The porous medium equation in a two-component domain

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Abstract. We consider a system of two porous medium equations defined on two different components of the real line, which are connected by the nonlinear contact condition

$$u_x = v_x \ , \qquad v = \psi(u) \qquad \text{on the contact line } S.$$

First we prove existence and uniqueness of a solution (u, v) on a bounded domain. Furthermore, we are interested in the behaviour of the interface of the porous medium equation when it crosses the contact line S between the two components. To this end we solve the Cauchy problem on unbounded components, consider self similar solutions for special $\psi(u) = Mu^{\omega}$ and derive a formula for the shape of the interface in that case.

1. Introduction

Let $0 < m, \sigma < 1$ be given. Consider the equations

$$(u^m)_t - u_{xx} = 0 \qquad -\ell < x < 0 \ , \ 0 < t < T, \tag{1.1}$$

$$(v^{\sigma})_t - v_{xx} = 0 \qquad 0 < x < \ell \ , \ 0 < t < T \tag{1.2}$$

for large $\ell > 0$, where nonnegative $u = u(x, t)$, $v = v(x, t)$ satisfy the following contact conditions for $x = 0$

$$u_x(0, t) = v_x(0, t) \qquad 0 < t < T, \tag{1.3}$$

$$v(0, t) = M\, u^{\omega}(0, t) \qquad 0 < t < T \tag{1.4}$$

for given $0 < M, \omega < \infty$, and boundary and initial conditions, respectively,

$$u_x(-\ell, t) = v_x(\ell, t) = 0 \qquad 0 < t < T \ , \tag{1.5}$$

$$u^m(x, 0) = \frac{1}{\lambda(0)} \left[1 - \left\{ \frac{x + 2\lambda(0)}{\lambda(0)} \right\}^2 \right]_+^{m/(1-m)} \qquad -\ell < x < 0 \ , \tag{1.6}$$

$$v^{\sigma}(x, 0) = 0 \qquad 0 \le x < \ell \ , \tag{1.7}$$

*Supported by the VEGA Grant Agency of the Slovak Republic No. 1/0260/03

where

$$\lambda(t) = \left\{ \frac{2(m+1)}{m(1-m)}(t+1) \right\}^{m/(1+m)} , \quad [\,\cdot\,]_+ = \max\{\cdot, 0\} \ .$$

It is well known, see [6], ([2]), that the equation (1.1) with the initial function (1.6) has the explicit solution

$$u(x,t) = \frac{1}{\lambda^{1/m}(t)} \left[1 - \left\{ \frac{x + 2\lambda(0)}{\lambda(t)} \right\}^2 \right]_+^{1/(1-m)} \tag{1.8}$$

and thus, u given by (1.8) and $v \equiv 0$ is the solution of our problem (1.1)–(1.7) until

$$\xi(t) = \lambda(t) - 2\lambda(0) \tag{1.9}$$

reaches 0, i.e. for $T = T^* \equiv 2^{(1+m)/m} - 1$. Let us formally define $\xi(t)$ for $t > T^*$ as $\xi(t) \equiv \sup\{x \in [0, \ell] : \ v(x,t) > 0\}$ and call it the (right hand) interface.

The main concern of this paper is to deal with the following questions:

– Is problem (1.1)–(1.7) well possed also for $T > T^*$?

– What is the behaviour of the interface $x = \xi(t)$ when it crosses $x = 0$?

We introduce a proper notion of a weak solution to Problem (1.1)–(1.7) and we prove the existence and comparison principle for arbitrarily large $0 < T < \infty$. To prove uniqueness we apply the method of *variable doubling*, a tool which has been introduced by Kruzkov [13], according to F. Otto [17]. This is done for more general initial functions and contact relations, respectively.

To study qualitative behaviour of the interface we restrict ourselves to a special value of the exponent ω in (1.4)

$$\omega \ = \ \frac{m+1}{\sigma+1}$$

and we set $\ell = \infty$. Thus we shall analyze the Cauchy problem with a particular choice of initial functions given by

$$u_0(x) = (-ax)^{1/(1-m)} \ , \ a > 0$$

for $x \leq 0$ and $v_0(x) = 0$ for any $x \geq 0$, which reflect the behaviour of the solution (1.8) in a neighbourhood of the interface $\xi(T^*)$ when it crosses $x = 0$.
We establish an existence result for the Cauchy problem, but we have not succeeded to prove its uniqueness yet.

Just the uniqueness is the principal part in proving the existence of a selfsimilar solution to the Cauchy problem. Assuming the uniqueness we prove that

$$u(x,t) = t^{1/(1-m)} w\left(\frac{x}{t}\right) , \qquad v(x,t) = t^{\omega/(1-m)} h\left(\frac{x}{t^\alpha}\right)$$

for ω given above and

$$\alpha = \frac{1 - m\sigma}{(1+\sigma)(1-m)} \ .$$

Here w, h are weak solutions of the following problem

$$\left.\begin{array}{ll} w''(x) + x(w^m(x))' - \frac{m}{1-m} w^m(x) = 0 & \text{for } x < 0 \ , \\ h''(x) + \alpha x(h^\sigma(x))' - \frac{\omega\sigma}{1-m} h^\sigma(x) = 0 & \text{for } x > 0 \ , \\ w'(0) = h'(0) \ , \quad h(0) = M w^\omega(0) \ , \quad w(x) \sim (-a\, x)^{1/(1-m)} & \text{as } \ x \to -\infty \ . \end{array}\right\}$$

We prove that there exists $\zeta > 0$ such that $h(x) = 0 \ \forall x \geq \zeta$ and $h(x) > 0$ for $x \in [0, \zeta)$. Hence

$$\xi(t) \ = \ \zeta t^\alpha \ .$$

Our study was motivated by the mathematical modelling of dermal and transdermal drug delivery [14]. The enthalpy formulation of a free boundary problem in a two-component domain in one space dimension reads in [14] as follows.

$$\begin{array}{ll} (b^-(u))_t - u_{xx} = 0 & -\ell < x < 0 \ , 0 < t < T, \\ (b^+(u))_t - v_{xx} = 0 & 0 < x < \ell \ , 0 < t < T \end{array}$$

with the contact conditions

$$u_x(0,t) = v_x(0,t) \ , \ v(0,t) = \psi(u(0,t)) \qquad 0 < t < T,$$

complemented by boundary and initial conditions as above. Here b^-, b^+ and ψ are supposed to consist of continuous monotone functions and a step function. We have tried to regularize the problem and we replaced b^-, b^+ and ψ by continuous functions. In order to preserve the property of the finite speed of propagation we have chosen the porous media type approximation and we arrived at our problem.

The reader is referred to the papers of D.G. Aronson [2]-[4] and J.L. Vázquez [18] for a wide source of references concerning the porous medium equation.

Let us finish this section by introducing some notation. We write u^m instead of $|u|^m \operatorname{sign} u$. To keep the notation short we set $Q^- = (-\ell, 0) \times (0,T)$, $Q^+ = (0,\ell) \times (0,T)$, $Q = Q^+ \cup Q^-$, $S = \{0\} \times (0,T)$, $\Omega^- = (-\ell, 0)$, $\Omega^+ = (0, \ell)$, $\Omega = \Omega^+ \cup \Omega^-$, and $L^\infty(Q) = L^\infty(Q^-) \times L^\infty(Q^+)$, $V = H^1(\Omega^-) \times H^1(\Omega^+)$ with its subspace $\tilde{V} = \{\varphi \in V : \varphi^-(0) = \varphi^+(0)\}$.

2. Problem on bounded components

Given a continuous and strongly monotone increasing function $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(0) = 0$, consider now

Problem (P)

$$
\begin{aligned}
&(u^m)_t = u_{xx} && \text{in } Q^-, && (2.1)\\
&(v^\sigma)_t = v_{xx} && \text{in } Q^+, && (2.2)\\
&u_x = v_x && \text{on } S, && (2.3)\\
&v = \psi(u) && \text{on } S, && (2.4)\\
&u_x(-\ell,\cdot) = v_x(\ell,\cdot) = 0 && \text{on } (0,T), && (2.5)\\
&u(\cdot,0) = u_0 && \text{on } \Omega^-, && (2.6)\\
&v(\cdot,0) = v_0 && \text{on } \Omega^+, && (2.7)
\end{aligned}
$$

where u_0 and v_0 are given nonnegative and bounded functions.

We refer to **Problem (P$_L$)** if conditions (2.3) and (2.4) on S are replaced by the approximation

$$u_x + L\big(\psi(u) - v\big) = 0\ , \qquad -v_x + L\big(v - \psi(u)\big) = 0 \quad \text{on } S. \tag{2.8}$$

This condition preserves (2.3), moreover it is expected that (2.4) is approximated as $L \to \infty$.

We continue this section by making precise the meaning of a solution of the problem (2.1)–(2.7).

Definition 2.1 *(a) A couple (u,v) is called subsolution of Problem (P) with initial data (u_0,v_0) if the following three conditions are fulfilled:*

(i) $(u,v) \in L^2(0,T;V) \cap L^\infty(Q)$*;*

(ii) the weak differential inequality

$$\int_{Q^-} \big((u_0^m - u^m)\varphi_t^- \ + \ u_x\varphi_x^-\big)\, dx\, dt \\ + \int_{Q^+} \big((v_0^\sigma - v^\sigma)\varphi_t^+ \ + \ v_x\varphi_x^+\big)\, dx\, dt + \int_S g(\varphi^+ - \varphi^-)dt \le 0 \tag{2.9}$$

holds for some $g \in L^2(0,T)$ and for all nonnegative $\varphi = (\varphi^-,\varphi^+) \in L^2(0,T;V)$ with $\varphi_t \in L^\infty(Q)$, $\varphi(\cdot,T) = 0$;

(iii) the contact condition

$$v = \psi(u) \tag{2.10}$$

is satisfied almost everywhere on S.

(b) A couple (u,v) is called supersolution of Problem (P) with initial data (u_0,v_0) if the following three conditions are fulfilled:

(i) $(u,v) \in L^2(0,T;V) \cap L^\infty(Q)$*;*

(ii) the weak differential inequality

$$\int_{Q^-} \left((u_0^m - u^m)\varphi_t^- \ + \ u_x\varphi_x^-\right) dx\, dt$$
$$+ \int_{Q^+} \left((v_0^\sigma - v^\sigma)\varphi_t^+ \ + \ v_x\varphi_x^+\right) dx\, dt + \int_S g(\varphi^+ - \varphi^-)dt \geq 0 \quad (2.11)$$

holds for some $g \in L^2(0,T)$ *and for all nonnegative* $\varphi = (\varphi^-, \varphi^+) \in L^2(0,T;V)$ *with* $\varphi_t \in L^\infty(Q)$, $\varphi(\cdot,T) = 0$;

(iii) the contact condition

$$v = \psi(u) \quad (2.10)$$

is satisfied almost everywhere on S.

(c) A couple (u,v) *is called solution of Problem (P) if* (u,v) *is both sub- and supersolution.*

Note, that condition (2.3) is already included in the definition of a weak solution if (2.9) and (2.11) are supposed to hold only for test functions with $\varphi^+(0,t) = \varphi^-(0,t)$ (cf. [11]). Then g disappears. Hence our definition requires some more regularity at $x = 0$, which is proven in Theorem 4.3. Especially, if u_x, v_x have a trace at $x = 0$, then (2.9) and (2.11) mean $u_x \leq g \leq v_x$ or $u_x \geq g \geq v_x$, respectively, on S. For a solution in that case the definition implies $g = u_x(0,\cdot) = v_x(0,\cdot) \in L^2(0,T)$. For a solution according to Definition 2.1 this holds in a weak sense:

Proposition 2.1 *There are equivalent:*

- *(a)* (u,v) *is a solution of Problem (P) in the sense of Definition 2.1.*
- *(b)* (u,v) *fulfils (i) and (iii) of Definition 2.1 and satisfies the relation*

$$\int_{Q^-} \left((u_0^m - u^m)\varphi_t^- \ + \ u_x\varphi_x^-\right) dx\, dt + \int_{Q^+} \left((v_0^\sigma - v^\sigma)\varphi_t^+ \ + \ v_x\varphi_x^+\right) dx\, dt = 0 \quad (2.12)$$

for all $\varphi = (\varphi^-, \varphi^+) \in L^2(0,T;\tilde{V})$ *with* $\varphi_t \in L^\infty(Q)$, $\varphi(\cdot,T) = 0$. *Moreover, there is a* $g \in L^2(0,T)$ *such that*

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_0^T \int_{-\delta}^0 u_x(x,t)\,\omega(x,t)\,dxdt = \int_0^T g(t)\,\omega(0,t)\,dt\,. \quad (2.13)$$

holds for all $\omega \in C_0^1(\mathbb{R} \times (-\infty,T))$.

Proof: (a) $\Rightarrow$ (b): Obviously, the integral on S in (ii) disappears if $\varphi \in L^2(0,T;\tilde{V})$. This yields (2.12). Furthermore, we obtain (2.13) if we test the relation for (u,v) with $\varphi = ((1-\chi_\delta)\omega\,,\,0)$ where $\chi_\delta(x) = \min\{\frac{1}{\delta}|x|, 1\}$ $(\delta > 0)$, and pass $\delta \to 0$.

(b) $\Rightarrow$ (a): For any $(\omega^-,\omega^+) \in C_0^1(\mathbb{R}\times(-\infty,T))^2$, $\varphi = \chi_\delta\,\omega$ is an admissible test function in relation (2.12), which yields

$$\left.\begin{aligned} &\int_{Q^-} \big((u_0^m - u^m)\chi_\delta\,\omega_t^- \;+\; u_x\omega_x^-\chi_\delta\big)\,dxdt \\ +&\int_{Q^+} \big((v_0^\sigma - v^\sigma)\chi_\delta\,\omega_t^+ \;+\; v_x\omega_x^+\chi_\delta\big)\,dx\,dt \end{aligned}\right\}\quad (I_\delta)$$

$$-\frac{1}{\delta}\int_0^T\int_{-\delta}^0 u_x(x,t)\omega^-(x,t)\,dxdt + \frac{1}{\delta}\int_0^T\int_0^\delta v_x(x,t)\omega^+(x,t)\,dxdt = 0\,. \qquad (2.14)$$

Let first $\omega^+ \in C_0^1(\mathbb{R}\times(-\infty,T))$ be given and choose some $\omega^- \in C_0^1(\mathbb{R}\times(-\infty,T))$ with $\omega^-(0,t) = \omega^+(0,t)$. Then by Lebesgue's theorem and relation (2.12) we have $(I_\delta)\to 0$ as $\delta\to 0$. By means of (2.13) this yields

$$\begin{aligned} \lim_{\delta\to 0}\frac{1}{\delta}\int_0^T\int_0^\delta v_x(x,t)\,\omega^+(x,t)\,dxdt &= \lim_{\delta\to 0}\frac{1}{\delta}\int_0^T\int_{-\delta}^0 u_x(x,t)\,\omega^-(x,t)\,dxdt \\ &= \int_0^T g(t)\,\omega^-(0,t)\,dt = \int_0^T g(t)\,\omega^+(0,t)\,dt\,. \quad (2.15)\end{aligned}$$

Take now ω^- and ω^+ which do not necessary coincide on S. Then properties (2.13) and (2.15) imply condition (ii) of Definition 2.1 by a limit process $\delta\to 0$ in (2.14), first for C^1-test functions, but by a density argument also for all test functions from the definition. □

We finish this section with the corresponding explanation of a solution of Problem (P_L).

Definition 2.2 *A couple (u,v) is called subsolution (supersolution, solution) of Problem (P_L), if (i) and (ii) of Definition 2.1 hold with $g = L\big(v(0,\cdot) - \psi(u(0,\cdot))\big)$.*

3. Uniqueness due to F. Otto

In the paper [11] the authors prove a comparison theorem for problems (P_L) and (P) with regularized data by means of solving a dual problem. It was not possible to extend this method to degenerated equations like porous medium with contact conditions (2.3), (2.4). Now we are able to prove comparison theorems for our problems (P) and (P_L) by the method of doubling of variables, which was introduced by S.N. Kružkov [13] for conservation laws and was developed by J. Carrillo [7],[8] and F. Otto [17] to prove comparison theorems and uniqueness for degenerate parabolic equations. The basis for our results is an adaption of the theorem of Felix Otto (cf. [17, formula (15)]) to our problem. Our contribution is to manage the contact conditions between the two components.

Consider the following two cases

(C1) Let (u_1, v_1) and (u_2, v_2) fulfil condition (i) of Definition 2.1 and the relation

$$\int_{Q^-} \Big((u_{i0}^m - u_i^m)\, \varphi_t^- + u_{ix} \varphi_x^- \Big) dxdt \\ + \int_{Q^+} \Big((v_{i0}^\sigma - v_i^\sigma)\, \varphi_t^+ + v_{ix} \varphi_x^+ \Big) dxdt + \int_S g_i\, (\varphi^+ - \varphi^-) dt \\ = \int_Q F_i \varphi\, dxdt + \int_S (f_i^+ \varphi^+ - f_i^- \varphi^-) dt \quad (3.1)$$

for all $\varphi = (\varphi^-, \varphi^+) \in L^2(0,T;V)$ with $\varphi_t \in L^\infty(Q)$, $\varphi(\cdot, T) = 0$, some $g \in L^2(0,T)$, and given functions $F_i \in L^\infty((0,T); L^1(\Omega))$, $f_i^\pm \in L^\infty(0,T)$, $i = 1, 2$.

(C2) Let (u_1, v_1) and (u_2, v_2) be sub- resp. supersolution of Problem (P_L) or (P) with initial data (u_{10}, v_{10}) resp. (u_{20}, v_{20}). Then we set $F_i \equiv 0$, $f_i \equiv 0$.

Case (C1) is needed in the proof of Lemma 4.1. In the following we use the notation

$$[w]_+ = \max\{w, 0\} \qquad \text{and} \qquad \mathrm{sign}^+(w) = \begin{cases} 1 & \text{for } w > 0 \\ 0 & \text{for } w \le 0 \end{cases}.$$

Theorem 3.1 *Assume (C1) or (C2). Then*

$$\int_{Q^-} \Big([u_{10}^m - u_{20}^m]_+ - [u_1^m - u_2^m]_+ \Big)\, \alpha'(t)\, dxdt + \int_{Q^+} \Big([v_{10}^\sigma - v_{20}^\sigma]_+ - [v_1^\sigma - v_2^\sigma]_+ \Big)\, \alpha'(t)\, dxdt \\ + \int_S (g_1 - g_2) \big(sign^+(v_1 - v_2) - sign^+(u_1 - u_2) \big)\, \alpha\, dt \\ \le \int_Q |F_1 - F_2|\, \alpha\, dxdt + \int_S \big(|f_1^- - f_2^-| + |f_1^+ - f_2^+| \big)\, \alpha\, dt \quad (3.2)$$

holds for all nonnegative $\alpha \in C_0^\infty((-\infty, T))$.

Proof: Since the proof in a wide range follows the proof of the theorem in Otto [17] we only sketch some ideas and pay attention especially on the additional items.

We start with some smooth nondecreasing approximation $\eta_\delta : \mathbb{R} \to \mathbb{R}$ of $[\cdot]_+$ and define

$$q_\delta^-(z, z_0) = \int_{z_0}^{z} \eta_\delta'(\zeta - z_0)\, d(\zeta^m) = \int_{z_0^m}^{z^m} \eta_\delta'(\xi^{1/m} - z_0)\, d\xi\,,$$

$$q_\delta^+(z, z_0) = \int_{z_0}^{z} \eta_\delta'(\zeta - z_0)\, d(\zeta^\sigma) = \int_{z_0^\sigma}^{z^\sigma} \eta_\delta'(\xi^{1/\sigma} - z_0)\, d\xi\,.$$

Let $(u, v) = (u_1, v_1)$ be a subsolution (case(C2)) with initial data (u_0, v_0) or fulfil relation (3.1) (case(C1)), and $(\bar{u}, \bar{v}) \in V$ be fixed. Then we test the relation with

$$\varphi^- = \eta_\delta'(u - \bar{u})\gamma^-\,, \qquad \varphi^+ = \eta_\delta'(v - \bar{v})\gamma^+\,,$$

where $\gamma^\pm \in C_0^\infty(\mathbb{R}\times(0,T))$, which has to be regularized with respect to t in order to be an admissible test function. Otto proves in [17, Lemma 1] the chain rule for

$$\langle \partial_t u^m \,,\, \eta_\delta'(u-\bar{u})\gamma^-\rangle + \langle \partial_t v^\sigma \,,\, \eta_\delta'(v-\bar{v})\gamma^+\rangle \,,$$

which leads to

$$\begin{aligned}
&\int_{Q^-} \Big(- q_\delta^-(u,\bar{u})\gamma_t^- + u_x \big(\eta_\delta'(u-\bar{u})\gamma^-\big)_x \Big)\, dxdt \\
+ &\int_{Q^+} \Big(- q_\delta^+(v,\bar{v})\gamma_t^+ + v_x \big(\eta_\delta'(v-\bar{v})\gamma^+\big)_x \Big)\, dxdt \\
+ &\int_{S} g_1 \Big(\eta_\delta'(v-\bar{v})\gamma^+ - \eta_\delta'(u-\bar{u})\gamma^-\Big)\, dt \\
\leq &\int_{Q^-} F_1^-\, \eta_\delta'(u-\bar{u})\gamma^-\, dxdt + \int_{Q^+} F_1^+\, \eta_\delta'(v-\bar{v})\gamma^+\, dxdt \\
+ &\int_{S} \Big(f_1^+\, \eta_\delta'(v-\bar{v})\gamma^+ - f_1^-\, \eta_\delta'(u-\bar{u})\gamma^- \Big)\, dt
\end{aligned} \tag{3.3}$$

for all nonnegative $\gamma^\pm \in C_0^\infty(\mathbb{R}\times(0,T))$. An similar inequality holds for a supersolution.

Now we carry out the doubling of time variable. Let t_1 denote the time variable of (u_1, v_1) and t_2 denote the time variable of (u_2, v_2), then we extend (u_1, v_1) and (u_2, v_2) to $\tilde{Q}^\pm = \Omega^\pm \times (0,T)^2$ by $(u_i(x,t_1,t_2), v_i(x,t_1,t_2)) = (u_i(x,t_i), v_i(x,t_i))$, $i = 1,2$. Then we insert $(\bar{u},\bar{v}) = (u_2(\cdot,t_2), v_2(\cdot,t_2))$ with $\gamma = \gamma(x,t_1,t_2)$ into (3.3), and integrate the inequality over $t_2 \in (0,T)$. The same is done with the inequality corresponding to (3.3) for (u_2, v_2): set $(\bar{u},\bar{v}) = (u_1(\cdot,t_1), v_1(\cdot,t_1))$, $\gamma = \gamma(x,t_1,t_2)$, and integrate over $t_1 \in (0,T)$. Taking the difference and a space-independent test function $\gamma(x,t_1,t_2) = \gamma(t_1,t_2)$ we arrive at

$$\begin{aligned}
&\int_{\tilde{Q}^-} -\big(q_\delta^-(u_1,u_2)\,\gamma_{t_1} + \tilde{q}_\delta^-(u_2,u_1)\,\gamma_{t_2}\big)\, dxdt_1dt_2 \\
+ &\int_{\tilde{Q}^+} -\big(q_\delta^+(v_1,v_2)\,\gamma_{t_1} + \tilde{q}_\delta^+(v_2,v_1)\,\gamma_{t_2}\big)\, dxdt_1dt_2 \\
+ &\int_{\tilde{Q}^-} (u_1-u_2)_x\, \eta_\delta'(u_1-u_2)_x\, \gamma\, dxdt_1dt_2 + \int_{\tilde{Q}^+} (v_1-v_2)_x\, \eta_\delta'(v_1-v_2)_x\, \gamma\, dxdt_1dt_2 \\
+ &\int_{S^2} (g_1-g_2)\big(\eta_\delta'(v_1-v_2) - \eta_\delta'(u_1-u_2)\big)\,\gamma\, dt_1dt_2 \\
\leq &\int_{\tilde{Q}^-} (F_1^- - F_2^-)\, \eta_\delta'(u_1-u_2)\,\gamma\, dxdt_1dt_2 + \int_{\tilde{Q}^+} (F_1^+ - F_2^+)\, \eta_\delta'(v_1-v_2)\,\gamma\, dxdt_1dt_2 \\
+ &\int_{S^2} \Big((f_1^+ - f_2^+)\, \eta_\delta'(v_1-v_2) - (f_1^- - f_2^-)\, \eta_\delta'(u_1-u_2) \Big)\,\gamma\, dt_1dt_2,
\end{aligned}$$

where $\tilde{q}_\delta(z,z_0)$ is defined like $q_\delta(z,z_0)$ with $\tilde{\eta}_\delta(w) = \eta_\delta(-w)$ instead of η_δ. The items containing second order derivatives of η_δ are nonnegative and can be omitted. Next we want to let $\delta \to 0$. Then

$$\eta'_\delta(z_1 - z_2) \to \text{sign}^+(z_1 - z_2)$$

and

$$q_\delta^-(u_1,u_2),\ \tilde{q}_\delta^-(u_2,u_1) \to [u_1^m - u_2^m]_+ \ , \qquad q_\delta^+(v_1,v_2),\ \tilde{q}_\delta^+(v_2,v_1) \to [v_1^\sigma - v_2^\sigma]_+ \ ,$$

hence we obtain the analogue to [17, formula (36)],

$$\begin{aligned}
&\int_{\tilde{Q}^-} -[u_1(x,t_1)^m - u_2(x,t_2)^m]_+ (\gamma_{t_1} + \gamma_{t_2})\, dx\, dt_1 dt_2 \\
&+ \int_{\tilde{Q}^+} -[v_1(x,t_1)^\sigma - v_2(x,t_2)^\sigma]_+ (\gamma_{t_1} + \gamma_{t_2})\, dx\, dt_1 dt_2 \\
&+ \int_{S^2} (g_1(t_1) - g_2(t_2))\big(\text{sign}^+(v_1 - v_2) - \text{sign}^+(u_1 - u_2)\big)\, \gamma\, dt_1 dt_2 \\
&\le \int_{\tilde{Q}} |F_1(x,t_1) - F_2(x,t_2)|\, \gamma\, dx\, dt_1 dt_2 \\
&+ \int_{S^2} \Big(|f_1^+(t_1) - f_2^+(t_2)| + |f_1^-(t_1) - f_2^-(t_2)|\Big)\, \gamma\, dt_1 dt_2 \ .
\end{aligned}$$

The last step is to choose nonnegative $\alpha \in C_0^\infty((-\infty,T))$ and $\phi \in C_0^\infty(\mathbb{R})$ with unit mass and insert

$$\gamma(t_1,t_2) = \frac{1}{\varepsilon}\, \phi\left(\frac{t_1 - t_2}{\varepsilon}\right) \alpha\left(\frac{t_1 + t_2}{2}\right)$$

into the above inequality. In order to pass to the limit $\varepsilon \to 0$ it is appropriate to substitute $\tau = t_1 - t_2$. Since all items are bounded with respect to t and the shift operator $S_\tau w(t) = w(t-\tau)$ is continuous in $L^p(0,T)$, $1 \le p < \infty$, we have no problems to send $\tau \to 0$. Thus, it is straightforward now to come to (3.2). □

Now we are able to prove comparison results and L_1-contraction for problems (P_L) and (P).

Theorem 3.2 *(i) Let (u_1,v_1) and (u_2,v_2) be weak solutions of Problem (P_L) or Problem (P), resp., with initial functions (u_{10},v_{10}) and (u_{20},v_{20}), respectively. Then for almost all $t \in [0,T]$,*

$$\begin{aligned}
&\int_{\Omega^-} |u_1(x,t)^m - u_2(x,t)^m|\, dx + \int_{\Omega^+} |v_1(x,t)^\sigma - v_2(x,t)^\sigma|\, dx \\
&\qquad \le \int_{\Omega^-} |u_{10}(x)^m - u_{20}(x)^m|\, dx + \int_{\Omega^+} |v_{10}(x)^\sigma - v_{20}(x)^\sigma|\, dx. \qquad (3.4)
\end{aligned}$$

(ii) Let $(\underline{u},\underline{v})$ be a subsolution and $(\overline{u},\overline{v})$ a supersolution of Problem (P_L) or Problem (P), resp., with initial data $(\underline{u}_0,\underline{v}_0)$ and $(\overline{u}_0,\overline{v}_0)$, respectively. Then if $\underline{u}_0 \le \overline{u}_0$ on Ω^- and $\underline{v}_0 \le \overline{v}_0$ on Ω^+ it follows that

$$\underline{u}(x,t) \le \overline{u}(x,t) \quad \text{a.e. on } Q^- \ , \qquad \underline{v}(x,t) \le \overline{v}(x,t) \quad \text{a.e. on } Q^+. \qquad (3.5)$$

Proof: (ii): The crucial point to prove the result by means of Theorem 3.1 is the conclusion that the integral on S on the left hand side of (3.2) is nonnegative and can be omitted. Indeed, if $(\underline{u},\underline{v})$ and $\overline{u},\overline{v})$ are sub- resp. supersolutions of Problem (P), then it satisfies (2.10), hence

$$\mathrm{sign}^+(\underline{v}-\overline{v}) - \mathrm{sign}^+(\underline{u}-\overline{u}) = \mathrm{sign}^+(\psi(\underline{u})-\psi(\overline{u})) - \mathrm{sign}^+(\underline{u}-\overline{u}) = 0 \quad \text{on } S$$

because of the monotonicity of ψ, and the corresponding integral disappears. Otherwise, if $(\underline{u},\underline{v})$ and $\overline{u},\overline{v})$ are sub- resp. supersolutions of Problem (P$_L$), due to Definition 2.2 the corresponding integral has the form

$$\int_S L\big((\underline{v}-\psi(\underline{u})) - (\overline{v}-\psi(\overline{u})\big)\big(\mathrm{sign}^+(\underline{v}-\overline{v}) - \mathrm{sign}^+(\underline{u}-\overline{u})\big)\,\alpha\,dt$$
$$= \int_S L\big((\underline{v}-\overline{v})) - (\psi(\underline{u}-\psi(\overline{u})\big)\big(\mathrm{sign}^+(\underline{v}-\overline{v}) - \mathrm{sign}^+(\underline{u}-\overline{u})\big)\,\alpha\,dt.$$

It is easy to check that this integral is nonnegative for nonnegative α because of monotonicity of ψ again. Hence, since $F_i \equiv 0$, $f_i \equiv 0$, Theorem 3.1 yields

$$\int_{Q^-}\Big([\underline{u}_0^m - \overline{u}_0^m]_+ - [\underline{u}^m - \overline{u}^m]_+\Big)\,\alpha'(t)\,dxdt + \int_{Q^+}\Big([\underline{v}_0^\sigma - \overline{v}_0^\sigma]_+ - [\underline{v}^\sigma - \overline{v}^\sigma]_+\Big)\,\alpha'(t)\,dxdt \le 0$$

for all nonnegative $\alpha \in C_0^\infty((-\infty,T))$ or, equivalently,

$$-\int_0^T\Big(\int_{\Omega^-}[\underline{u}^m - \overline{u}^m]_+\,dx + \int_{\Omega^+}[\underline{v}^\sigma - \overline{v}^\sigma]_+\,dx\Big)\,\alpha'(t)\,dt$$
$$\le \Big(\int_{\Omega^-}[\underline{u}_0^m - \overline{u}_0^m]_+\,dx + \int_{\Omega^+}[\underline{v}_0^\sigma - \overline{v}_0^\sigma]_+\,dx\Big)\,\alpha(0).$$

Testing now with a smooth approximation of $\alpha_\delta(t) = \min\{1, \frac{1}{\delta}(T-t)\}$ and passing $\delta \to 0$, for almost every $t \in [0,T]$ we obtain

$$\int_{\Omega^-}[\underline{u}(x,t)^m - \overline{u}(x,t)^m]_+\,dx + \int_{\Omega^+}[\underline{v}(x,t)^\sigma - \overline{v}(x,t)^\sigma]_+\,dx$$
$$\le \int_{\Omega^-}[\underline{u}_0^m - \overline{u}_0^m]_+\,dx + \int_{\Omega^+}[\underline{v}_0^\sigma - \overline{v}_0^\sigma]_+\,dx. \quad (3.6)$$

Since the right hand side vanishes this proves (ii).

(i) is now an easy consequence of (3.6). □

Corollary 3.1 *Let (u,v) be a weak solutions of Problem (P$_L$) or Problem (P), resp., with initial values (u_0,v_0) and c^-, c^+ be nonnegative constants with $c^+ = \psi(c^-)$. Then*

$$0 \le u_0 \le c^-\ ,\ 0 \le v_0 \le c^+ \qquad \textit{implies} \qquad 0 \le u \le c^-\ ,\ 0 \le v \le c^+ \qquad \textit{a.e. in } Q.$$

Clearly, $(0,0)$ is a subsolution and (c^-,c^+) is a supersolution.

4. Existence

Existence of solutions to more general nonlinear parabolic problems in a multi-component domain with contact condition (2.3),(2.4) is investigated by the authors in their paper [11]. However, there is a restriction $0 < \kappa \le \psi' \le K$ which we want to overcome in order to deal with the special $\psi(u) = Mu^\omega$ from the introduction. This special kind of ψ is needed for the rescaling method in the last section. Moreover, the assumptions in [11] on the parabolic nonlinearity $b(u)_t$, despite including free boundary problems, do not really cover the case $b(u) = u^m$. Hence, we have to introduce some new ideas, but in general we follow the concept of [11] to prove existence of solutions to our problems.

We start with Problem (P_L). This problem approximates the contact condition of Problem (P) and yields a solution of Problem (P) as $L \to \infty$. Therefore, in the proof of the following theorem we have to care that the bounds of the a priori estimates are independent on L. Although it is irrelevant for the proof we restrict oneself to nonnegative solutions.

Theorem 4.1 *Let (u_0, v_0) be nonnegative and bounded. Then there is a solution (u, v) of Problem (P_L) in the sense of Definition 2.2.*

Proof: Step 1: We regularize (P_L). For given $0 < \varepsilon \ll 1$ let $b_\varepsilon^-(\cdot)$, $b_\varepsilon^+(\cdot)$ and $\psi_\varepsilon(\cdot)$ are monotone increasing functions, a.e. differentiable and

$$0 < \varepsilon \le (b_\varepsilon^\pm)', (\psi_\varepsilon)' \le K_\varepsilon$$

for a positive constant K_ε, $b_\varepsilon^\pm(0) = \psi_\varepsilon^i(0) = 0$. Moreover, let

$$b_\varepsilon^-(u) \longrightarrow u^m\ , \quad b_\varepsilon^+(v) \longrightarrow v^\sigma\ , \quad \psi_\varepsilon(u) \longrightarrow \psi(u) \qquad \text{as } \varepsilon \to 0$$

uniformly on compact subsets of $\mathbb{R}$. The corresponding problem replacing u^m, v^σ and $\psi(u)$ by $b_\varepsilon^-(u)$, $b_\varepsilon^+(v)$ and $\psi_\varepsilon(u)$, respectively, with initial data $(u_{0\varepsilon}, v_{0\varepsilon}) \in V$ we denote by Problem (P_L^ε). Existence of a bounded nonnegative solution $(u_\varepsilon, v_\varepsilon)$ to (P_L^ε) is proved in [11, Theorem 3.2]. The bounds are uniform with respect to ε since a comparison result such as Theorem 3.2 and the resulting Corollary 3.1 may also be derived for Problem (P_L^ε). In the next step we derive estimates for the limit process $\varepsilon \to 0$.

Step 2: For simplicity we omit the ε and write (u, v) instead of $(u_\varepsilon, v_\varepsilon)$ again. We indicate dependence on ε and L whenever it is important. First we define special test functions

$$\phi_l^-(u) = \begin{cases} u & \text{for } l = 1 \\ \psi_\varepsilon(u) & \text{for } l = 2 \end{cases} \qquad \text{and} \qquad \phi_l^+(v) = \begin{cases} \psi_\varepsilon^{-1}(v) & \text{for } l = 1 \\ v & \text{for } l = 2\ . \end{cases}$$

This definition of $\phi_l^\pm$ is chosen in a way that

$$\phi_l^-(u) = \phi_l^+(\psi_\varepsilon(u)) \tag{4.1}$$

for both $l = 1, 2$. Moreover, the items

$$G_l^{\pm}(s) = \int_0^s \phi_l^{\pm}\big((b_\varepsilon^{\pm})^{-1}(r)\big)\, dr$$

fulfil $G_l'(b(u)) = \phi_l(u)$. This leads to the chain rule (see Carrillo [8, Lemma 4])

$$\int_\Omega G_l(b_\varepsilon(u(x,t)))\, dx - \int_\Omega G_l(b_\varepsilon(u_0))\, dx = \int_0^t \langle b_\varepsilon(u)_t, \phi_l(u)\rangle d\tau$$

for almost every $t \in [0, T]$, where $\langle \cdot, \cdot \rangle$ denotes the duality between V^* and V. Testing now the weak relation to $(\mathrm{P}_L^\varepsilon)$ with ϕ_l, $l = 1, 2$, we obtain

$$\begin{aligned}
&\int_{\Omega^-} G_l^-(b_\varepsilon^-(u(x,t)))\, dx + \int_{\Omega^+} G_l^+(b_\varepsilon^+(v(x,t)))\, dx \\
&- \int_{\Omega^-} G_l^-(b_\varepsilon^-(u_0))\, dx - \int_{\Omega^+} G_l^+(b_\varepsilon^+(v_0))\, dx \\
&+ \int_0^t \int_{\Omega^-} \phi_l^-(u)_x\, u_x\, dx d\tau + \int_0^t \int_{\Omega^+} \phi_l^+(v)_x\, v_x\, dx d\tau \\
&+ L \int_0^t \big(v - \psi_\varepsilon(u)\big)\big(\phi_l^+(v) - \phi_l^-(u)\big)(0, \tau)\, d\tau = 0.
\end{aligned} \tag{4.2}$$

The two integrals on the first line are nonnegative and can be omitted, the integrals on the second line are uniformly bounded with respect to ε. Fix now $l = 1$. In view of monotonicity of ϕ_1^+ and (4.1) the items

$$\phi_1^+(v)_x\, v_x \quad \text{and} \quad \big(v - \psi_\varepsilon(u)\big)\big(\phi_1^+(v) - \phi_1^-(u)\big) = \big(v - \psi_\varepsilon(u)\big)\big(\phi_1^+(v) - \phi_1^+(\psi_\varepsilon(u))\big)$$

are nonnegative. The remaining integral of (4.2) then yields

$$\int_{Q^-} \big((u_\varepsilon)_x\big)^2\, dxdt \le C. \tag{4.3}$$

Fixing $l = 2$ in (4.2) in the same way we obtain the estimates

$$\int_{Q^+} \big((v_\varepsilon)_x\big)^2\, dxdt \le C, \tag{4.4}$$

$$\int_S \big(v_\varepsilon - \psi_\varepsilon(u_\varepsilon)\big)^2\, dt \le \frac{C}{L}, \tag{4.5}$$

with constants independent of ε and L.

Step 3: Before we can go to the limit $\varepsilon \to 0$ we need some compactness with respect to t. Here we follow the concept of Alt, Luckhaus [1]. Testing $(\mathrm{P}_L^\varepsilon)$ for fixed

$t \in [0, T-h]$, $h > 0$ with $(\chi_{[t,t+h]}(u(x,t+h) - u(x,t))\,,\,\chi_{[t,t+h]}(v(x,t+h) - v(x,t)))$, where $\chi_{[t,t+h]}$ is the characteristic function on $[t, t+h]$, we obtain

$$\int_0^{T-h}\int_{\Omega^-} \big(b_\varepsilon^-(u_\varepsilon(x,t+h)) - b_\varepsilon^-(u_\varepsilon(x,t))\big)\big(u_\varepsilon(x,t+h) - u_\varepsilon(x,t)\big)\,dxdt + \\ \int_0^{T-h}\int_{\Omega^+} \big(b_\varepsilon^+(v_\varepsilon(x,t+h)) - b_\varepsilon^+(v_\varepsilon(x,t))\big)\big(v_\varepsilon(x,t+h) - v_\varepsilon(x,t)\big)\,dxdt \le C(L)\,h \tag{4.6}$$

(cf. [1, Theorem 1.7] or [11, Lemma 4.3]). The item on S is treated there like the elliptic part using uniform boundedness w.r.t. ε of $(u_\varepsilon, v_\varepsilon)$. We intend to apply Lemma 1.9 of [1], however, in our case the b_ε in (4.6) depend on ε, too. Note therefore, that for $0 < m, \sigma < 1$

$$|u_1^m - u_2^m|^{\frac{m+1}{m}} \le (u_1^m - u_2^m)(u_1 - u_2) \quad \text{and} \quad |v_1^\sigma - v_2^\sigma|^{\frac{\sigma+1}{\sigma}} \le (v_1^\sigma - v_2^\sigma)(v_1 - v_2).$$

If the regularization b_ε is chosen in such a way that $|b_\varepsilon^-(u_1) - b_\varepsilon^-(u_2)| \le |u_1^m - u_2^m|$ and $|b_\varepsilon^+(v_1) - b_\varepsilon^+(v_2)| \le |v_1^\sigma - v_2^\sigma|$, this property is preserved with $b_\varepsilon^-(u)$ and $b_\varepsilon^+(v)$ instead of u^m and v^σ, respectively. Then from (4.6) follows

$$\left(\int_0^{T-h}\int_{\Omega^-} \big|b_\varepsilon^-(u_\varepsilon(x,t+h)) - b_\varepsilon^-(u_\varepsilon(x,t))\big|\,dxdt\right)^{\frac{m+1}{m}} \\ + \left(\int_0^{T-h}\int_{\Omega^+} \big|b_\varepsilon^+(v_\varepsilon(x,t+h)) - b_\varepsilon^+(v_\varepsilon(x,t))\big|\,dxdt\right)^{\frac{\sigma+1}{\sigma}} \le C(L)\,h \tag{4.7}$$

with constant $C(L)$ independent of ε. This inequality replaces [1, Lemma 1.8] which may not be uniform w.r.t. ε if $b = b_\varepsilon$. Now the ideas of [1, Lemma 1.9] are applicable, which in view of uniform boundedness in $L^\infty(Q)$, together with (4.3) and (4.4), even yield

$$(u_\varepsilon\,,\,v_\varepsilon) \rightharpoonup (u, v) \qquad \text{in } L^2(0,T;V), \tag{4.8}$$

$$(b_\varepsilon^-(u_\varepsilon)\,,\,b_\varepsilon^+(v_\varepsilon)) \to (u^m, v^\sigma) \qquad \text{in } L^p(Q)\,,\ 1 \le p < \infty, \tag{4.9}$$

for a subsequence $\varepsilon \to 0$ where L remains fixed. Since (4.9) also implies strong convergence of $(u_\varepsilon, v_\varepsilon) \to (u, v)$ in $L^p(Q)$, by interpolation we obtain strong convergence in $L_2(S)$, too. Now it is not difficult to verify that (u, v) is a solution of Problem (P_L) in the sense of Definition 2.2, which concludes the proof. □

Our aim is now to let $L \to \infty$. While the estimates in step 2 of the proof of Theorem 4.1 are independent of L, i.e. these estimates hold uniformly for all L, the estimate in step 3 is useless for this aim. If the initial data have some more regularity, however, we are able to derive uniform Lipschitz continuity of (u^m, v^σ) w.r.t. t in $L^1(\Omega)$.

Lemma 4.1 *Suppose $(u_0, v_0) \in V$ possessing second order derivatives $((u_0)_{xx}, (v_0)_{xx}) \in L^1(\Omega)$. Then the appropriate solution (u, v) of Problem (P_L) for a.a. $t_1, t_2 \in [0, T]$ satisfies*

$$\int_{\Omega^-} |u(x,t_1)^m - u(x,t_2)^m|\, dx + \int_{\Omega^+} |v(x,t_1)^\sigma - v(x,t_2)^\sigma|\, dx \le C\, |t_1 - t_2|. \qquad (4.10)$$

If, additionally, $v_0(0) = \psi(u_0(0))$ then the constant C is independent of L.

Proof: For given $h > 0$ we define

$$(u_1, v_1) = (u, v)\ , \qquad (u_2(\cdot,t), v_2(\cdot,t)) = \begin{cases} (u(\cdot, t-h), v(\cdot, t-h)) & \text{for } t \in [h, T] \\ (u_0, v_0) & \text{for } t \in [0, h]. \end{cases}$$

Obviously, (u_2, v_2) is a solution of (P_L) in $\Omega \times (h, T)$. We consider the most important case $v_0 = \psi(u_0)$ and extend the equation for (u, v) to $\Omega \times (0, T)$,

$$\begin{aligned}
&\int_{Q^-} \big((u_0^m - u_2^m)\varphi_t^- \ +\ (u_2)_x \varphi_x^-\big)\, dx\, dt \\
&\quad + \int_{Q^+} \big((v_0^\sigma - v_2^\sigma)\varphi_t^+ \ +\ (v_2)_x \varphi_x^+\big)\, dx\, dt + L \int_S \big(v_2 - \psi(u_2)\big)(\varphi^+ - \varphi^-)\, dt \\
&= \int_0^h \left(\int_{\Omega^-} (u_0)_x \varphi_x^-\, dx + \int_{\Omega^+} (v_0)_x \varphi_x^+\, dx + L\big(v_0 - \psi(u_0)\big)(\varphi^+ - \varphi^-)\big|_S \right) dt \\
&= -\int_0^h \left(\int_{\Omega^-} (u_0)_{xx} \varphi^-\, dx + \int_{\Omega^+} (v_0)_{xx} \varphi^+\, dx + \big((v_0)_x \varphi^+ - (u_0)_x \varphi^-\big)\big|_S \right) dt \\
&= \int_Q F_2\, \varphi\, dx dt + \int_S (f_2^+ \varphi^+ - f_2^- \varphi^-)\, dt,
\end{aligned}$$

where

$$\begin{aligned}
F_2^-(x,t) &= -\chi_{[0,h]}(t)\, u_0''(x), \qquad & f_2^-(t) &= -\chi_{[0,h]}(t)\, u_0'(0), \\
F_2^+(x,t) &= -\chi_{[0,h]}(t)\, v_0''(x), \qquad & f_2^+(t) &= -\chi_{[0,h]}(t)\, v_0'(0).
\end{aligned}$$

Otherwise, if $v_0 \neq \psi(u_0)$, we get an additional item for $f_2^\pm$. The above relation is just (3.1). (u_1, v_1) fulfils an analogue relation with same initial function but $F_1 \equiv 0$, $f_1^- = f_1^+ \equiv 0$. Now we apply Theorem 3.1 Note that $(g_1 - g_2)\big(\operatorname{sign}^+(v_1 - v_2) - \operatorname{sign}^+(u_1 - u_2)\big)$ on the left hand side of (3.2) remains nonnegative for $g_i = v_i - \psi(u_i)$, $i = 1, 2$. Then Theorem 3.1 yields an weak Gronwall inequality which implies

$$\begin{aligned}
&\int_{\Omega^-} [u_1(x,t)^m - u_2(x,t)^m]_+\, dx + \int_{\Omega^+} [v_1(x,t)^\sigma - v_2(x,t)^\sigma]_+\, dx \\
&\qquad\qquad \le \int_0^t \left(\int_\Omega |F_2|\, dx + \big(|f_2^-| + |f_2^+|\big) \right) dt \le C\, h
\end{aligned}$$

for a.e. $t \in [0, T]$. Interchanging (u_1, v_1) and (u_2, v_2) and remember of its definition we arrive at (4.10). □

Theorem 4.2 *Let (u_0, v_0) be nonnegative and bounded. Then there is a solution (u, v) of Problem (P) fulfilling (i) and (iii) of Definition 2.1 and relation (2.12) of Proposition 2.1.*

Proof: First we regularize the initial data by smooth functions $(u_{0\delta}, v_{0\delta})$, uniformly bounded w.r.t. δ and possessing the regularity supposed in Lemma 4.1 including the compatibility condition $v_{0\delta} = \psi(u_{0\delta})$ on S. Of course, this does not provide any restriction of our original initial data.

Now we start with a solution $(u_{\delta L}, v_{\delta L})$ of Problem (P_L) with initial data $(u_{0\delta}, v_{0\delta})$ which satisfies

$$\int_{Q^-} \big((u_{0\delta}^m - u_{\delta L}^m)\varphi_t^- \;+\; (u_{\delta L})_x \varphi_x^-\big)\, dx\, dt + \int_{Q^+} \big((v_{0\delta}^\sigma - v_{\delta L}^\sigma)\varphi_t^+ \;+\; (v_{\delta L})_x \varphi_x^+\big)\, dx\, dt + L \int_S \big(v_{\delta L} - \psi(u_{\delta L})\big)(\varphi^+ - \varphi^-) dt = 0 \quad (4.11)$$

for all $\varphi \in L^2(0, T; V)$ with $\varphi_t \in L^\infty(Q)$, $\varphi(\cdot, T) = 0$. Due to Corollary 3.1 these solutions are uniformly bounded with respect to δ and L. Furthermore, remember that the constants in (4.3)-(4.5) only depend on the L^∞-bound of the initial data. Hence $(u_{\delta L}, v_{\delta L})$ fulfil the estimates (4.3)-(4.5) again for each $L > 0$, and the estimate of Lemma 4.1 together with boundedness yields

$$\begin{aligned} &(u_{\delta L}, v_{\delta L}) \rightharpoonup (u_\delta, v_\delta) && \text{in } L^2(0, T; V), \\ &((u_{\delta L})^m, (v_{\delta L})^\sigma) \to (u_\delta^m, v_\delta^\sigma) && \text{in } L^p(Q)\ ,\ 1 \le p < \infty, \end{aligned}$$

for a subsequence as $L \to \infty$. If we use test functions $\varphi \in L^2(0, T; \tilde{V})$, i.e. $\varphi^- = \varphi^+$ on S, the integral on S disappears in relation (4.11), hence by means of a limit process $L \to \infty$ in (4.11) we see that (u_δ, v_δ) fulfils relation (2.12). Moreover, in view of (4.5) it satisfies the contact condition (2.10). Thus we have proved the theorem for the regularized initial values.

It remains to overcome the regularization. Consider a sequence

$$(u_{0\delta_n}, v_{0\delta_n}) \to (u_0, v_0) \quad \text{in } L^\infty(\Omega) \qquad \text{as } \delta_n \to 0.$$

By Theorem 3.2 the associated $(u_{\delta_n L}, v_{\delta_n L})$ have L^1-contraction property (3.4). Integrating (3.4) over $t \in [0, T]$, due to the above convergence property as $L \to \infty$ we obtain

$$\int_{Q^-} |(u_{\delta_k})^m - (u_{\delta_l})^m|\, dxdt + \int_{Q^+} |(v_{\delta_k})^\sigma - (v_{\delta_l})^\sigma|\, dxdt \le T \int_{\Omega^-} |(u_{0\delta_k})^m - (u_{0\delta_l})^m|\, dx + T \int_{\Omega^+} |(v_{0\delta_k})^\sigma - (v_{0\delta_l})^\sigma|\, dx$$

for the solutions of Problem (P). Observing the uniform a priori estimates (4.3) and (4.5) as well as uniform boundedness again, this yields

$$\begin{aligned} (u_\delta, v_\delta) &\rightharpoonup (u,v) && \text{in } L^2(0,T;V), \\ (u_\delta^m, v_\delta^\sigma) &\to (u^m, v^\sigma) && \text{in } L^p(Q)\ ,\ 1 \le p < \infty, \end{aligned}$$

if $\delta \to 0$. Since (u_δ, v_δ) fulfils (2.10) and (2.12) it does the limit (u,v), too. □

Note that in Theorem 4.2 we have proved existence of a solution to Problem (P) in a weaker sense than proposed in Definition 2.1. However, the comparison results of Section 3 only hold for such slightly stronger solutions. Therefore it arises the question whether there is also a solution according to Definition 2.1. The answer is 'yes' if we assume the regularity of the initial data from Lemma 4.1.

Theorem 4.3 *Assume that (u_0, v_0) has the regularity supposed in Lemma 4.1 and fulfils the compatibility condition $v_0(0) = \psi(u_0(0))$. Then there is a solution of Problem (P) according to Definition 2.1.*

Proof: Let (u,v) be the solution according to Theorem 4.2. The assertion is proved if we show property (2.13) of Proposition 2.1 (a). Notice first that there is a function $g \in L^\infty(0,T)$ such that

$$\int_0^T \int_{\Omega^+} \big(v_0(x)^\sigma - v(x,t)^\sigma\big) dx\, \alpha'(t)\, dt + \int_0^T g(t)\alpha(t)\, dt = 0 \tag{4.12}$$

for all $\alpha \in C_0^1(-\infty, T)$. Indeed, for every $h > 0$ the approximating solutions (u_L, v_L) of problem (P_L) fulfil the identity

$$\int_0^T \int_{\Omega^+} \big(v_0(x)^\sigma - v_L(x,t)^\sigma\big) dx\, \frac{\alpha(t+h) - \alpha(t)}{h}\, dt \\ + \int_0^T \underbrace{\int_{\Omega^+} \frac{v_L(x,t-h)^\sigma - v_L(x,t)^\sigma}{h}\, dx}_{=:[v_L^\sigma]_h(t)}\, \alpha(t)\, dt = 0.$$

Due to Lemma 4.1 the item $[v_L^\sigma]_h$ is uniformly bounded with respect to h and L in $L^\infty(0,T)$. If now $h = \frac{1}{L} \to 0$ there is a subsequence with

$$[v_L^\sigma]_h \rightharpoonup g \quad \text{weak}^* \text{ in } L^\infty(0,T),$$

which yields (4.12).

Let now $\omega \in C_0^1(\mathbb{R} \times (-\infty, T))$ be given and $\chi_\delta(x) = \max\{\min\{\frac{1}{\delta}(x+\delta)\,,\,1\}\}$. Then we test relation (2.12) with $\varphi^- = \chi_\delta(x)\omega(x,t)$, $\varphi^+ = \omega(0,t)$ and obtain

$$\int_{Q^-} \Big(\big(u_0^m - u^m\big)\,\omega_t + u_x \omega_x\Big) \chi_\delta(x)\, dxdt \\ + \frac{1}{\delta}\int_0^T \int_{-\delta}^0 u_x\, \omega\, dxdt + \int_{Q^+} \big(v_0^\sigma - v^\sigma\big) dx\, \omega_t(0,t)\, dt = 0$$

Because of (4.12) we can replace the last integral by $-\int_0^T g(t)\,\omega(0,t)\,dt$. Then (2.13) follows from $\delta \to 0$. □

Remark 4.1 *Relation (4.12) indicates that* $g(t) = \partial_t \|v_0^\sigma - v(\cdot,t)^\sigma\|_{L^1(\Omega^+)}$. *This has a physical interpretation: It means that the mass flux through the contact surface between the two components, represented by g, is equal to the change of the total mass within the second component* Ω^+. *Since we have a no flux condition on the right hand boundary this is an evident property.*

We finish this section with a return to the introduction. The initial function u_0 given by (1.6) has just the required regularity for Theorem 4.3 even on the interface $x = \xi(0) = -\lambda(0)$ since $\frac{1}{1-m} > 1$. Moreover, in that case we have compatibility $v_0(0) \equiv \psi(u_0(0)) \equiv 0$. Hence there is a unique solution of this introductory problem in the sense of Definition 2.1. More general, the propagation speed of the interface of the porous medium equation is given by

$$\dot{\xi}(t) = -\frac{1}{1-m} \lim_{x \to \xi(t)-0} (u(x,t)^{1-m})_x \; .$$

Since it is finite for slow diffusion ($m < 1$) it follows that $u_x = \left((u^{1-m})^{1/(1-m)}\right)_x = 0$ at the point of degeneration $u(\xi,t) = 0$. Hence, the supposed regularity of the initial data appears to be not too restrictive.

5. Cauchy Problem

In order to investigate the qualitative behaviour of the interface at the contact line S in the succeeding section now we have to solve the Cauchy problem for our equations. Existence and uniqueness of a solution of the Cauchy problem to porous medium equation on a connected unbounded strip is proven by Oleinik, Kalashnikov and Yui-Lin [15] for bounded initial data. In the following we prove existence of a weak solution of the Cauchy problem including our contact condition but with unbounded initial data. Note that our initial trace u_0 belongs to the admissible growth class of [5].

To this end, for the remaining two sections of this paper we use the notations $\Omega^- = (-\infty,0)$, $\Omega^+ = (0,\infty)$, $Q^- = (-\infty,0)\times(0,\infty)$, $Q^+ = (0,\infty)\times(0,\infty)$, $S = \{0\}\times(0,\infty)$. If necessary, we refer to the bounded sets introduced at the end of Section 1 by $\Omega_\ell^\pm$, $Q_{\ell,T}^\pm$, and S_T, respectively. Moreover, on the unbounded domain we use the notation X_{loc} for the usual spaces of local integrability, i.e. $f \in X_{loc}$ iff $f\varphi \in X$ for every smooth test function φ with bounded support. Then we consider

Cauchy Problem (CP)

$$
\begin{array}{lll}
(u^m)_t = u_{xx} & \text{in} & Q^-, \\
(v^\sigma)_t = v_{xx} & \text{in} & Q^+, \\
u_x = v_x & \text{on} & S, \\
v = \psi(u) & \text{on} & S, \\
u(\cdot,0) = u_0 & \text{on} & \Omega^-, \\
v(\cdot,0) = 0 & \text{on} & \Omega^+.
\end{array}
$$

We assume here $v_0 \equiv 0$ because we are interested in the situation when the interface just arrives at $x = 0$. The initial function u_0, however, should not be bounded or vanish at infinity but its growth is limited by the condition

$$
0 \le u_0(x) \le (-ax)^{1/(1-m)}, \qquad x \in \Omega^-, \tag{5.1}
$$

with some positive constant a. Assume moreover, that there are monotone increasing functions $\psi^-, \psi^+ \in C^1([0,\infty))$ with

$$
0 \le (\psi^\pm)' \le K_l \qquad \text{on every bounded interval } [0,l] \tag{5.2}
$$

such that

$$
v = \psi(u) \quad \Leftrightarrow \quad \psi^+(v) = \psi^-(u), \qquad u, v \in \mathbb{R}_+. \tag{5.3}
$$

For instance, if $\psi(u) = Mu^\omega$ for some $M > 0$, $0 < \omega < 1$, we may choose $\psi^+(v) = v^{1/\omega}$, $\psi^-(u) = M^{1/\omega}u$. Obviously, (5.3) transfers strong monotonicity of ψ to ψ^- and ψ^+.

In this section we prove existence of a weak solution of this problem in the following sense:

Definition 5.1 *A couple (u,v) is called solution of Problem (CP) with initial data $(u_0, 0)$ if the following three conditions are fulfilled:*

(i) $(u,v) \in L^2_{loc}(0,T;V_{loc}) \cap L^\infty_{loc}(Q)$;

(ii) the weak relation

$$
\int_{Q^-} \left((u_0^m - u^m)\varphi_t^- \ + \ u_x\varphi_x^-\right) dx\,dt + \int_{Q^+} \left(-v^\sigma \varphi_t^+ \ + \ v_x\varphi_x^+\right) dx\,dt = 0 \tag{5.4}
$$

holds for all $\varphi = (\varphi^-, \varphi^+) \in L^2(0,T;V)$ with bounded support where $\varphi_t \in L^\infty(Q)$, $\varphi^- = \varphi^+$ on S;

(iii) the contact condition

$$
v = \psi(u)
$$

is satisfied almost everywhere on S.

The idea to obtain a weak solution of (CP) is to approximate the initial function u_0 by approximates $u_{0\nu}$ with finite support, solve the corresponding problems (P) on bounded domains by (u_ν, v_ν) and show that (u_ν, v_ν) converge to a solution of (CP) as $\nu \to \infty$. Hence, let $\big(u_{0\nu}\big)_{\nu>0}$ be approximates of u_0 with

$$\operatorname{supp} u_{0\nu} \subset \overline{\Omega_\nu^-}\ , \qquad u_{0\nu} \to u_0 \quad \text{in } L^1_{loc}(\Omega^-) \quad \text{as } \nu \to \infty, \tag{5.5}$$

fulfilling (5.1) and the assumptions of Lemma 4.1. Note that compatibility condition $\psi(u_{0\nu}(0)) = v_{0\nu}(0) = 0$ is fulfilled, too. Due to Theorem 4.3 then we have:

Proposition 5.1 *For every $\nu > 0$, $T > 0$, on the bounded domain $Q_{\nu,T}$ there is a solution (u_ν, v_ν) of Problem (P) with initial data $(u_{0\nu}, 0)$ in the sense of Definition 2.1.*

In order to prove convergence of (u_ν, v_ν) we need a priori estimates. However, by (5.1) and (5.5), these solutions will not be uniformly bounded for all $\nu > 0$. But by means of the comparison results of Section 3 we are able to derive local estimates which are uniform with respect to ν on bounded domains.

Lemma 5.1 *Let $R, T > 0$ be given and $\nu \geq R$. Then there is a constant $C = C(R,T)$ independent of ν such that*

$$0 \leq u_\nu(x,t), v_\nu(x,t) \leq C(R,T)$$

holds for $-R \leq x \leq \nu$ and for a.e. $t \in [0,T]$.

Proof: We look for a supersolution. However, we were not succeeded in finding a supersolution to our contact problem for all $x \in \mathbb{R}, t > 0$ which satisfies the right conditions on the contact line S. But we may construct a function which is a supersolution on a given bounded domain. Let

$$h_\rho(x) = \begin{cases} -ax & \text{for } -\infty < x \leq -\rho, \\ \frac{a}{2}\big(\frac{x^2}{\rho} + \rho\big) & \text{for } -\rho \leq x \leq 0 \end{cases}$$

for some $\rho > 0$ and

$$\begin{aligned} \bar{u}_\rho(x,t) &= \Big[\Big(\tfrac{2}{m} + \tfrac{1}{1-m}\Big)a^2 t + h_\rho(x)\Big]^{1/(1-m)}, && -\infty < x \leq 0, 0 \leq t \leq T, \\ \bar{v}_\rho(x,t) &= \psi\left(\bar{u}_\rho(0,t)\right), && 0 \leq x < \infty, 0 \leq t \leq T. \end{aligned}$$

$(\bar{u}_\rho, \bar{v}_\rho)$ is a supersolution to Problem (P) in $(-\nu, \nu) \times (0,T)$. In fact, it holds

$$\partial_x \bar{u}_\rho(0,t) = 0 = \partial_x \bar{v}_\rho(0,t) \qquad \text{and} \qquad \bar{v}_\rho(0,t) = \psi(\bar{u}_\rho(0,t))$$

on the contact line S as well as

$$-\partial_x \bar{u}_\rho(-\nu,t) \geq -\tfrac{1}{1-m}\, \bar{u}_\rho(-\nu,t)^m\, h'_\rho(-\nu) \geq 0\ , \qquad \partial_x \bar{v}_\rho(\nu,t) \geq 0$$

on the outer boundaries. It remains to check the integral inequality. For $x < 0, t > 0$ one calculates

$$(\bar{u}_\rho^m)_t - (\bar{u}_\rho)_{xx} = \left(\tfrac{2a^2}{1-m} + \tfrac{ma^2}{(1-m)^2} - \tfrac{m}{(1-m)^2}\, h'_\rho(x)^2 - \tfrac{1}{1-m}\, h''_\rho(x)\, \bar{u}_\rho^{1-m}\right) \cdot \bar{u}_\rho^{2m-1}.$$

Because of $h'_\rho(x)^2 \le a^2$ the item is nonnegative if

$$2a^2 - h''_\rho(x) \left[\left(\tfrac{2}{m} + \tfrac{1}{1-m}\right)a^2 t + h_\rho(x)\right] \ge 0 \tag{5.6}$$

holds. Obviously, (5.6) holds for $-\infty < x < -\rho$. For $-\rho < x < 0$ we have $h''_\rho(x) = \frac{a}{\rho}$ and $h_\rho(x) \le a\rho$. Then (5.6) holds if

$$\left(\tfrac{2}{m} + \tfrac{1}{1-m}\right)a^2 t \le a\rho. \tag{5.7}$$

Choose now $\rho = \rho(T)$ such that (5.7) is fulfilled for all $t \in [0, T]$. Then

$$(\bar{u}_\rho^m)_t - (\bar{u}_\rho)_{xx} \ge 0$$

holds for all $x < 0$ and $t \in (0, T)$.

To verify this for the weak formulation, take the first integral of relation (2.12) in Definition 2.1 and integrate by parts on the domain $Q_\delta^- = (-\nu, -\delta) \times (0, T)$ for some $\delta > 0$ in order to do not run into difficulties because of the singularity of $(\bar{u}_\rho)_{xx}$ at $(0, 0)$. The additional integral $\int_0^T \partial_x \bar{u}_\rho(-\delta, t)\varphi^-\, dt$ is non-singular and tends to 0 as $\delta \to 0$.

The corresponding inequality for $\bar{v}_\rho$ is easy to check. Thus $(\bar{u}_\rho, \bar{v}_\rho)$ is a supersolution to Problem (P) on the bounded domain $Q_{\nu,T}$ in the sense of Definition 2.1 where $g = 0$. Moreover, $(\underline{u}, \underline{v}) = (0, 0)$ is a subsolution. Since furthermore the initial data fulfil the estimates

$$0 \le u_{0\nu}(x) \le (-ax)^{1/(1-m)} \le \bar{u}_\rho(x, 0)\,, \qquad 0 = v_\nu(x, 0) \le \bar{v}_\rho(x, 0)\,,$$

the comparison result Theorem 3.2 (ii) applied on $Q_{\nu,T}$ yields

$$\begin{aligned} 0 \le u_\nu(x,t) &\le \left[\left(\tfrac{2}{m} + \tfrac{1}{1-m}\right)a^2 T + h_\rho(-R)\right]^{1/(1-m)} =: C_u(R,T), \qquad -R \le x \le 0, \\ 0 \le v_\nu(x,t) &\le \psi\left(\left[\left(\tfrac{2}{m} + \tfrac{1}{1-m}\right)a^2 T + \tfrac{a\rho}{2}\right]^{1/(1-m)}\right) =: C_v(T), \qquad 0 \le x \le \nu, \end{aligned}$$

for all $t \in [0, T]$. This estimate is independent of ν. □

Note that the boundedness proved in the above lemma holds for solutions of the corresponding problems (P_L), too. Namely, the supersolution $(\bar{u}_\rho, \bar{v}_\rho)$ is also a supersolution to (P_L) since $\bar{v}_\rho = \psi(\bar{u}_\rho)$.

Lemma 5.2 *Let $R \ge 1$, $T > 0$ be given and $\nu \ge R$. Then there are constants $C(R,T)$ and $C(T)$ independent of ν such that*

$$\int_0^T \int_{-R+1}^0 |(u_\nu)_x(x,t)|^2\, dxdt \le C(R,T)\,, \qquad \int_0^T \int_0^\nu |(v_\nu)_x(x,t)|^2\, dxdt \le C(T).$$

Proof: For the proof of this lemma we return to the solutions $(u_{\nu,L}, v_{\nu,L})$ of problems (P_L) which are constructed in Theorem 4.1 as approximates to (u_ν, v_ν) on the fixed domain $Q_{\nu,T}$. These functions satisfy the estimates (4.3) and (4.4), however the bounds there depend on the bounds of the initial functions on $\Omega = \Omega_\nu$. To overcome this dependence on ν we introduce a cut-off function $\mu_R(x) = \min\big\{ \max\{0, R+x\}, 1 \big\}$ and repeat step 2 of the proof of Theorem 4.1 with $\mu_R(x)\phi_l^-(u)$ instead of $\phi_l^-(u)$. Remember that $v_{0\nu} \equiv 0$. Then, instead of (4.2) we obtain

$$\int_{\Omega_\nu^-} \mu_R(x)\, G_l^-(b_\varepsilon^-(u_\varepsilon(x,t)))\, dx + \int_{\Omega_\nu^+} G_l^+(b_\varepsilon^+(v_\varepsilon(x,t)))\, dx - \int_{\Omega_\nu^-} \mu_R(x)\, G_l^-(b_\varepsilon^-(u_{0\nu}))\, dx$$
$$+ \int_0^t \int_{\Omega_\nu^-} \mu_R(x)\, \phi_l^-(u_\varepsilon)_x\, (u_\varepsilon)_x\, dx d\tau + \int_0^t \int_{-R}^{-R+1} \phi_l^-(u_\varepsilon)\, (u_\varepsilon)_x\, dx d\tau$$
$$+ \int_0^t \int_{\Omega_\nu^+} \phi_l^+(v_\varepsilon)_x\, (v_\varepsilon)_x\, dx d\tau + L \int_0^t \big(v_\varepsilon - \psi_\varepsilon(u_\varepsilon)\big)\big(\phi_l^+(v_\varepsilon) - \phi_l^-(u_\varepsilon)\big)(0,\tau)\, d\tau = 0.$$

Fix now $l = 1$. Omitting some nonnegative items and using the convergence properties as $\varepsilon \to 0$ at the beginning and the end of the proof of Theorem 4.1 we arrive at

$$\int_0^T \int_{-R+1}^0 \big((u_{\nu,L})_x\big)^2\, dx dt + \int_0^T \int_{-R}^{-R+1} u_{\nu,L}\, (u_{\nu,L})_x\, dx dt \le \int_{-R}^0 G_1^-((u_{0\nu})^m)\, dx$$

The right hand side is bounded by some constant $C(R)$ due to (5.1) and (5.5). The second integral on the left side may be transformed into

$$\int_0^T \int_{-R}^{-R+1} \frac{1}{2} \big(u_{\nu,L}(x,t)^2\big)_x\, dx dt = \int_0^T \Big(u_{\nu,L}(-R+1,t)^2 - u_{\nu,L}(-R,t)^2\Big)\, dt,$$

which is bounded independent of ν and L as a consequence of Lemma 5.1 and the note after it. This yields the assertion first for the approximates $u_{\nu,L}$, but due to independence of L of the bounds it holds also for u_ν.

To derive the corresponding estimate for v_ν we set $l = 2$ and obtain

$$\int_0^T \int_0^\nu \big((v_{\nu,L})_x\big)^2\, dx dt + \int_0^T \int_{-R}^{-R+1} \psi(u_{\nu,L})\, (u_{\nu,L})_x\, dx dt \le \int_{-R}^0 G_2^-((u_{0\nu})^m)\, dx.$$

For some fixed R, e.g. $R = 1$, the second integral can be transformed in the same way as above using the primitive of ψ. This implies boundedness, which concludes the proof. □

In the next lemma we prove convergence of a subsequence of (u_ν, v_ν) on a given bounded domain. To this aim for given $R > 2$, $T > 0$, define

$$X_{R,T} := \{(u,v) \in L_2\big(0,T; H^1(-R+2,0)\big) \times L_2\big(0,T; H^1(0,R-2)\big)\}$$

with weak topology with respect to the derivatives and strong L_2-topology with respect to u and v.

Lemma 5.3 *Let $R > 2$, $T > 0$ be given and $\nu \in \mathcal{V} \subset \mathbb{N}$, where $\mathcal{V}$ is an unbounded index set. Then there is a subsequence $(\nu_k)_{k=1,2,\dots} \subset \mathcal{V}$ such that $(u_{\nu_k}, v_{\nu_k}) \to (u, v)$ in $X_{R,T}$.*

Proof: In addition to the a priori estimates from Lemma 5.1 and Lemma 5.2 we need some compactness of the set (u_ν, v_ν) with respect to t. Let us suppress the subscript ν for the next few estimates. Assume $\nu > R$ and $h > 0$. For given $t \in [0, T-h]$ we test relation (2.12) with $\chi_{[t,t+h]}(\tau)w^\pm(x)$ where χ_A is the characteristic function of the set A and $w^- \in H^1(-\nu, 0)$, $w^+ \in H^1(0, \nu)$ with $w^- = w^+$ on S. This yields

$$\int_{-\nu}^{0} \big(u^m(x,t+h) - u^m(x,t)\big)w^-(x)\,dx + \int_0^\nu \big(v^\sigma(x,t+h) - v^\sigma(x,t)\big)w^+(x)\,dx$$
$$+ \int_t^{t+h}\int_{-\nu}^{0} u_x(x,\tau)w_x^-(x)\,dxd\tau + \int_t^{t+h}\int_0^\nu v_x(x,\tau)w_x^+(x)\,dxd\tau = 0.$$

We choose now

$$\begin{aligned} w^-(x) &= \mu_{R-1}(x)\cdot\big(\psi^-(u(x,t+h)) - \psi^-(u(x,t))\big)\\ w^+(x) &= \big(\psi^+(v(x,t+h)) - \psi^+(v(x,t))\big), \end{aligned}$$

where μ_R is the cut-off function defined in the proof of Lemma 5.2, integrate over $t \in [0, T-h]$ and obtain

$$\int_0^{T-h}\int_{-R+1}^{0} \mu_{R-1}(x)\big(u^m(x,t+h) - u^m(x,t)\big)\big(\psi^-(u(x,t+h)) - \psi^-(u(x,t))\big)\,dxdt$$
$$+ \int_0^{T-h}\int_0^\nu \big(v^\sigma(x,t+h) - v^\sigma(x,t)\big)\big(\psi^+(v(x,t+h)) - \psi^+(v(x,t))\big)\,dxdt$$
$$+ h\int_0^{T_h}\int_{-R+1}^{-R+2} \frac{1}{h}\int_t^{t+h} u_x(x,\tau)d\tau\cdot\big(\psi^-(u(x,t+h)) - \psi^-(u(x,t))\big)\,dxdt$$
$$+ h\int_0^{T_h}\int_{-R+1}^{0} \chi_{R-1}(x)\,\frac{1}{h}\int_t^{t+h} u_x(x,\tau)d\tau\cdot\big(\psi^-(u(x,t+h))_x - \psi^-(u(x,t))_x\big)\,dxdt$$
$$+ h\int_0^{T_h}\int_0^\nu \frac{1}{h}\int_t^{t+h} v_x(x,\tau)d\tau\cdot\big(\psi^+(v(x,t+h))_x - \psi^+(v(x,t))_x\big)\,dxdt = 0.$$

The last three integrals are bounded because of the local boundedness of (u, v) due to Lemma 5.1, assumption (5.2), and the estimates of Lemma 5.4. This yields

$$\int_0^{T-h}\int_{-R+2}^{0} \big(u_\nu^m(x,t+h) - u_\nu^m(x,t)\big)\big(\psi^-(u_\nu(x,t+h)) - \psi^-(u_\nu(x,t))\big)\,dxdt$$
$$+ \int_0^{T-h}\int_0^{R-2} \big(v_\nu^\sigma(x,t+h) - v_\nu^\sigma(x,t)\big)\big(\psi^+(v_\nu(x,t+h)) - \psi^+(v_\nu(x,t))\big)\,dxdt$$
$$\le C_{R,T}\,h. \tag{5.8}$$

We are prepared now to send $\nu \to \infty$. First, by Lemma 5.2 there is a subsequence (u_{ν_k}, v_{ν_k}) converging to (u, v) weakly in $X_{R,T}$. It remains to show strong convergence in $L_2(Q_{R-2,T})$. Denote for a moment

$$U_\nu = \psi^-(u_\nu),\ V_\nu := \psi^+(v_\nu) \quad \text{and} \quad b^-(U) := \big((\psi^-)^{-1}(U)\big)^m,\ b^+(V) := \big((\psi^+)^{-1}(V)\big)^\sigma.$$

Then, by (5.2) and Lemma 5.4, there is a subsubsequence (U_{ν_k}, V_{ν_k}) (we denote subsequences of (ν_k) by (ν_k) again) converging to (U, V) weakly in $X_{R,T}$, too. Hence, by the estimate (5.8) we are in the situation of [1, Lemma 1.9] which yields L_1-convergence of $(b^-(U_{\nu_k}), b^+(V_{\nu_k}))$. Since (U_{ν_k}, V_{ν_k}) are uniformly bounded on $Q_{R-2,T}$ there is a subsequence that converges almost everywhere. Finally, this implies convergence of (u_{ν_k}, v_{ν_k}) almost everywhere on $Q_{R-2,T}$ and, by uniform boundedness again, convergence in $L_2(Q_{R-2,T})$. □

Theorem 5.1 *Let $0 < m, \sigma < 1$, $u_0 : \mathbb{R}_- \to \mathbb{R}_+$ be a measurable function fulfilling the growth condition (5.1) for some $a > 0$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and strongly monotone increasing with $\psi(0) = 0$ having the property (5.2), (5.3). Then there is a weak solution of the Cauchy Problem (CP) in the sense of Definition 5.1.*

Proof: We use Lemma 5.3 and Cantors diagonal selection procedure. Choose monoton increasing sequences $R_n, T_n \to \infty$ and consider first the set of solutions $(u_\nu, v_\nu) \in X_{R_1,T_1}$ of Problem (P) defined at the beginning of this section with $\nu \geq R_1$. Due to Lemma 5.3 there is a subsequence $(u_{\nu_{1,k}}, v_{\nu_{1,k}})$ converging in X_{R_1,T_1} to some (u, v). Let now

$$\mathcal{V}_n = \Big\{\nu_{n,k} : \nu_{n,k} \geq R_{n+1}, \quad (u_{\nu_{n,k}}, v_{\nu_{n,k}}) \to (u, v) \quad \text{in } X_{R_n,T_n}\Big\}.$$

By Lemma 5.3 again, there is a subsequence $(\nu_{n+1,k})_{k=1,2,\ldots} \subset \mathcal{V}_n$ such that

$$(u_{\nu_{n+1,k}}, v_{\nu_{n+1,k}}) \to (\tilde{u}, \tilde{v}) \quad \text{in } X_{R_{n+1},T_{n+1}}.$$

Since we have the continuous embedding $X_{R_{n+1},T_{n+1}} \subset X_{R_n,T_n}$ it holds $(\tilde{u}, \tilde{v}) = (u, v)$ in X_{R_n,T_n}. Selecting now the diagonal sequence $(\nu_{n,n})_{n=1,2,\ldots}$ we have

$$(u_{\nu_{n,n}}, v_{\nu_{n,n}}) \to (u, v) \quad \text{in } X_{R,T}$$

for every fixed $R, T > 2$.

Finally it is easy to see from Definition 2.1 for (u_ν, v_ν) that (u, v) fulfils relation (5.4) and is a solution in the sense of Definition 5.1. □

It is well-known for the porous medium equation (1.1) in a single domain that the interface has finite propagation speed in our case of slow diffusion $0 < m < 1$. Of course, this property is preserved for v if the interface crosses the contact line S. We conclude this section with a proof of it.

Theorem 5.2 *Let (u,v) be the weak solution of Problem (CP) according to Theorem 5.1. Then the interface of v has finite propagation speed, i.e.*

$$\xi(t) := \sup\{x \in [0,\infty) : v(x,t) > 0\} < \infty \quad \textit{for a.a. } t > 0.$$

Proof: Consider the approximates (u_ν, v_ν) only on the right part $Q^+_{\nu,T}$ of the domain. Then v_ν is a weak solution to $(v^\sigma)_t - v_{xx} = 0$ with Dirichlet data $v_\nu(0,t)$ on S_T and homogeneous Neumann data on the outer boundary $x = \nu$, $0 < t < T$. By Lemma 5.1 there is a constant $K = K(T)$ independent of ν with $v_\nu(0,t) \leq K$ for $0 \leq t \leq T$. Hence, v_ν is a subsolution to the mixed Dirichlet-Neumann problem considered by F. Otto [17] with Dirichlet data $v_D = K$ on S_T and initial values $b_1^0 \equiv 0$. Moreover, for every fixed $c > 0$

$$\bar{v}(x,t) = \left[K + \tfrac{c^2 t}{1-\sigma} - cx\right]_+^{\frac{1}{1-\sigma}}$$

is a supersolution to the same problem with initial values $b_2^0 = [K - cx]_+^{\sigma/(1-\sigma)} \geq 0$ if ν is large enough such that $\bar{v}_x(\nu,t) = 0$ for all $t \leq T$. Then from the comparison result in [17, Theorem] follows

$$0 \leq v_\nu(x,t) \leq \bar{v}(x,t) \qquad \text{for all } x \in [0,\nu] \text{ and a.a. } t \in [0,T].$$

Here $T > 0$ is fixed but arbitrarily chosen. The above estimate especially implies

$$v_\nu(x,t) = 0 \qquad \forall x \geq \frac{K}{c} + \frac{ct}{1-\sigma}$$

for a.a. $t \in [0,T]$. Since the bound is independent of ν this is also valid for the limit v which concludes the proof. □

6. Selfsimilar solutions

Let us seek a nonnegative solution (u,v) of the Cauchy Problem (CP) for a given power law nonlinearity

$$\psi(u) = M u^\omega \tag{6.1}$$

invariant under the dilation scaling

$$u(x,t) \longmapsto \lambda^d\, u\left(\frac{x}{\lambda^b}, \frac{t}{\lambda}\right) \qquad \text{for } x \leq 0\,,$$

$$v(x,t) \longmapsto \lambda^c\, v\left(\frac{x}{\lambda^\alpha}, \frac{t}{\lambda}\right) \qquad \text{for } x \geq 0$$

so that

$$u(x,t) = \lambda^d\, u\left(\frac{x}{\lambda^b}, \frac{t}{\lambda}\right) \qquad \text{for any } x \leq 0\,,\ t > 0\,,$$

$$v(x,t) = \lambda^c\, v\left(\frac{x}{\lambda^\alpha}, \frac{t}{\lambda}\right) \qquad \text{for any } x \geq 0\,,\ t > 0$$

and for all $\lambda > 0$.

After tedious but not difficult formal manipulations we have arrived at the critical exponent ω in (6.1)

$$\omega \;=\; \frac{m+1}{\sigma+1}\;, \tag{6.2}$$

special initial data

$$u_0(x) = (-a\,x)^{1/(1-m)} \quad \text{for } x \le 0\,, \qquad v_0(x) \equiv 0 \quad \text{for } x \ge 0 \tag{6.3}$$

and

$$\left.\begin{array}{ccc} d = \dfrac{1}{1-m} & b = 1 & c = \dfrac{\omega}{1-m} \\[2ex] & \alpha = \dfrac{1-m\sigma}{(1+\sigma)(1-m)} & \end{array}\right\} \tag{6.4}$$

for which the following statement hold.

Proposition 6.1 *Let (u,v) be a weak solution of the (CP) for (u_0, v_0) and ω given by (6.2) and (6.3) above. Then also*

$$U_\lambda(x,t) \;=\; \lambda^{1/(1-m)}\, u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right) \qquad \textit{for } \; x \le 0\,,\; t > 0$$

$$V_\lambda(x,t) \;=\; \lambda^{\omega/(1-m)}\, v\left(\frac{x}{\lambda^\alpha}, \frac{t}{\lambda}\right) \qquad \textit{for } \; x \ge 0\,,\; t > 0$$

is a weak solution of (CP) with the same initial function. Note that α is given in (6.4) above and $\lambda > 0$ is arbitrary.

Proof: Recall first that (u,v) satisfies (5.4) for any $\varphi = (\varphi^-, \varphi^+)$ such that φ^- has a compact support in $(-\infty, 0] \times [0, \infty)$ and φ^+ has a compact support in $[0,\infty) \times [0,\infty)$. We change now variables setting

$$x = \frac{y}{\lambda} \quad \text{if } \; x < 0\,, \qquad x = \frac{y}{\lambda^\alpha} \quad \text{if } \; x > 0\,, \qquad t = \frac{\tau}{\lambda}$$

and define

$$\psi^-(y,\tau) \;\equiv\; \varphi^-\left(\frac{y}{\lambda}, \frac{\tau}{\lambda}\right)\,, \qquad \psi^+(y,\tau) \;\equiv\; \varphi^+\left(\frac{y}{\lambda^\alpha}, \frac{\tau}{\lambda}\right)\,.$$

(5.4) then yields

$$\int_0^\infty \int_{-\infty}^0 \lambda^{-1/(1-m)} \left[\left((-ay)^{m/(1-m)} - U_\lambda^m\right)\psi_\tau^- \;+\; (U_\lambda)_y \psi_y^-\right](y,\tau)\, dy\, d\tau$$

$$+ \int_0^\infty \int_0^\infty \left[-\lambda^{-\frac{\omega\sigma}{1-m}-\alpha}\, V_\lambda^\sigma \psi_\tau^+ \;+\; \lambda^{-\frac{\omega}{1-m}+\alpha-1}\,(V_\lambda)_y \psi_y^+\right](y,\tau)\, dy\, d\tau \;=\; 0\,.$$

As

$$\frac{1}{1-m} = \frac{\omega\sigma}{1-m} + \alpha = \frac{\omega}{1-m} + 1 - \alpha$$

we see that (U_λ, V_λ) satisfies the item (ii) of Definition 5.1. The contact condition for (6.1) is satisfied as well. □

We henceforth assume the following assumption

Hypothesis 6.1 *Problem (CP) is uniquely solvable in the class of solutions given by Definition 5.1 above.*

Remark 6.1 *We have tried to prove uniqueness but we have not succeeded yet.*

As a consequence of our Hypothesis 6.1 we have

$$u(x,t) = \lambda^{1/(1-m)}\, u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right) \qquad \text{for } x \le 0\,,\ t > 0$$

$$v(x,t) = \lambda^{\omega/(1-m)}\, v\left(\frac{x}{\lambda^\alpha}, \frac{t}{\lambda}\right) \qquad \text{for } x \ge 0\,,\ t > 0$$

for all $\lambda > 0$.
Setting

$$\lambda = t$$

we obtain

$$\left.\begin{aligned} u(x,t) &= t^{1/(1-m)}\, w\left(\frac{x}{t}\right)\,, & w(y) &\equiv u(y,1)\,, \\ v(x,t) &= t^{\omega/(1-m)}\, h\left(\frac{x}{t^\alpha}\right)\,, & h(y) &\equiv v(y,1)\,. \end{aligned}\right\} \tag{6.5}$$

In the next theorem we prove that w, h are solutions of the following problem:

$$\left.\begin{aligned} & w''(x) + x(w^m(x))' - \frac{m}{1-m} w^m(x) = 0 && \text{for } x < 0\,, \\ & h''(x) + \alpha x(h^\sigma(x))' - \frac{\omega\sigma}{1-m} h^\sigma(x) = 0 && \text{for } x > 0\,, \\ & w'(0) = h'(0)\,, \quad h(0) = M w^\omega(0) && \\ & w(x) \sim (-a\, x)^{1/(1-m)} && \text{as } x \to -\infty\,, \\ & \text{there exists } \zeta > 0 \text{ such that } h(x) = 0 && \forall x \ge \zeta\,, \end{aligned}\right\} \tag{6.6}$$

in the sense of the following

Definition 6.1 *A couple (w,h) is called a weak solution of Problem (6.6) if the following four properties are fulfilled:*
1. $w \in H^1(-R,0)$, $h \in H^1(0,R)$ *for all* $R>0$, $h(0) = M w^\omega(0)$ *;*
2.

$$\int_{-\infty}^{0} \left\{ w^m(y) \left[(yf(y))' + \frac{m}{1-m} f(y) \right] + f'(y)\, w'(y) \right\} dy$$

$$+ \int_{0}^{\infty} \left\{ h^\sigma(y) \left[\alpha (y\, g(y))' + \frac{\sigma\omega}{1-m} g(y) \right] + g'(y)\, h'(y) \right\} dy \ = 0$$

for any couple of test functions $(f,g) \in \tilde{V}$ *with compact support.*
3.

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{0} \left[(-ax)^{m/(1-m)} - \frac{1}{\varepsilon} \int_0^\varepsilon t^{m/(1-m)} w^m \left(\frac{x}{t} \right) dt \right] f(x)\, dx \ = \ 0$$

for any $f \in C_0^1((-\infty,0])$, $f(0)=0$ *and finally,*
4. there exists $\zeta > 0$ *such that* $h(x) = 0 \ \ \forall x \geq \zeta$ *and* $h(x) > 0$ *for* $x \in [0,\zeta)$.

Remark 6.2 *Item 3. of the above definition is a weak formulation of the asymptotic condition for* $x \to -\infty$ *in (6.6). Indeed, one can prove: If*

$$\frac{w(x)}{(-ax)^{1/(1-m)}} \to 1 \quad \text{as } x \to -\infty$$

then condition 3. of Definition 6.1 holds.

Theorem 6.1 *Assume that (u,v) is the weak solution of the Cauchy problem (CP) and let us suppose that Hypotheses 6.1 holds.*

Then the couple (w,h) being determined by (6.5) is the weak nonnegative solution of Problem (6.6) in the sense of Definition 6.1. Moreover, if we recall the notation

$$\xi(t) = \sup \{ x \in [0,\infty) \ : \ \ v(x,t) > 0 \} \ , \tag{6.7}$$

then

$$\xi(t) \ = \ \zeta\, t^\alpha \tag{6.8}$$

with

$$\alpha = \frac{1 - m\sigma}{(1+\sigma)(1-m)}$$

and $\zeta > 0$ is the given constant from the item 4. of Definition 6.1 above.

Proof: Let us recall (6.5) and insert

$$\varphi^-(x,t) = t^{-1/(1-m)} f\left(\frac{x}{t} \right) \chi_\varepsilon(t-1) \ , \quad \varphi^+(x,t) = t^{-\delta} g\left(\frac{x}{t^\alpha} \right) \chi_\varepsilon(t-1)$$

into (5.4), where f, g satisfy the assumptions from the item 2. of Definition 6.1, $\chi_\varepsilon \in C_0^\infty(\mathbb{R})$, supp $\chi_\varepsilon \subseteq [-\varepsilon, \varepsilon]$, $\displaystyle\int_{1-\varepsilon}^{1+\varepsilon} t^{-1}\chi_\varepsilon(t-1)dt = 1$ and

$$\delta = \alpha + \frac{\omega\sigma}{1-m}\ .$$

As $\varphi^-(x,0) = \varphi^+(x,0) = 0$, we arrive at

$$\begin{aligned}
&\int_{Q^-}\left[w^m\left(\frac{x}{t}\right)\left(\frac{x}{t}\, f'\left(\frac{x}{t}\right) + \frac{1}{1-m} f\left(\frac{x}{t}\right)\right) + f'\left(\frac{x}{t}\right) w'\left(\frac{x}{t}\right)\right] \frac{\chi_\varepsilon(t-1)}{t^2}\, dxdt \\
&+ \int_{Q^+}\left[h^\sigma\left(\frac{x}{t^\alpha}\right)\left(\alpha\frac{x}{t^\alpha}\, g'\left(\frac{x}{t^\alpha}\right) + \delta g\left(\frac{x}{t^\alpha}\right)\right) + g'\left(\frac{x}{t^\alpha}\right) h'\left(\frac{x}{t^\alpha}\right)\right] \frac{\chi_\varepsilon(t-1)}{t^{\alpha+1}}\, dxdt \\
= &\int_{Q^-} w^m\left(\frac{x}{t}\right) f\left(\frac{x}{t}\right) \chi_\varepsilon'(t-1)t^{-1}\, dxdt + \int_{Q^+} h^\sigma\left(\frac{x}{t^\alpha}\right) g\left(\frac{x}{t^\alpha}\right) \chi_\varepsilon'(t-1)t^{-\alpha}\, dxdt\ .
\end{aligned}$$

Now, setting $x = yt$ if $x < 0$ and $x = yt^\alpha$ if $x > 0$ we get

$$\begin{aligned}
&\int_{-\infty}^0 \left[w^m(y)\left(yf'(y) + \frac{1}{1-m} f(y)\right) + f'(y)w'(y)\right] dy \int_0^\infty \frac{\chi_\varepsilon(t-1)}{t}\, dt \\
&\quad + \int_0^\infty \left[h^\sigma(y)\left(\alpha y g'(y) + \delta g(y)\right) + g'(y)h'(y)\right] dy \int_0^\infty \frac{\chi_\varepsilon(t-1)}{t}\, dt \\
= &\int_{-\infty}^0 w^m(y)f(y)dy \int_0^\infty \chi_\varepsilon'(t-1)dt + \int_0^\infty h^\sigma(y)g(y)dy \int_0^\infty \chi_\varepsilon'(t-1)\, dt \ =\ 0
\end{aligned}$$

and the item 2. follows easily. To show the item 3. let us take

$$\varphi^-(x,t) = f(x)\varepsilon^{-1}(\varepsilon - t)_+\ , \qquad \varphi^+(x,t) \equiv 0$$

as a test function into (5.4). We obtain

$$\begin{aligned}
&\int_{-\infty}^0 \left[(-ax)^{m/(1-m)} - \frac{1}{\varepsilon}\int_0^\varepsilon t^{m/(1-m)} w^m\left(\frac{x}{t}\right) dt\right] f(x)\, dx \\
&\qquad = \ \varepsilon^{-1}\int_0^\varepsilon (\varepsilon - t)\int_{-\infty}^0 u_x(x,t)f'(x)dx\, dt\ . \qquad (6.9)
\end{aligned}$$

A test function f has a compact support, say on $[-R, 0]$ and $u \in L^2(0,T; H^1(-R,0))$, the right hand side of (6.9) tends to zero as $\varepsilon \to 0$.
Finally, we proved in Theorem 5.2

$$v(x,t) = 0 \qquad \forall\, x \geq \frac{K}{c} + \frac{ct}{1-\sigma}\ .$$

Hence, (6.5) yields the existence of a number d such that $h(x) = 0\ \forall\, x \geq d$. We claim that if $h(c) = 0$ for $c < d$, then $h(x) = 0\ \forall x \in [c, d]$. Indeed, $f \equiv 0$ and $g(y) = h(y)$

$\forall y \in [c,d]$ and zero outside is an admissible test function in item 2. of Definition 6.1 and we arrive, after some manipulations, at

$$\int_c^d \left[\frac{\sigma(\sigma+m+2)}{(1-m)(\sigma+1)^2} h^{\sigma+1}(y) + |h'(y)|^2 \right] dy \;=\; 0 \;.$$

This establishes the claim.
Define

$$\zeta = \inf\{c \in [0,\infty) \;:\; \quad h(c) = 0\;\} \;.$$

Then

$$\zeta \;>\; 0\;. \tag{6.10}$$

To prove this assertion, let us suppose $\zeta = 0$. Hence $h \equiv 0$ on $[0,\infty)$ and consequently

$$v(x,t) = 0 \qquad \forall x \in [0,\infty),\; t \geq 0, \tag{6.11}$$

i.e.

$$\int_{Q^-} \left(\left((-ax)^{m/(1-m)} - u^m \right) \varphi_t^- + u_x \varphi_x^- \right) dx\; dt \;=\; 0 \tag{6.12}$$

$\forall \varphi$ as in (5.4).
Set

$$U(x,t) = \begin{cases} u(x,t) & x \leq 0 \\ \\ u(-x,t) & x \geq 0 \end{cases} \quad , \quad \varphi(x,t) = \begin{cases} \varphi^-(x,t) & x \leq 0 \\ \\ \varphi^+(x,t) & x \geq 0 \end{cases} \;.$$

According to (6.12), U is nonnegative and

$$\int_0^\infty \int_{-\infty}^\infty \left(((a|x|)^{m/(1-m)} - U^m)\varphi_t + U_x \varphi_x \right) dx dt = 0 \;.$$

If we restrict ourselves to the bounded domain

$$(-R,R) \times (0,T)$$

for sufficiently large R, T, U is a supersolution to the following problem

$$\left.\begin{array}{ll} (u^m)_t = u_{xx} & x \in (-R,R), t \in (0,T) \\ \\ u(-R,t) = u(R,t) = 0 & \text{for } t > 0 \;, \\ \\ u(x,0) = u_0(x) & \end{array}\right\} \tag{6.13}$$

for u_0 given by (1.6).
Then, due to the Comparison principle of [17] we have

$$U(x,t) \geq \frac{1}{\lambda^{1/m}(t)} \left[1 - \left\{ \frac{x + 2\lambda(0)}{\lambda(t)} \right\}^2 \right]_+^{1/(1-m)} \;.$$

Consequently,

$$U(0,t) \geq \frac{1}{\lambda^{1/m}(t)} \left[1 - \left\{ \frac{2\lambda(0)}{\lambda(t)} \right\}^2 \right]_+^{1/(1-m)} > 0$$

if t is large enough, a contradiction to (6.1) and (6.11). □

Note that Theorem 6.1 gives some information on the speed of the interface $\xi(t)$ when it crosses the contact line S. We have

$$\begin{cases} \alpha > 1 & \text{if } \sigma < m \\ \alpha = 1 & \text{if } \sigma = m \\ \alpha < 1 & \text{if } \sigma > m \,, \end{cases} \qquad \text{hence} \quad \xi'(0+) = \begin{cases} 0 & \text{if } \sigma < m \\ \zeta > 0 & \text{if } \sigma = m \\ \infty & \text{if } \sigma > m \,, \end{cases}$$

that means the interface starts at the contact line with zero (infinity) speed if the nonlinear diffusion in the right domain is slower (faster) than the nonlinear diffusion in the left domain.

Acknowledgements. We thank Volkswagen-Stiftung for its support and hospitality through the RiP program at Oberwolfach.

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