| + |\mathfrak{f}(x_1)|] \in \frac{\alpha}{\varepsilon}\,|\mathfrak{f}(x_3) - \mathfrak{f}(x_1)| + C \subseteq \frac{\alpha}{\alpha + \varepsilon}\,[\mathfrak{f}(x_3) - \mathfrak{f}(x_1)] + C \subseteq \mathfrak{f}(x_2) - \mathfrak{f}(x_1) + C.$$

Recalling boundedness of $\mathfrak{f}$, we get

$$\frac{2}{\varepsilon}\,\|x_2 - x_1\|\, y \in \mathfrak{f}(x_2) - \mathfrak{f}(x_1) + C.$$

Since we can interchange x_1 and x_2, we conclude that $\mathfrak{f}$ is Lipschitz near x. ∎

Corollary 3.1 *The proof shows that $\mathfrak{f}$ is Lipschitz of (the same) rank, if there exists a global bound for $\mathfrak{f}$.*

4 Exact penalization via distance function

Consider
$$\mathfrak{g}_\alpha(.) := \mathfrak{f}(.) + d_X(.)\alpha k, \qquad \alpha \geq 1,$$
with $k \in C$. In this section, we show that the efficiency sets with respect to $\mathfrak{f}$ and $\mathfrak{g}_\alpha$ are identical. Note that we do not need convexity of the funtion $\mathfrak{f}$ or the set X.

Theorem 4.1 *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be Lipschitz of rank $k \in C$, $X \subset \mathfrak{X}$ be a closed subset of $\mathfrak{X}$ and suppose $\bar{x} \in X$. Then $\bar{x} \in \mathrm{Eff}(\mathfrak{f}(X), C)$ implies $\bar{x} \in \mathrm{Eff}(\mathfrak{g}_\alpha(\mathfrak{X}), C)$ for each $\alpha > 1$.*

Proof. Fix $\alpha > 1$ and consider $\bar{x} \notin \mathrm{Eff}(\mathfrak{g}_\alpha(\mathfrak{X}), C)$. Then there exists $x \in \mathfrak{X}$ such that
$$\mathfrak{g}_\alpha(\bar{x}) - \mathfrak{g}_\alpha(x) = \mathfrak{f}(\bar{x}) - \mathfrak{f}(x) - d_X(x)\alpha k \in C\backslash\{0\}. \tag{3}$$
We can assume $x \notin X$, i.e. $d_X(x) > 0$, avoiding a contradiction to $\bar{x} \in \mathrm{Eff}(\mathfrak{f}(X), C)$. So, we can choose $\tilde{x} \in X$ with $\|\tilde{x} - x\| \leq \alpha d_X(x)$ and get
$$\mathfrak{f}(\bar{x}) - \mathfrak{f}(\tilde{x}) \in \mathfrak{f}(x) - \mathfrak{f}(\tilde{x}) + d_X(x)\alpha k + C\backslash\{0\} \in \mathfrak{f}(x) - \mathfrak{f}(\tilde{x}) + \|\tilde{x} - x\| k + C\backslash\{0\} \in C\backslash\{0\},$$
in contradiction to $\bar{x} \in \mathrm{Eff}(\mathfrak{f}(X), C)$. ■

Remark 4.1 If $\operatorname{int} C \neq \emptyset$, the same result can be stated for weak efficiency. Since equation (3) can be replaced by
$$\mathfrak{f}(\bar{x}) - \mathfrak{f}(x) - (d_X(x) + \varepsilon)\, k \in C\backslash\{0\}$$
with sufficiently small $\varepsilon > 0$, the assertions of the theorem remain valid for $\alpha = 1$.

Theorem 4.2 *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be Lipschitz of rank $k \in C$, $X \subset \mathfrak{X}$ be a closed subset of $\mathfrak{X}$. Then $\tilde{x} \in \mathrm{Eff}(\mathfrak{g}_\alpha(\mathfrak{X}), C)$ for some $\alpha > 1$ implies $\tilde{x} \in X$ and therefore $\tilde{x} \in \mathrm{Eff}(\mathfrak{f}(X), C)$.*

Proof. Assume $\tilde{x} \in \mathrm{Eff}(\mathfrak{g}_\alpha(X), C)$ for some $\alpha > 1$. If $\tilde{x} \in X$ we have nothing to do, so let us study the case $\tilde{x} \notin X$. Then,
$$S := \{x \in X : \ \|\tilde{x} - x\| < \alpha d_X(\tilde{x})\} \neq \emptyset.$$
For $x \in S$ we have by the efficiency of $\tilde{x}$
$$\mathfrak{g}_\alpha(\tilde{x}) - \mathfrak{g}_\alpha(x) = \mathfrak{f}(\tilde{x}) + d_X(\tilde{x})\alpha k - \mathfrak{f}(x) \notin C\backslash\{0\} \tag{4}$$
and since f is Lipschitz
$$\mathfrak{f}(\tilde{x}) + d_X(\tilde{x})\alpha k - \mathfrak{f}(x) \in \mathfrak{f}(\tilde{x}) + \|\tilde{x} - x\| k - \mathfrak{f}(x) + C \subseteq C \tag{5}$$
By the combination of the inclusions (4) and (5) we derive $\mathfrak{f}(\tilde{x}) + d_X(\tilde{x})\alpha k = \mathfrak{f}(x)$ for $x \in S$. Inserting this equation into the Lipschitz condition yields $(\|\tilde{x}-x\| - \alpha d_X(\tilde{x}))\, k \in C$, thus $\|\tilde{x}-x\| \geq \alpha d_X(\tilde{x})$ for $x \in S$, in contradiction to the definition of set S. Therefore, $\tilde{x} \notin X$ is not possible, and by $\mathfrak{f}(x) = \mathfrak{g}_\alpha(x)$ for $x \in X$ our assertion holds. ■

Remark 4.2 If $\operatorname{int} C \neq \emptyset$, the same result holds for weakly efficient points. Even more, the results hold for each cone $C' \subset \mathfrak{Y}$, $C' \cap -C' = \emptyset$, $C' + k \subset C'$, i.e. for most of the cones defining proper efficiency.

References

[1] Clarke, F. H. (1990): *Optimization and Nonsmooth Analysis*. Classics in Applied Mathematics, 5. Reprint, SIAM.

[2] Jahn, J. (1986): *Mathematical vector optimization in partially ordered linear spaces*. Methoden und Verfahren der Mathematischen Physik [Methods and Procedures in Mathematical Physics], 31. Verlag Peter D. Lang, Frankfurt am Main.

[3] Kusraev, A. (1978): On necessary conditions for an extremum of nonsmooth vector-valued mappings. *Soviet Mathematics, Doklady 19*, 1057–1060. Transl. from Dokl. Akad. Nauk SSSR 242, 44-47 (1978).

[4] Papageorgiou, N. S. (1983): Nonsmooth analysis on partially ordered vector spaces. I. Convex case. *Pacific Journal of Mathematics 107 (2)*, 403–458.

[5] Papageorgiou, N. S. (1986): Nonsmooth analysis on partially ordered vector spaces: The subdifferential theory. *Nonlinear Analysis, Theory, Methods & Applications, Series A 10 (7)*, 615–637.

[6] Peressini, A. L. (1967): *Ordered topological vector spaces*. Harper & Row Publishers, New York.

[7] Reiland, T. W. (1989): Generalized differentiability for a class of nondifferentiable operators with applications to nonsmooth optimization. *Australian Mathematical Society, Series A 47 (1)*, 114–132.

[8] Reiland, T. W. (1992): Nonsmooth analysis and optimization on partially ordered vector spaces. *International Journal of Mathematics and Mathematical Sciences 15 (1)*, 65–81.

[9] Roberts, A. W. and Varberg, D. E. (1974): Another proof that convex functions are locally Lipschitz. *The American Mathematical Monthly 81*, 1014–1016.

[10] Staib, T. (1992): Necessary optimality conditions for nonsmooth multicriterial optimization problems. *SIAM Journal on Optimization 2 (1)*, 153–171.

[11] Thibault, L. (1980): Subdifferentials of compactly Lipschitzian vector-valued functions. *Annali di Matematica Pura ed Applicata, Serie 4, 125*, 157–192.

[12] Thibault, L. (1982a): On generalized differentials and subdifferentials of Lipschitz vector-valued functions. *Nonlinear Analysis, Theory, Methods & Applications, Series A 6 (10)*, 1037–1053.

[13] Thibault, L. (1982b): Subdifferentials of nonconvex vector-valued functions. *Journal of Mathematical Analysis and Applications 86 (2)*, 319–344.