Quasilinear parabolic integro-differential equations with nonlinear boundary conditions

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Abstract

We study the L_p-theory of a class of quasilinear parabolic partial integro-differential equations with nonlinear boundary conditions. The main objective here is to prove existence and uniqueness of local (in time) strong solutions of these problems. Our approach relies on linearization and the contraction mapping principle. To make this work we establish optimal regularity estimates of type L_p for associated linear problems with inhomogeneous boundary data, using here recent results on maximal L_p-regularity for abstract parabolic Volterra equations.

1 Introduction

The purpose of this paper is to establish local (in time) strong L_p-well-posedness for a class of quasilinear parabolic partial integro-differential equations with nonlinear boundary conditions. To describe the problems to be studied, let Ω be a bounded domain in $\mathbb{R}^n$ with C^2-smooth boundary Γ which decomposes according to $\Gamma = \Gamma_D \cup \Gamma_N$ with $\mathrm{dist}(\Gamma_D, \Gamma_N) > 0$. For the unknown $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, we consider the subsequent problem:

$$\left\{ \begin{array}{rll} \partial_t u + dk * (\mathcal{A}(u) : \nabla^2 u) = F(u) + dk * G(u), & t \geq 0,\, x \in \Omega \\ \mathcal{B}_D(u) = 0, & t \geq 0,\, x \in \Gamma_D \\ \mathcal{B}_N(u) = 0, & t \geq 0,\, x \in \Gamma_N \\ u|_{t=0} = u_0, & x \in \Omega. \end{array} \right. \tag{1}$$

Here, $(dk * w)(t,x) = \int_0^t dk(\tau) w(t-\tau, x)$, $t \geq 0$, $x \in \Omega$, $\partial_t u$ means the partial derivative of u w.r.t. t, $\nabla u = \nabla_x u$ is the gradient of u w.r.t. the spatial variables, $\nabla^2 u$ denotes its Hessian matrix, and $B : C = \sum_{i=1,\, j=1}^n B_{ij} C_{ij}$, $B, C \in \mathbb{R}^{n \times n}$. Furthermore, we have the substitution operators

$$\begin{aligned} \mathcal{A}(u)(t,x) &= -a(t,x,u(t,x),\nabla u(t,x)),\ t \geq 0,\, x \in \Omega, \\ F(u)(t,x) &= f(t,x,u(t,x),\nabla u(t,x)),\ t \geq 0,\, x \in \Omega, \\ G(u)(t,x) &= g(t,x,u(t,x),\nabla u(t,x)),\ t \geq 0,\, x \in \Omega, \\ \mathcal{B}_D(u)(t,x) &= b^D(t,x,u(t,x)),\ t \geq 0,\, x \in \Gamma_D, \\ \mathcal{B}_N(u)(t,x) &= b^N(t,x,u(t,x),\nabla u(t,x)),\ t \geq 0,\, x \in \Gamma_N, \end{aligned} \tag{2}$$

where a is $\mathbb{R}^{n \times n}$-valued, and f, g, b^D, b^N are all scalar functions. The scalar-valued kernel k is of bounded variation on each compact interval $[0,T]$ with $k(0) = 0$, and belongs to a certain kernel class with parameter $\alpha \in [0,1)$ which contains, roughly speaking, all 'regular' kernels that behave like t^α for $t\,(>0)$ near zero. Note that

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this formulation includes the special case $k(t) = 1$, $t > 0$, in which (1) amounts to the quasilinear initial-boundary value problem

$$\left\{\begin{array}{rl} \partial_t u + \mathcal{A}(u) : \nabla^2 u = H(u), & t \geq 0,\, x \in \Omega \\ \mathcal{B}_D(u) = 0, & t \geq 0,\, x \in \Gamma_D \\ \mathcal{B}_N(u) = 0, & t \geq 0,\, x \in \Gamma_N \\ u|_{t=0} = u_0, & x \in \Omega, \end{array}\right. \tag{3}$$

where $H(u) = F(u) + G(u)$. Observe further that the case $k(t) = t$, $t \geq 0$, is not admissible; in our setting, this kernel would lead to a hyperbolic problem.

Equations of the form (1) appear in a variety of applied problems. They typically arise in mathematical physics by some constitutive laws pertaining to materials with memory when combined with the usual conservation laws such as balance of energy or balance of momentum. Important applications are heat conduction with memory (see e.g. [12], [13], [22], [25], [26], [27]) and the theory of viscoelasticity (cf. e.g. [3], [5] [6], [27], [31]).

Although there is a wide literature on problems of the form (1), not much seems to be known towards an L_p-theory in the integro-differential case with *nonlinear boundary conditions*, even in the linear situation with inhomogeneous Dirichlet and/or Neumann boundary conditions. Before commenting on available results in the literature we describe our main result to (1), which is stated in Theorem 4.2. For $T > 0$ and $1 < p < \infty$, set $J = [0, T]$ and define the space Z^T by

$$Z^T = H_p^{1+\alpha}(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)).$$

Here $H_p^s(J; L_p(\Omega))$ $(s > 0)$ means the vector-valued Bessel potential space of functions on J taking values in the Lebesgue space $L_p(\Omega)$. We assume $n + 2/(1+\alpha) < p < \infty$, a condition which ensures that the embedding $Z^T \hookrightarrow C(J; C^1(\overline{\Omega}))$ is valid. Theorem 4.2 now asserts that under suitable assumptions on the nonlinearities and the initial data, problem (1) admits a unique local in time strong solution in the following sense: there exists $T > 0$ such that there is one and only one function $u \in Z^T$ that satisfies (1), the integro-differential equation a.e. on $J \times \Omega$, the initial and boundary conditions being fulfilled pointwise on the entire sets considered.

As to literature, there has been a substantial amount of work on nonlinear Volterra and integro-differential equations. We can only mention some of the main results here. Using maximal C^α-regularity for linear parabolic differential equations, in 1985 Lunardi and Sinestrari [24] were able to prove local existence and uniqueness in spaces of Hölder continuity for a large class of fully nonlinear integro-differential equations with a homogeneous linear boundary condition. However, to make their approach work, they assume (in our terminology) that the kernel k has a jump at $t = 0$, a property which is not required in this paper. Concerning C^α-theory for Volterra and integro-differential equations, we further refer to [14], [21], [23], [27]; for the case of fractional differential equations see also [7], [8], [9], and [11]. The standard reference for parabolic partial differential equations in this context is [20].

In the L_p-setting, quasilinear integro-differential equations were first studied by Prüss [30]. He also employs the method of maximal regularity, now in spaces of integrable functions, to obtain existence and uniqueness of strong solutions of the scalar problem

$$\left\{\begin{array}{l} \partial_t u(t,x) = \int_0^t dk(\tau)\{\operatorname{div} g(x, \nabla u(t-\tau, x)) + f(t-\tau, x)\}, \quad t \in J,\, x \in \Omega \\ u(t,x) = 0, \quad t \in J,\, x \in \partial\Omega \\ u(0,x) = u_0(x), \quad x \in \Omega \end{array}\right. \tag{4}$$

in the class $H_p^1(J; L_q(\Omega)) \cap L_p(J; H_q^2(\Omega))$ provided that either T or the data u_0, f are sufficiently small. The kernel $k \in BV_{loc}(\mathbb{R}_+)$ involved is assumed to be 1-regular in the sense of [30, p. 405] and to fulfill an angle condition of the form $|\arg \widehat{dk}(\lambda)| \leq \theta < \frac{\pi}{2}$, $\operatorname{Re}\lambda > 0$, where the hat indicates Laplace transform.

For $\Omega=(0,1)$, g not depending on x, and with $k=1*k_1$, that is $dk*w=k_1*w$, global existence of strong solutions of (4) ($J=\mathbb{R}_+$) with $u\in L_{2,loc}(\mathbb{R}_+;H_2^2([0,1]))$ was established by Gripenberg [17], [18] under different assumptions on g and the kernel k_1. Engler [16] extended the results of the latter work by treating also higher space dimensions and by allowing for a larger class of nonlinear functions g.

Our approach to (1) is based on linearization and the contraction mapping principle and is inspired by Clément and Li [10], who employed this method for obtaining local strong L_p-well-posedness for abstract quasilinear parabolic evolution equations, see also [29]. In order to reformulate (1) as a fixed point problem in an appropriate subset of Z^T, we need optimal regularity estimates of type L_p for associated linear problems with *inhomogeneous boundary data*. Applying recent results from [34] on maximal L_p-regularity for abstract parabolic Volterra equations, we establish these crucial estimates first in the full and half space case with constant coefficients and differential operators consisting only of their principal parts. Well-known techniques like localization, perturbation, and coordinate transformation then allow to transfer these results to the corresponding problems on smooth domains with variable coefficients and including lower order terms.

The plan for this paper is as follows. In Section 2 we fix some notations and describe and motivate the class of kernels used in this paper. Section 3 is devoted to the linear problems associated with (1). Finally, in Section 4 we prove the main result described above concerning the nonlinear problem (1).

2 Preliminaries

We begin by fixing some notations. By $f*g$ we mean the convolution defined by $(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau$, $t\geq 0$, of two functions f, g supported on the halfline. The Laplace and Fourier transform of a function f will be denoted by $\hat{f}$ and $\tilde{f}$, respectively.

The symbol $\mathcal{B}(X,Y)$ means the space of all bounded linear operators from X to Y, we write $\mathcal{B}(X)=\mathcal{B}(X,X)$ for short. Recall that a Banach space X belongs to the class $\mathcal{HT}$, if the Hilbert transform is bounded on $L_2(\mathbb{R};X)$.

For an interval $J\subset\mathbb{R}$, a Banach space X, $s>0$ and $1<p<\infty$, by $H_p^s(J;X)$ and $B_{pp}^s(J;X)$ we mean the vector-valued Bessel potential space resp. Sobolev-Slobodeckij space of X-valued functions on J, see [2], [32], [33]. If $\mathcal{F}$ is any of the aforementioned function spaces with $J=[0,T]$, then ${}_0\mathcal{F}$ consists of all functions $f\in\mathcal{F}$ with the property that for any $i\in\mathbb{N}\cup\{0\}$, $\partial_t^i f$ vanishes at $t=0$ whenever this trace exists.

If X is a complex Banach space and A is a closed linear operator in X, then A is called *sectorial* if the following two conditions are fullfilled: (i) the null space $\mathcal{N}(A)=\{0\}$, and the domain $\mathcal{D}(A)$ as well as the range $\mathcal{R}(A)$ of A are dense in X; (ii) $(-\infty,0)$ is contained in the resolvent set $\rho(A)$ of A and the resolvent estimate $|t(t+A)^{-1}|_{\mathcal{B}(X)}\leq M$, $t>0$, holds, for some constant $M>0$. The class of sectorial operators in X is denoted by $\mathcal{S}(X)$. Putting $\Sigma_\theta=\{\lambda\in\mathbb{C}\setminus\{0\}:|\arg\lambda|<\theta\}$ the *spectral angle* $\phi_A\in[0,\pi)$ of $A\in\mathcal{S}(X)$ is defined by $\phi_A=\inf\{\phi:\rho(-A)\supset\Sigma_{\pi-\phi},\sup\{\lambda\in\Sigma_{\pi-\phi}:|\lambda(\lambda+A)^{-1}|<\infty\}\}$.

We say that an operator $A\in\mathcal{S}(X)$ admits bounded imaginary powers and write $A\in\mathcal{BIP}(X)$ if the imaginary powers A^{is} form a bounded C_0-group on X. The type θ_A of this group is called the *power angle* of A. For more details on sectorial operators and the class $\mathcal{BIP}(X)$ we refer to [1] and [15].

A function $a\in L_{1,loc}(\mathbb{R}_+)$ is called to be of *subexponential growth* if for all $\varepsilon>0$, $\int_0^\infty e^{-\varepsilon t}|a(t)|\,dt<\infty$. Following [27, Def. 3.3] we say that a kernel $a\in L_{1,loc}(\mathbb{R}_+)$ of subexponential growth is 1-*regular* if there is a constant $c>0$ such that $|\lambda\hat{a}'(\lambda)|\leq c\,|\hat{a}(\lambda)|$ for all $\operatorname{Re}\lambda>0$. A kernel $a\in L_{1,loc}(\mathbb{R}_+)$ of subexponential growth satisfying $\hat{a}(\lambda) \neq 0$, $\operatorname{Re}\lambda > 0$, is called *$\theta$-sectorial* ($\theta > 0$) if $|\arg \hat{a}(\lambda)| \leq \theta$ for all $\operatorname{Re}\lambda > 0$ (cp. [27, Def. 3.2]). The subsequent class of kernels will be basic to the problems studied in this paper.

Definition 2.1 *Let $a \in L_{1,loc}(\mathbb{R}_+)$ be of subexponential growth, and let $\theta_a > 0$, and $\alpha \geq 0$. Then a is said to belong to the class $\mathcal{K}^1(\alpha, \theta_a)$ if a is 1-regular and θ_a-sectorial, and satisfies*

$$\limsup_{\mu\to\infty} |\,\hat{a}(\mu)|\,\mu^\alpha < \infty, \ \liminf_{\mu\to\infty} |\,\hat{a}(\mu)|\,\mu^\alpha > 0, \ \liminf_{\mu\to 0} |\,\hat{a}(\mu)| > 0.$$

To give an example, consider the so-called *standard kernel* $a(t) = t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$, with $\alpha > 0$. Its Laplace transform is given by $\hat{a}(\lambda) = \lambda^{-\alpha}$, $\operatorname{Re}\lambda > 0$. Hence $a \in \mathcal{K}^1(\alpha, \alpha\frac{\pi}{2})$.

The reason for considering the above class of kernels lies in the following two fundamental theorems from the monograph Prüss [27] on the inversion of the convolution in $L_p(\mathbb{R}; X)$.

Theorem 2.1 ([27, Thm. 8.6]) *Let $X \in \mathcal{HT}$, and $p \in (1, \infty)$. Assume $a \in L_{1,loc}(\mathbb{R}_+)$ is of subexponential growth, and that a is 1-regular and θ-sectorial, where $\theta < \pi$. Then there is a unique operator $B \in \mathcal{S}(L_p(\mathbb{R}; X))$ such that $(Bf)\tilde{}(\rho) = \hat{a}(i\rho)^{-1}\tilde{f}(\rho)$, $\rho \in \mathbb{R}$, $\tilde{f} \in C_0^\infty(\mathbb{R}\setminus\{0\}; X)$. Moreover, B is causal, commutes with the group of translations, and belongs to the class $\mathcal{BIP}(L_p(\mathbb{R}; X))$ with $\theta_B = \phi_B = \sup\{|\arg \hat{a}(\lambda)| \, : \, \operatorname{Re}\lambda > 0\}$.*

Proposition 2.1 ([27, Cor. 8.1]) *Assume $X \in \mathcal{HT}$, $p \in (1, \infty)$, $a \in \mathcal{K}^1(\alpha, \theta)$ with $\theta < \pi$, and let B be the operator from Theorem 2.1. Then $\mathcal{D}(B) = H_p^\alpha(\mathbb{R}; X)$.*

As a consequence of these two results one obtains (see [34, Cor. 2.1])

Corollary 2.1 *Let $X \in \mathcal{HT}$, $p \in (1, \infty)$, and $J = [0, T]$. Suppose $a \in \mathcal{K}^1(\alpha, \theta)$ with $\theta < \pi$. Then the restriction $\mathcal{B} := B|_{L_p(J;X)}$ of the operator B constructed in Theorem 2.1 to $L_p(J; X)$ is well-defined. The operator $\mathcal{B}$ belongs to $\mathcal{BIP}(L_p(J; X))$ with $\theta_{\mathcal{B}} \leq \theta_B = \theta_a$ and is invertible satisfying $\mathcal{B}^{-1}w = a * w$ for all $w \in L_p(J; X)$. Moreover $\mathcal{D}(\mathcal{B}) = {}_0H_p^\alpha(J; X)$.*

Example 2.1 For $a_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$, $\alpha \in (0, 1)$, the operator $\mathcal{B}$ in Corollary 2.1 takes the form $\mathcal{B}u(t) = d/dt \int_0^t a_{1-\alpha}(t-s)u(s)\,ds$, $t > 0$, $u \in {}_0H_p^\alpha(J; X)$, thus coincides with $(d/dt)^\alpha$, the *derivation operator of order α*.

Suppose we are in the situation of Corollary 2.1. Assume that $\alpha \in (0, 2) \setminus \{1/p, 1 + 1/p\}$ and let $\mu > 0$. For $f \in H_p^\alpha(J; X)$, we put with some abuse of language

$$|f|^{(a,\mu)}_{H_p^\alpha(J;X)} = \begin{cases} |\mathcal{B}f|_{L_p(J;X)} & : \alpha \in (0, \frac{1}{p}) \\ |\mathcal{B}(f - f(0))|_{L_p(J;X)} + \mu|f(0)|_X & : \alpha \in (\frac{1}{p}, 1 + \frac{1}{p}) \\ |\mathcal{B}(f - f(0) - t\dot{f}(0))|_{L_p(J;X)} + \mu|f(0)|_X + \mu|\dot{f}(0)|_X & : \alpha \in (1 + \frac{1}{p}, 2). \end{cases} \tag{5}$$

This is a well-defined expression in view of Sobolev's embedding theorem. It is not difficult to prove the following result, see [35, Corollary 2.8.2].

Corollary 2.2 *Let the assumptions of Corollary 2.1 hold. Let $\mu > 0$, and assume that $\alpha \in (0, 2) \setminus \{1/p, 1 + 1/p\}$. Then (5) defines an equivalent norm for $H_p^\alpha(J; X)$.*

3 Linear Problems

3.1 Full space problems

In this subsection we study the Volterra equation

$$v + k * \mathcal{A}(\cdot, x, D_x)v = f, \quad t \in J,\ x \in \mathbb{R}^n, \tag{6}$$

in $L_p(J; L_p(\mathbb{R}^n))$. Here $J = [0,T]$ and $\mathcal{A}(t,x,D_x)$ is a differential operator of second order with variable coefficients:

$$\mathcal{A}(t,x,D_x) = -a(t,x) : \nabla_x^2 + b_1(t,x) \cdot \nabla_x + b_0(t,x),\ t \in J,\ x \in \mathbb{R}^n. \tag{7}$$

We denote the principal part of the differential operator (7) by $\mathcal{A}_\#(t,x,D_x)$, that is $\mathcal{A}_\#(t,x,D_x) = -a(t,x) : \nabla_x^2$, and we write $\mathcal{A}(t,x,D_x) = \mathcal{A}_\#(t,x,D_x) + \mathcal{A}_R(t,x,D_x)$.

For an unbounded domain $\Omega \subset \mathbb{R}^n$ and a Banach space X, we set

$$C_{ul}(J \times \overline{\Omega}; X) = \{g \in C(J \times \overline{\Omega}; X) : \lim_{|x| \to \infty} g(t,x) \text{ exists uniformly for all } t \in J\}.$$

The symbol $\mathrm{Sym}\{n\}$ stands for the space of n-dimensional real symmetric matrices.

The goal of this subsection is to prove the following result.

Theorem 3.1 *Let $1 < p < \infty$ and $n \in \mathbb{N}$. Suppose the differential operator $\mathcal{A}(t,x,D_x)$ is given by (7). Assume the following properties.*

(H1) $k \in \mathcal{K}^1(\alpha, \theta)$, where $\alpha \in (0,2) \setminus \{1/p, 1+1/p\}$, $\theta < \pi$;

(H2) $a \in C_{ul}(J \times \mathbb{R}^n, \mathrm{Sym}\{n\})$, $b_1 \in L_\infty(J \times \mathbb{R}^n, \mathbb{R}^n)$, $b_0 \in L_\infty(J \times \mathbb{R}^n)$;

(H3) $\exists a_0 > 0 : a(t,x)\xi \cdot \xi \geq a_0 |\xi|^2$, $t \in J$, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$.

Then the Volterra equation (6) admits a unique solution in the space

$$Z := H_p^\alpha(J; L_p(\mathbb{R}^n)) \cap L_p(J; H_p^2(\mathbb{R}^n))$$

if and only if the function f is subject to the subsequent conditions.

(i) $\quad f \in H_p^\alpha(J; L_p(\mathbb{R}^n))$;

(ii) $\quad f|_{t=0} \in \gamma_0 Z := B_{pp}^{2 - \frac{2}{p\alpha}}(\mathbb{R}^n)$, *if* $\alpha > 1/p$;

(iii) $\quad \partial_t f|_{t=0} \in \gamma_1 Z := B_{pp}^{2(1 - \frac{1}{\alpha} - \frac{1}{p\alpha})}(\mathbb{R}^n)$, *if* $\alpha > 1 + 1/p$.

Remark 3.1 Note that Theorem 3.1 and the subsequent theorems on linear problems have to be understood in the following sense: whenever necessary resp. sufficient conditions are stated in terms of regularity classes, this, by convention, means that the corresponding *a priori* estimates hold true.

Proof. We start with the necessity part. Suppose that $v \in Z$ solves (6). By assumption (H2), we immediately verify that $\mathcal{A}(t,x,D_x)v \in L_p(J; L_p(\mathbb{R}^n))$. Thus, in view of (H1) and by Corollary 2.1, $f = v - k * \mathcal{A}(\cdot, x, D_x)v \in H_p^\alpha(J; L_p(\mathbb{R}^n))$, i.e. condition (i) is satisfied. Note that $L_p(\mathbb{R}^n)$ is an $\mathcal{HT}$-space for any $1 < p < \infty$, cf. [4]. To see (ii) and (iii), apply [34, Theorem 3.6] to the space $L_p(\mathbb{R}^n)$ and the operator $A = -\Delta$ with domain $H_p^2(\mathbb{R}^n)$, and use $(L_p(\mathbb{R}^n), H_p^2(\mathbb{R}^n))_{s,p} = B_{pp}^{2s}(\mathbb{R}^n)$, $s \in (0,1)$, together with $\partial_t^j v|_{t=0} = \partial_t^j f|_{t=0}$ in case $\alpha > j + 1/p$, $j = 0, 1$. Note that $A \in \mathcal{BIP}(L_p(\mathbb{R}^n))$ (see e.g. [15, Theorem 5.5]), and hence [34, Theorem 3.6] is applicable.

The sufficiency part is more involved. Suppose f satisfies (i)-(iii). In order to prove existence of a unique solution of (6) in Z, we use localization and perturbation to reduce (6) to related equations with constant coefficients.

Given $\eta > 0$, assumption (H2) allows us to select a large ball $B_{r_0}(0) \subset \mathbb{R}^n$ such that
$$|a(t,x) - a(t,\infty)|_{\mathcal{B}(\mathbb{R}^{n\times n})} \le \frac{\eta}{2}, \quad \text{for all } t \in J,\, x \in \mathbb{R}^n, |x| \ge r_0.$$
Putting $U_0 = \mathbb{R}^n \setminus \overline{B_{r_0}(0)}$ we can further cover $\overline{B_{r_0}(0)}$ by finitely many balls $U_j = B_{r_j}(x_j)$, $j = 1, \dots, N$, and choose a partition $0 =: T_0 < T_1 < \ldots < T_{M-1} < T_M := T$ such that for all $i = 0, \dots, M-1$, $j = 1, \dots, N$,
$$|a(t,x) - a(T_i, x_j)|_{\mathcal{B}(\mathbb{R}^{n\times n})} \le \eta, \quad t \in [T_i, T_{i+1}],\, x \in \overline{B_{r_j}(x_j)},$$
and
$$|a(t,\infty) - a(T_i,\infty)|_{\mathcal{B}(\mathbb{R}^{n\times n})} \le \frac{\eta}{2}, \quad t \in [T_i, T_{i+1}].$$
Define coefficients of spatially local operators $\mathcal{A}^j_{\#}(t,x,D_x) = -a^j(t,x) : \nabla_x^2$ e.g. by reflection, that is
$$a^0(t,x) := \begin{cases} a(t,x) & : t \in J,\, x \notin B_{r_0}(0) \\ a(t, r_0^2 \frac{x}{|x|^2}) & : t \in J,\, x \in B_{r_0}(0) \end{cases}$$
and
$$a^j(t,x) := \begin{cases} a(t,x) & : t \in J,\, x \in B_{r_j}(x_j) \\ a(t, x_j + r_j^2 \frac{x - x_j}{|x - x_j|^2}) & : t \in J,\, x \notin B_{r_j}(x_j) \end{cases}$$
for each $j = 1, \dots, N$. With $x_0 = \infty$, we then have
$$|a^j(t,x) - a(T_i, x_j)|_{\mathcal{B}(\mathbb{R}^{n\times n})} \le \eta, \quad t \in [T_i, T_{i+1}],\, x \in \mathbb{R}^n,$$
for all $i = 0, \dots, M-1$, $j = 0, \dots, N$.

The strategy for solving (6) is now as follows. First we determine the solution of (6) on the interval $[0, T_1]$. Let us denote it by v^1. Now suppose we already know the solution v^i of (6) on the interval $[0, T_i]$, where $1 \le i \le M-1$. We then seek the solution v^{i+1} of (6) on the larger interval $[0, T_{i+1}]$ which equals v^i on $[0, T_i]$. The last step is repeated as long as $i < M$. Proceeding in this way we finally obtain the solution of (6) on the entire interval $[0, T]$. This is the basic idea concerning the localization in *time*. Besides, we will also localize (6) with respect to the *space* variable. Following [15, Chap. II], this will be done in each single time step by means of a partition of unity $\{\varphi_j\}_{j=0}^N \subset C^\infty(\mathbb{R}^n)$ which enjoys the properties $\sum_{j=0}^N \varphi_j \equiv 1$, $0 \le \varphi_j(x) \le 1$ and $\operatorname{supp} \varphi_j \subset U_j$. We will also make use of a fixed family $\{\psi_j\}_{j=0}^N \subset C^\infty(\mathbb{R}^n)$ that satisfies $\psi_j \equiv 1$ on an open set V_j containing $\operatorname{supp} \varphi_j$, and $\operatorname{supp} \psi_j \subset U_j$.

Suppose we are in the $(i+1)$th time step of the above procedure. Set ${}_{(0)}Z_{i+1} = {}_{(0)}H^\alpha_p([0, T_{i+1}]; L_p(\mathbb{R}^n)) \cap L_p([0, T_{i+1}]; H^2_p(\mathbb{R}^n))$. If $i = 0$ (initial time step), we have to find v^1 in the space
$$Z_1(v^0) := \{w \in Z_1 : \partial_t^m w|_{t=0} = \partial_t^m f|_{t=0}, \text{ if } \alpha > m + 1/p,\, m = 0, 1\}.$$
If $i > 0$, we assume that $v^i =: \mathcal{V}_i^{-1} f$ lies in Z_i. Here $\mathcal{V}_i$ refers to the operator $I + k * \mathcal{A}(\cdot, x, D_x)$ on $[0, T_i]$. Using the notation
$$Z_{i+1}(\tilde{w}) := \{w \in Z_{i+1} : w|_{[0,T_i]} = \tilde{w}\} \quad \text{for } \tilde{w} \in Z_i, \tag{8}$$
our aim is then to determine v^{i+1} in the space $Z_{i+1}(v^i)$. To achieve this, observe that with $d(w_1, w_2) = |w_1 - w_2|_{Z_{i+1}}$, $(Z_{i+1}(v^i), d)$ is a complete metric space. Our

plan is to transform (6) to an appropriate fixed point equation in $Z_{i+1}(v^i)$ and to apply the contraction mapping principle.

We first derive the local equations associated with $\{\varphi_j\}_{j=0}^N$. Note that (6) is equivalent to

$$v + k * \mathcal{A}_\#(\cdot, x, D_x)v = f - k * \mathcal{A}_R(\cdot, x, D_x)v,$$

which when multiplied by φ_j becomes

$$\begin{aligned}\varphi_j v + k * \mathcal{A}^j_\#(\cdot, x, D_x)\varphi_j v &= \varphi_j f - k * \varphi_j \mathcal{A}_R(\cdot, x, D_x)v \\ &\quad + k * [\mathcal{A}_\#(\cdot, x, D_x), \varphi_j]v. \end{aligned} \tag{9}$$

We freeze the coefficients of the local operator $\mathcal{A}^j_\#(t, x, D_x)$ at the point (T_i, x_j) to get the homogeneous differential operator with constant coefficients $\mathcal{A}^{ij}(D_x) := \mathcal{A}^j_\#(T_i, x_j, D_x)$. Then (9) can be written as

$$\begin{aligned}\varphi_j v + k * \mathcal{A}^{ij}(D_x)\varphi_j v &= \varphi_j f - k * \varphi_j \mathcal{A}_R(\cdot, x, D_x)v + k * [\mathcal{A}_\#(\cdot, x, D_x), \varphi_j]v \\ &\quad + k * (\mathcal{A}^j_\#(T_i, x_j, D_x) - \mathcal{A}^j_\#(\cdot, x, D_x))\varphi_j v. \end{aligned} \tag{10}$$

Let A^{ij} be the L_p-realization of the differential operator $\mathcal{A}^{ij}(D_x)$. For $l \in \{1, \ldots, M\}$, we put $X_l = L_p([0, T_l]; L_p(\mathbb{R}^n))$ and define the space Ξ_l as the set of all functions $g \in H^\alpha_p([0, T_l]; L_p(\mathbb{R}^n))$ satisfying $g|_{t=0} \in \gamma_0 Z$ in case $\alpha > 1/p$, and $\partial_t g|_{t=0} \in \gamma_1 Z$ in case $\alpha > 1 + 1/p$. Ξ_l endowed with the norm

$$|g|_{\Xi_l} = |g|^{(k,1)}_{H^\alpha_p([0,T_l];L_p(\mathbb{R}^n))} + \chi_{(\frac{1}{p},2)}(\alpha)|g|_{t=0}|_{\gamma_0 Z} + \chi_{(1+\frac{1}{p},2)}(\alpha)|\partial_t g|_{t=0}|_{\gamma_1 Z}$$

is a Banach space, by Corollary 2.2. Recall that for $g \in {}_0\Xi_l := {}_0H^\alpha_p([0, T_l]; L_p(\mathbb{R}^n))$, one has $|g|^{(k,1)}_{H^\alpha_p([0,T_l];L_p(\mathbb{R}^n))} = |\mathcal{B}_k g|_{X_l}$, $\mathcal{B}_k$ denoting the inverse convolution operator in X_l associated with the kernel k (see Corollary 2.1). For the spaces Z_l, $l = 1, \ldots, M$, we choose the norm

$$|w|_{Z_l} = |w|^{(k,1)}_{H^\alpha_p([0,T_l];L_p(\mathbb{R}^n))} + |\nabla^2_x w|_{X_l^{n^2}}.$$

Claim 1: Let $i \in \{0, \ldots, M-1\}$, $j \in \{0, \ldots, N\}$, and $l \in \{1, \ldots, M\}$. Then

$$w + k * A^{ij} w = g, \quad t \in [0, T_l],\, x \in \mathbb{R}^n, \tag{11}$$

possesses a unique solution $w =: \mathcal{L}^{ij}_l g$ in the space Z_l if and only if $g \in \Xi_l$. Further, there exists a constant $C > 0$ not depending on i, j, l such that

$$|\mathcal{L}^{ij}_l g|_{Z_l} \le C|g|_{\Xi_l}, \quad \forall g \in {}_0\Xi_l. \tag{12}$$

Claim 1 is an immediate consequence of [34, Theorem 3.4]. Note that after a rotation and a stretch of the spatial coordinates, the elliptic operator A^{ij} becomes the negative Laplacian. The constant C in (12) can be selected to be independent of i and j, due to the uniform ellipticity assumption (H3); the independence on l is clear, because in (12), functions g in the subspace ${}_0\Xi_l$ are considered, only.

By applying the solution operator $\mathcal{L}^{ij}_{i+1}$ to (10) we get

$$(I - S^{ij})\varphi_j v = \mathcal{L}^{ij}_{i+1}(\varphi_j f) + \mathcal{L}^{ij}_{i+1}\, k * \mathcal{C}_j(\cdot, x, D_x)v =: h^{ij}(f, v), \tag{13}$$

where

$$S^{ij} w = \mathcal{L}^{ij}_{i+1}\, k * (\mathcal{A}^j_\#(T_i, x_j, D_x) - \mathcal{A}^j_\#(\cdot, x, D_x))w, \tag{14}$$

and

$$\mathcal{C}_j(t,x,D_x) = [\mathcal{A}_\#(t,x,D_x), \varphi_j] - \varphi_j \mathcal{A}_R(t,x,D_x). \tag{15}$$

One immediately verifies that $S^{ij} \in \mathcal{B}(Z_{i+1})$. Furthermore, S^{ij} enjoys the subsequent properties, which will be shown at the end of this proof.

Claim 2: There exists $\eta_0 > 0$ such that whenever $\eta \le \eta_0$, $i \in \{0,\dots,M-1\}$, and $j \in \{0,\dots,N\}$,

(a) $|S^{ij}w|_{Z_{i+1}} \le \frac{1}{2}|w|_{Z_{i+1}}$ for all $w \in Z_{i+1}(0)$ $(Z_1(0) := {}_0Z_1)$;

(b) if $w \in Z_{i+1}(0)$ and $w_0 = (I - S^{ij})w$, then $|w|_{Z_{i+1}} \le 2|w_0|_{Z_{i+1}}$;

(c) for each $g \in \Xi_{i+1}$ and $v \in Z_{i+1}(\mathcal{V}_i^{-1}g)$, the equation $(I - S^{ij})w = h^{ij}(g,v)$ admits a unique solution $w =: (I-S^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}_i^{-1}g)} h^{ij}(g,v)$ in $Z_{i+1}(\varphi_j \mathcal{V}_i^{-1}g)$. Here $Z_1(\mathcal{V}_0^{-1}g) := \{v \in Z_1 : \partial_t^m v|_{t=0} = \partial_t^m g|_{t=0},$ if $\alpha > m + 1/p$, $m = 0,1\}$.

Let $\eta \le \eta_0$. By employing the operators $(I - S^{ij})|^{-1}_{Z_{i+1}(\varphi_j v^i)}$ described in Claim 2 we infer from (13) that

$$\varphi_j v = (I - S^{ij})|^{-1}_{Z_{i+1}(\varphi_j v^i)} h^{ij}(f,v).$$

Since $\psi_j \equiv 1$ on $\operatorname{supp}\varphi_j$, we may multiply this equation by ψ_j resulting in

$$\varphi_j v = \psi_j (I - S^{ij})|^{-1}_{Z_{i+1}(\varphi_j v^i)} h^{ij}(f,v).$$

Summing over j then yields

$$v = \sum_{j=0}^{N} \psi_j (I - S^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}_i^{-1} f)} \mathcal{L}^{ij}_{i+1}(\varphi_j f + k * \mathcal{C}_j(\cdot,x,D_x)v) =: \mathcal{G}(v), \tag{16}$$

which is a fixed point equation for $v \in Z_{i+1}(v^i)$.

Due to Claim 2(c), $\mathcal{G}$ is a self-mapping of $Z_{i+1}(v^i)$. Thus the contraction principle is applicable to (16), if we can verify that $\mathcal{G}$ is a strict contraction. It turns out that this can be achieved by choosing a yet finer partition $0 = T_0 < T_1 < \ldots < T_{M-1} < T_M = T$, more precisely, by making $\delta := \max_i |T_{i+1} - T_i|$ sufficiently small. The following observation is crucial in this connection.

Suppose $u \in Z_{i+1}(0)$. By causality, it is clear that $\mathcal{B}_k u|_{[0,T_i]} = 0$. So we can write

$$u = k * \mathcal{B}_k u = (k\chi_{[0,T_{i+1}-T_i]}) * \mathcal{B}_k u.$$

Thus, by Young's inequality,

$$|u|_{X_{i+1}} \le |k|_{L_1(0,T_{i+1}-T_i)} |\mathcal{B}_k u|_{X_{i+1}} \le |k|_{L_1(0,\delta)} |u|_{Z_{i+1}} \quad \forall u \in Z_{i+1}(0). \tag{17}$$

Another observation concerns the operators $\mathcal{C}_j(t,x,D_x)$. Since those are at most of first order and have bounded coefficients, for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|\mathcal{C}_j(t,x,D_x)w|_{X_{i+1}} \le \varepsilon |\nabla_x^2 w|_{X^{n^2}_{i+1}} + C_\varepsilon |w|_{X_{i+1}} \tag{18}$$

for all $i = 0,\dots,M-1$, $j = 0,\dots,N$, and $w \in L_p([0,T_{i+1}]; H^2_p(\mathbb{R}^n))$.

We are now ready to prove the contractivity of $\mathcal{G}$. Let v, $\bar v \in Z_{i+1}(v^i)$. In view of (12), Claim 2(b), (17), and (18), we may estimate

$$\begin{aligned}
|\mathcal{G}(v)-\mathcal{G}(\bar v)|_{Z_{i+1}} &= |\sum_{j=0}^{N} \psi_j (I-S^{ij})|^{-1}_{Z_{i+1}(0)} \mathcal{L}^{ij}_{i+1} k * \mathcal{C}_j(\cdot,x,D_x)(v-\bar v)|_{Z_{i+1}} \\
&\le C_0 \sum_{j=0}^{N} |\mathcal{L}^{ij}_{i+1}\, k * \mathcal{C}_j(\cdot,x,D_x)(v-\bar v)|_{Z_{i+1}} \\
&\le C_0 C_1 \sum_{j=0}^{N} |\mathcal{C}_j(\cdot,x,D_x)(v-\bar v)|_{X_{i+1}} \\
&\le C_0 C_1 \sum_{j=0}^{N} (\varepsilon |\nabla_x^2 (v-\bar v)|_{X^{n^2}_{i+1}} + C_\varepsilon |(v-\bar v)|_{X_{i+1}}) \\
&\le C_0 C_1 (N+1)\Big(\varepsilon + C_\varepsilon |k|_{L_1(0,\delta)}\Big) |v-\bar v|_{Z_{i+1}} =: \kappa(\varepsilon,\delta)|v-\bar v|_{Z_{i+1}}, \qquad (19)
\end{aligned}$$

with C_0, C_1 and N being independent of δ. This shows existence of a *left* inverse $\mathcal{Q}_{i+1}$ for the operator

$$\mathcal{V}_{i+1} = I + k * \mathcal{A}(\cdot,x,D_x) : Z_{i+1}(v^i) \to \Xi_{i+1}(\mathcal{V}_i v^i), \qquad (20)$$

provided that $\kappa < 1$, that is, if the numbers ε and δ are selected sufficiently small. The symbol $\Xi_{i+1}(\mathcal{V}_i v^i)$ in (20) has to be understood like the corresponding one for Z defined in (8).

We still have to show that $v^{i+1} = \mathcal{Q}_{i+1} f$ indeed solves (6) on $[0, T_{i+1}]$, i.e. that $\mathcal{V}_{i+1}$ is a surjection. To this purpose we define the linear operator $\mathcal{K}_{i+1} : \Xi_{i+1} \to {}_0\Xi_{i+1}$ by

$$\mathcal{K}_{i+1} g = k * \sum_{j=0}^{N} [\mathcal{A}_\#(\cdot,x,D_x),\psi_j](I-S^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}_i^{-1} g)} h^{ij}(g, \mathcal{Q}_{i+1} g).$$

The commutators $[\mathcal{A}_\#(t,x,D_x),\psi_j]$ are differential operators of order ≤ 1. Thus for δ sufficiently small, we see that the mapping $g \mapsto f - \mathcal{K}_{i+1} g$ is a strict contraction in the space $\{g \in \Xi_{i+1} : \partial_t^m g|_{t=0} = \partial_t^m f|_{t=0},\ \text{if } \alpha > m + 1/p,\ m = 0,1\}$. That means for such δ, there exists $g \in \Xi_{i+1}$ satisfying $g + \mathcal{K}_{i+1} g = f$. We apply now $\mathcal{V}_{\#,\,i+1} := I + k * \mathcal{A}_\#(\cdot,x,D_x)$ to $v = Q_{i+1} g$ in (16) with f replaced by g. This gives

$$\begin{aligned}
\mathcal{V}_{\#,\,i+1}\mathcal{Q}_{i+1} g &= \sum_{j=0}^{N} \mathcal{V}_{\#,\,i+1}\psi_j (I-S^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}_i^{-1} g)} h^{ij}(g,\mathcal{Q}_{i+1} g) \\
&= \sum_{j=0}^{N} \psi_j(\varphi_j g + k * \mathcal{C}_j(\cdot,x,D_x)\mathcal{Q}_{i+1} g) + \mathcal{K}_{i+1} g.
\end{aligned}$$

From $\sum_j \varphi_j = 1$ we infer that $\sum_j [\mathcal{A}_\#(t,x,D_x),\varphi_j] = 0$. Using this, together with the fact that $\psi_j \equiv 1$ on $\operatorname{supp}\varphi_j$, we see that

$$\sum_{j=0}^{N} \psi_j(\varphi_j g + k * \mathcal{C}_j(\cdot,x,D_x)\mathcal{Q}_{i+1} g) = g - k * \mathcal{A}_R(\cdot,x,D_x)\mathcal{Q}_{i+1} g.$$

Therefore

$$\mathcal{V}_{i+1}\mathcal{Q}_{i+1} g = g + \mathcal{K}_{i+1} g = f. \qquad (21)$$

Hence $\mathcal{V}_{i+1}$ is surjective, provided that δ is sufficiently small.

Concluding, we have proven that (6) admits a unique solution $v \in Z$.

It remains to prove Claim 2. Let $w \in Z_{i+1}(0)$. Thanks to Claim 1 we may estimate

$$\begin{aligned}
|S^{ij}w|_{Z_{i+1}} &= |\mathcal{L}^{ij}_{i+1}\, k * (\mathcal{A}^j_\#(T_i, x_j, D_x) - \mathcal{A}^j_\#(\cdot, x, D_x))w|_{Z_{i+1}} \\
&\le C|(\mathcal{A}^j_\#(t, x, D_x) - \mathcal{A}^j_\#(T_i, x_j, D_x))w|_{X_{i+1}} \\
&\le C\,|(a^j(\cdot,\cdot) - a(T_i, x_j)) : \nabla^2_x w|_{L_p([T_i,T_{i+1}];L_p(\mathbb{R}^n))} \\
&\le C\eta\,|\nabla^2_x w|_{X^{n^2}_{i+1}} \\
&\le C\eta\,|w|_{Z_{i+1}} \le \tfrac{1}{2}|w|_{Z_{i+1}}, \qquad (22)
\end{aligned}$$

provided that $\eta \le \eta_0 := 1/2C$. This shows (a). Suppose now that $w \in Z_{i+1}(0)$ and $w_0 = (I - S^{ij})w$. Then in view of (a)

$$|w|_{Z_{i+1}} \le |S^{ij}w|_{Z_{i+1}} + |w_0|_{Z_{i+1}} \le \tfrac{1}{2}|w|_{Z_{i+1}} + |w_0|_{Z_{i+1}}.$$

Hence (b) holds.

Turning to (c), let $g \in \Xi_{i+1}$ and $v \in Z_{i+1}(\mathcal{V}_i^{-1}g)$. Evidently $h^{ij}(g,v) \in Z_{i+1}$. If $i > 0$, we further have

$$\begin{aligned}
&h^{ij}(g,v)|_{[0,T_i]} = (\mathcal{L}^{ij}_{i+1}(\varphi_j g + k * \mathcal{C}_j(\cdot, x, D_x)v))|_{[0,T_i]} \\
&\quad = \mathcal{L}^{ij}_i(\varphi_j g|_{[0,T_i]} + k * \mathcal{C}_j(\cdot, x, D_x)\mathcal{V}_i^{-1}g) \qquad \text{(here $*$ is meant on $[0,T_i]$)} \\
&\quad = \varphi_j\mathcal{V}_i^{-1}g - \mathcal{L}^{ij}_i k * (\mathcal{A}^j_\#(\cdot, x, D_x) - \mathcal{A}^j_\#(T_i, x_j, D_x))\varphi_j\mathcal{V}_i^{-1}g.
\end{aligned}$$

Thus, $w \in Z_{i+1}(\varphi_j\mathcal{V}_i^{-1}g)$ implies $F(w) := S^{ij}w + h^{ij}(g,v) \in Z_{i+1}(\varphi_j\mathcal{V}_i^{-1}g)$. In fact,

$$\begin{aligned}
(F(w))|_{[0,T_i]} &= \mathcal{L}^{ij}_i k * (\mathcal{A}^j_\#(\cdot, x, D_x) - \mathcal{A}^j_\#(T_i, x_j, D_x))\varphi_j\mathcal{V}_i^{-1}g + h^{ij}(g,v)|_{[0,T_i]} \\
&= \varphi_j\mathcal{V}_i^{-1}g.
\end{aligned}$$

So F is a self-mapping of $Z_{i+1}(\varphi_j\mathcal{V}_i^{-1}g)$. This is true for $i = 0$, too. Compare the initial values of all terms occurring above to see this. In view of (22), F is also a strict contraction. Hence the assertion follows by the contraction mapping principle. $\square$

3.2 Half space problems

Let $J = [0,T]$, $n \ge 2$, and $\mathbb{R}^n_+ = \{x := (x', y) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\, y > 0\}$.

Define the operators $\mathcal{A}(t, x, D_x)$ and $\mathcal{B}(t, x', D_x)$ by

$$\mathcal{A}(t, x, D_x) = -a(t,x) : \nabla^2_x + a_1(t,x)\cdot\nabla_x + a_0(t,x),\ t \in J,\ x \in \mathbb{R}^n_+,$$

and

$$\mathcal{B}(t, x', D_x) = -\partial_y + b(t,x')\cdot\nabla_{x'} + b_0(t,x'),\ t \in J,\ x' \in \mathbb{R}^{n-1}, \qquad (23)$$

respectively. We consider the two separate problems

$$\left\{\begin{aligned} v + k * \mathcal{A}(\cdot, x, D_x)v = f, &\quad t \in J,\ x \in \mathbb{R}^n_+, \\ v = g, &\quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y = 0 \end{aligned}\right., \qquad (24)$$

$$\left\{\begin{aligned} v + k * \mathcal{A}(\cdot, x, D_x)v = f, &\quad t \in J,\ x \in \mathbb{R}^n_+, \\ \mathcal{B}(t, x', D_x)v = h, &\quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y = 0 \end{aligned}\right., \qquad (25)$$

where the kernel k is as in the preceding subsection. We seek unique solutions in the maximal regularity class

$$Z = H^\alpha_p(J; L_p(\mathbb{R}^n_+)) \cap L_p(J; H^2_p(\mathbb{R}^n_+)).$$

Concerning (24), we have the following result.

Theorem 3.2 *Let $1 < p < \infty$, $J = [0,T]$, $n \geq 2$, and $k \in \mathcal{K}^1(\alpha,\theta)$, where $\theta < \pi$, and $\alpha \in (0,2)\backslash\left\{\frac{1}{p},\ \frac{2}{2p-1},\ 1+\frac{1}{p},\ 1+\frac{3}{2p-1}\right\}$. Suppose $a \in C_{ul}(J\times\overline{\mathbb{R}^n_+}, \mathrm{Sym}\{n\})$, $a_1 \in L_\infty(J\times\mathbb{R}^n_+,\mathbb{R}^n)$, $a_0 \in L_\infty(J\times\mathbb{R}^n_+)$, and assume further that there exists $c_0 > 0$ such that $a(t,x)\xi\cdot\xi \geq c_0|\xi|^2$, $t\in J$, $x\in\overline{\mathbb{R}^n_+}$, $\xi\in\mathbb{R}^n$.*

Then (24) has a unique solution in the space Z if and only if the data f and g are subject to the subsequent conditions.

$$
\begin{array}{ll}
(i) & f\in H^\alpha_p(J;L_p(\mathbb{R}^n_+));\\
(ii) & g\in B^{\alpha(1-\frac{1}{2p})}_{pp}(J;L_p(\mathbb{R}^{n-1}))\cap L_p(J;B^{2-\frac{1}{p}}_{pp}(\mathbb{R}^{n-1}));\\
(iii) & f|_{t=0}\in B^{2-\frac{2}{p\alpha}}_{pp}(\mathbb{R}^n_+),\ \text{if}\ \alpha>\frac{1}{p};\\
(iv) & f|_{t=0,\,y=0}=g|_{t=0},\ \text{if}\ \alpha>\frac{2}{2p-1};\\
(v) & \partial_t f|_{t=0}\in B^{2(1-\frac{1}{\alpha}-\frac{1}{p\alpha})}_{pp}(\mathbb{R}^n_+),\ \text{if}\ \alpha>1+\frac{1}{p};\\
(vi) & \partial_t f|_{t=0,\,y=0}=\partial_t g|_{t=0},\ \text{if}\ \alpha>1+\frac{3}{2p-1}.
\end{array}
$$

In order to formulate the corresponding result for (25), we put $s_1 = \alpha(\frac{1}{2}-\frac{1}{2p})$, $s_2 = 1-\frac{1}{p}$, as well as $Y = Y_1\cap Y_2$, where $Y_1 = B^{s_1}_{pp}(J;L_p(\mathbb{R}^{n-1}))$ and $Y_2 = L_p(J;B^{s_2}_{pp}(\mathbb{R}^{n-1}))$.

Theorem 3.3 *Let $1 < p < \infty$, $J = [0,T]$, $n \geq 2$, and $k \in \mathcal{K}^1(\alpha,\theta)$, where $\theta < \pi$, and $\alpha \in (0,2)\setminus\left\{\frac{1}{p},\ \frac{2}{p-1},\ 1+\frac{1}{p}\right\}$. Let further a, a_1, and a_0 be as in Theorem 3.2. Assume $b \in C_{ul}(J\times\mathbb{R}^{n-1})$, as well as $(b,b_0)\in Y^n$, if $p > n+2/\alpha$, and $(b,b_0) \in (C^{r_1}(J;C(\mathbb{R}^{n-1}))\cap C(J;C^{r_2}(\mathbb{R}^{n-1})))^n$ with some $r_i > s_i$, $i=1,2$, otherwise.*

Then (25) has a unique solution in the space Z if and only if the data f and h satisfy the following conditions.

$$
\begin{array}{ll}
(i) & f\in H^\alpha_p(J;L_p(\mathbb{R}^n_+));\\
(ii) & h\in B^{\alpha(\frac{1}{2}-\frac{1}{2p})}_{pp}(J;L_p(\mathbb{R}^{n-1}))\cap L_p(J;B^{1-\frac{1}{p}}_{pp}(\mathbb{R}^{n-1}));\\
(iii) & f|_{t=0}\in B^{2-\frac{2}{p\alpha}}_{pp}(\mathbb{R}^n_+),\ \text{if}\ \alpha>\frac{1}{p};\\
(iv) & \mathcal{B}(0,x',D_x)f|_{t=0,\,y=0}=h|_{t=0},\ \text{if}\ \alpha>\frac{2}{p-1};\\
(v) & \partial_t f|_{t=0}\in B^{2(1-\frac{1}{\alpha}-\frac{1}{p\alpha})}_{pp}(\mathbb{R}^n_+),\ \text{if}\ \alpha>1+\frac{1}{p}.
\end{array}
$$

We will prove Theorem 3.2 and 3.3 here only in the special case of constant coefficients and differential operators consisting only of their principal parts. Having achieved this, the general case can be obtained by means of localization and perturbation arguments like in the proof of Theorem 3.1; here one also uses certain pointwise multiplication properties of the function spaces occurring on the boundary. For the complete proofs see [35, Section 4.2].

In what follows $a \in \mathrm{Sym}\{n\}$ is positive definite, and $b\in\mathbb{R}^n$. In place of (24) and (25) we separately consider the problems

$$
\left\{\begin{aligned} u-k*a:\nabla_x^2 u &= f, & t\in J,\ x\in\mathbb{R}^n_+,\\ u &= g, & t\in J,\ x'\in\mathbb{R}^{n-1},\ y=0,\end{aligned}\right. \tag{26}
$$

$$
\left\{\begin{aligned} u-k*a:\nabla_x^2 u &= f, & t\in J,\ x\in\mathbb{R}^n_+,\\ -\partial_y u+b\cdot\nabla_{x'}u &= h, & t\in J,\ x'\in\mathbb{R}^{n-1},\ y=0.\end{aligned}\right. \tag{27}
$$

Proof of Theorem 3.2 for (26). By means of a variable transformation of the form $\bar{x} = Q^T\Lambda Qx$, where Q is a rotation matrix and Λ is diagonal with $\Lambda_{ii} = 1/\sqrt{\lambda_i}$ ($\lambda_1,\ldots,\lambda_n$ denoting the positive eigenvalues of the matrix a), problem (26) can be reduced to a problem of the same structure but with $a = I_n$. The assertion follows then from [34, Theorem 5.1] applied to $X = L_p(\mathbb{R}^{n-1})$ and the operator $A = -\Delta_{x'}$, which belongs to $\mathcal{BIP}(X)$ and has power angle 0, and from the fact that all function spaces occurring in that theorem are preserved under the above variable transformation. □

Proof of Theorem 3.3 for (27). Use the variable transformation described in the above proof and normalize the coefficient in front of the normal derivative to reduce (27) to a problem of the same structure with $a = I_n$. The assertion is then a consequence of [34, Theorem 5.2] applied to $X = L_p(\mathbb{R}^{n-1})$, $A = -\Delta_{x'}$, and $D = b \cdot \nabla_{x'}$ $(b \in \mathbb{R}^{n-1})$. Note that $D \in \mathcal{BIP}(\overline{\mathcal{R}(D)})$ with power angle $\theta_D \leq \pi/2$ (cp. Prüss [28, Section 3]) and thus $\theta_D < \pi - \theta/2 = \pi - \max\{\theta, \theta_A\}/2$ showing the second angle condition in Theorem 5.2 in [34]. □

3.3 Problems in domains

In this subsection let $\Omega \subset \mathbb{R}^n$ be a domain with compact C^2-boundary Γ which decomposes according to $\Gamma = \Gamma_D \cup \Gamma_N$ and $\operatorname{dist}(\Gamma_D, \Gamma_N) > 0$. Let further $1 < p < \infty$ and $J = [0, T]$. We consider the problem

$$\left\{ \begin{aligned} v + k * \mathcal{A}(\cdot, x, D_x)v = f, &\quad t \in J,\ x \in \Omega, \\ v = g, &\quad t \in J,\ x \in \Gamma_D, \\ \mathcal{B}(t, x, D_x)v = h, &\quad t \in J,\ x \in \Gamma_N. \end{aligned} \right. \tag{28}$$

Here the differential operators $\mathcal{A}(t, x, D_x)$ and $\mathcal{B}(t, x, D_x)$ are of the form

$$\mathcal{A}(t, x, D_x) = -a(t, x) : \nabla_x^2 + a_1(t, x) \cdot \nabla_x + a_0(t, x),\ t \in J,\ x \in \Omega, \tag{29}$$

respectively

$$\mathcal{B}(t, x, D_x) = b(t, x) \cdot \nabla_x + b_0(t, x),\ t \in J,\ x \in \Gamma_N.$$

Put $Y = B^{s_1}_{pp}(J; L_p(\Gamma_N)) \cap L_p(J; B^{s_2}_{pp}(\Gamma_N))$ with $s_1 = \alpha(\frac{1}{2} - \frac{1}{2p})$, $s_2 = 1 - \frac{1}{p}$, as well as $M = \bigcup_{r_i > s_i} C^{r_1}(J; C(\Gamma_N)) \cap C(J; C^{r_2}(\Gamma_N))$. Let further $\nu(x)$ denote the outer unit normal of Ω at $x \in \Gamma$. Then our assumptions read as follows.

(H1) (*kernel*): $k \in \mathcal{K}^1(\alpha, \theta)$ with $\alpha \in (0, 2) \setminus \left\{ \frac{1}{p},\ \frac{2}{2p-1},\ \frac{2}{p-1},\ 1 + \frac{1}{p},\ 1 + \frac{3}{2p-1} \right\}$ and $\theta < \pi$;
(H2) (*smoothness of coefficients*): $a \in C_{ul}(J \times \overline{\Omega}, \mathrm{Sym}\{n\})$, $a_1 \in L_\infty(J \times \Omega, \mathbb{R}^n)$, $a_0 \in L_\infty(J \times \Omega)$, and $(b, b_0) \in Y^{n+1}$ if $p > n + 2/\alpha$, $(b, b_0) \in M^{n+1}$, otherwise;
(H3) (*uniform ellipticity*): $\exists c_0 > 0$ s.t. $a(t, x)\xi \cdot \xi \geq c_0 |\xi|^2$, $t \in J$, $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$;
(H4) (*normality*): $b(t, x) \cdot \nu(x) \neq 0$, $t \in J$, $x \in \Gamma_N$.

Concerning (28) we have the subsequent result.

Theorem 3.4 *Let $1 < p < \infty$, $J = [0, T]$, and $\Omega \subset \mathbb{R}^n$ be a domain with compact C^2-boundary Γ which decomposes as $\Gamma = \Gamma_D \cup \Gamma_N$ with dist$(\Gamma_D, \Gamma_N) > 0$. Suppose the assumptions (H1)-(H4) are satisfied. Then (28) admits a unique solution in the space*

$$H^\alpha_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega))$$

if and only if the functions f, g, h are subject to the following conditions.

$$
\begin{array}{rl}
(i) & f \in H_p^{\alpha}(J; L_p(\Omega)); \\
(ii) & g \in B_{pp}^{\alpha(1-\frac{1}{2p})}(J; L_p(\Gamma_D)) \cap L_p(J; B_{pp}^{2-\frac{1}{p}}(\Gamma_D)); \\
(iii) & h \in B_{pp}^{\alpha(\frac{1}{2}-\frac{1}{2p})}(J; L_p(\Gamma_N)) \cap L_p(J; B_{pp}^{1-\frac{1}{p}}(\Gamma_N)); \\
(iv) & f|_{t=0} \in B_{pp}^{2-\frac{2}{p\alpha}}(\Omega), \ \text{if}\ \alpha > \frac{1}{p}; \\
(v) & \partial_t f|_{t=0} \in B_{pp}^{2(1-\frac{1}{\alpha}-\frac{1}{p\alpha})}(\Omega), \ \text{if}\ \alpha > 1+\frac{1}{p}; \\
(vi) & f|_{t=0} = g|_{t=0}\ \text{on}\ \Gamma_D, \ \text{if}\ \alpha > \frac{2}{2p-1}; \\
(vii) & \mathcal{B}(0,x,D_x) f|_{t=0} = h|_{t=0}\ \text{on}\ \Gamma_N, \ \text{if}\ \alpha > \frac{2}{p-1}; \\
(viii) & \partial_t f|_{t=0} = \partial_t g|_{t=0}\ \text{on}\ \Gamma_D, \ \text{if}\ \alpha > 1+\frac{3}{2p-1}.
\end{array}
$$

Theorem 3.4 follows from Theorem 3.2 and 3.3 by means of localization, perturbation and coordinate transformations. For the details see [35, Section 4.3].

4 Nonlinear Problems

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary Γ which decomposes as $\Gamma = \Gamma_D \cup \Gamma_N$ with $\mathrm{dist}(\Gamma_D, \Gamma_N) > 0$. Let further $J_0 = [0, T_0]$ be a compact time-interval and $U_0 \subset \mathbb{R}$, $U_1 \subset \mathbb{R}^n$ be nonempty open convex sets. With the functions $a : J_0 \times \overline{\Omega} \times U_0 \times U_1 \to \mathbb{R}^{n\times n}$, $f,\, g : J_0 \times \Omega \times U_0 \times U_1 \to \mathbb{R}$, $b^D : J_0 \times \Gamma_D \times U_0 \to \mathbb{R}$, and $b^N : J_0 \times \Gamma_N \times U_0 \times U_1 \to \mathbb{R}$, we consider the substitution operators defined by (2) for functions $u : J_0 \times \overline{\Omega} \to \mathbb{R}$ in the space $C(J_0; C^1(\overline{\Omega}))$ which are subject to $u(t,x) \in U_0$ and $\nabla u(t,x) \in U_1$, for all $t \in J_0$, $x \in \overline{\Omega}$.

Let further $k \in BV_{loc}(\mathbb{R}_+) \cap \mathcal{K}^1(1+\alpha, \theta)$ with $k(0) = 0$, $\theta < \pi$ and $\alpha \in [0,1)$. Then the problem under consideration reads as

$$
\left\{
\begin{array}{rl}
\partial_t u + dk * (\mathcal{A}(u) : \nabla^2 u) = F(u) + dk * G(u), & t \in J_0,\ x \in \Omega \\
\mathcal{B}_D(u) = 0, & t \in J_0,\ x \in \Gamma_D \\
\mathcal{B}_N(u) = 0, & t \in J_0,\ x \in \Gamma_N \\
u|_{t=0} = u_0, & x \in \Omega.
\end{array}
\right. \tag{30}
$$

Our goal is to prove unique existence of a local strong solution, more precisely, we are looking for an interval $J = [0,T]$ with $0 < T \le T_0$ and a unique solution u of (30) on J in the space

$$Z^T := H_p^{1+\alpha}(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)).$$

This will be achieved under appropriate assumptions by means of maximal L_p-regularity of a linear problem related to (30) and the contraction mapping principle.

To fix notation, we denote the independent variables by $t \in J_0$, $x \in \overline{\Omega}$, $\xi \in U_0$, and $\eta \in U_1$. For what is to follow we need the spaces

$$X^T = L_p(J; L_p(\Omega)), \quad X_1^T = H_p^{\alpha}(J; L_p(\Omega)), \tag{31}$$

$$Z_\nabla^T = H_p^{\frac{1}{2}(1+\alpha)}(J; L_p(\Omega)) \cap L_p(J; H_p^1(\Omega)),$$

$$Y_D^T = B_{pp}^{(1+\alpha)(1-\frac{1}{2p})}(J; L_p(\Gamma_D)) \cap L_p(J; B_{pp}^{2-\frac{1}{p}}(\Gamma_D)), \tag{32}$$

$$Y_N^T = B_{pp}^{(1+\alpha)(\frac{1}{2}-\frac{1}{2p})}(J; L_p(\Gamma_N)) \cap L_p(J; B_{pp}^{1-\frac{1}{p}}(\Gamma_N)), \tag{33}$$

and

$$Y_0 = B_{pp}^{2-\frac{2}{p(1+\alpha)}}(\Omega), \quad Y_1 = B_{pp}^{2-\frac{2}{1+\alpha}-\frac{2}{p(1+\alpha)}}(\Omega),$$

$\alpha > 1/p$ being assumed in the definition of Y_1.

If $u \in Z^T$, then this corresponds, as we know from Theorem 3.4, to the regularity classes

$$\nabla u \in (Z_\nabla^T)^n, \nabla^2 u \in (X^T)^{n\times n}, u|_{\Gamma_D} \in Y_D^T, \nabla u|_{\Gamma_N} \in (Y_N^T)^n, u|_{t=0} \in Y_0, \partial_t u|_{t=0} \in Y_1.$$

Consequently, Y_0 is the natural space for u_0, and if $\alpha > 1/p$ and $u \in Z^T$ is a solution of (30), then we have to ensure that $u_1 := \partial_t u|_{t=0}$, which is given by

$$u_1(x) = f(0, x, u_0(x), \nabla u_0(x)),\ x \in \Omega,$$

belongs to Y_1. Of course, we have to assume that

$$u_0(x) \in U_0, \quad \nabla u_0(x) \in U_1, \quad x \in \overline{\Omega}. \tag{34}$$

Observe also that

$$Z^T \hookrightarrow C(J; Y_0) \hookrightarrow C(J \times \overline{\Omega})$$

as well as

$$Z_\nabla^T \hookrightarrow C(J; B_{pp}^{1-\frac{2}{p(1+\alpha)}}(\Omega)) \hookrightarrow C(J \times \overline{\Omega}),$$

provided that $1 - 2/p(1+\alpha) > n/p$, which is equivalent to

$$p > \tfrac{2}{1+\alpha} + n \tag{35}$$

and which will be assumed throughout this section.

Notice further that we have to take into account the three compatibility conditions

$$\begin{aligned} b^D(0, x, u_0(x)) &= 0,\ x \in \Gamma_D, \\ b^N(0, x, u_0(x), \nabla u_0(x)) &= 0,\ x \in \Gamma_N, \\ b_t^D(0, x, u_0(x)) + b_\xi^D(0, x, u_0(x))u_1(x) &= 0,\ x \in \Gamma_D,\ \text{if } \alpha > \tfrac{3}{2p-1}. \end{aligned}$$

Here and in the subsequent lines we assume that b^D and b^N are as smooth as needed to make the formulas meaningful. Precise regularity statements will be given later on.

We now set out to reformulate (30) as a fixed point problem in an appropriate subset of Z^T. To this end, we first put

$$\mathcal{A}_0(x) = -a(0, x, u_0(x), \nabla u_0(x)),\ x \in \overline{\Omega}.$$

Further we fix a function $\phi \in Z^{T_0}$ which satisfies $\phi|_{t=0} = u_0$ and $\partial_t \phi|_{t=0} = u_1$, the latter being demanded in case that $\alpha > 1/p$. In view of (34) and (35), we see that, for T sufficiently small, say $T \le T_1 \le T_0$, we have $\phi(t,x) \in U_0$ and $\nabla\phi(t,x) \in U_1$ for all $t \in J$ and $x \in \overline{\Omega}$. So for such T we may define operators $\mathcal{B}_K^\circ(\phi)$ and $\mathcal{R}_K^\phi$, $K = D$, N, by means of

$$\begin{aligned} \mathcal{B}_D^\circ(\phi)u(t,x) &= b_\xi^D(t, x, \phi(t,x))u(t,x),\ t \in J, x \in \Gamma_D, \\ \mathcal{B}_N^\circ(\phi)u(t,x) &= b_\xi^N(t, x, \phi(t,x), \nabla\phi(t,x))u(t,x) \\ &\quad + b_\eta^N(t, x, \phi(t,x), \nabla\phi(t,x)) \cdot \nabla u(t,x),\ t \in J, x \in \Gamma_N, \end{aligned}$$

and

$$\mathcal{R}_K^\phi(u) = \mathcal{B}_K(u) - \mathcal{B}_K(\phi) - \mathcal{B}_K^\circ(\phi)(u - \phi), \quad K = D, N.$$

Obviously, (30) restricted to J is equivalent to

$$\left\{\begin{array}{rll} \partial_t u + dk * \mathcal{A}_0 : \nabla^2 u = & F(u) + dk * G(u) & \\ & +dk * ((\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u) & (J \times \Omega) \\ \mathcal{B}_D^\circ(\phi) u = & -\mathcal{B}_D(\phi) + \mathcal{B}_D^\circ(\phi)\phi - \mathcal{R}_D^\phi(u) & (J \times \Gamma_D) \\ \mathcal{B}_N^\circ(\phi) u = & -\mathcal{B}_N(\phi) + \mathcal{B}_N^\circ(\phi)\phi - \mathcal{R}_N^\phi(u) & (J \times \Gamma_N) \\ u|_{t=0} = & u_0 & (\Omega). \end{array}\right. \tag{36}$$

In other words, $u \in Z^T$ solves a problem of the form

$$\left\{\begin{array}{rll} \partial_t v + dk * \mathcal{A}_0 : \nabla^2 v = & h & (J \times \Omega) \\ \mathcal{B}_D^\circ(\phi) v = & \psi^D & (J \times \Gamma_D) \\ \mathcal{B}_N^\circ(\phi) v = & \psi^N & (J \times \Gamma_N) \\ v|_{t=0} = & v_0 & (\Omega). \end{array}\right. \tag{37}$$

with certain functions on the right-hand side. Given data h, ψ^D, ψ^N and v_0, (37) is a *linear* problem. For our construction, it is essential to understand this problem, in particular, one needs a precise characterization of unique solvability of (37) in Z^T in terms of regularity and compatibility conditions for the data.

Let us assume for the moment that we have such a result at our disposal - possibly on a yet smaller time-interval - and that the right-hand sides of the following both problems (38) and (40), which are of the form (37), fulfill the conditions needed to get a unique solution in Z^T in either case. Then it makes sense to define the *reference function* $w \in Z^T$ as solution of the linear problem

$$\left\{\begin{array}{rll} \partial_t w + dk * \mathcal{A}_0 : \nabla^2 w = & F(\phi) + dk * G(\phi) & (J \times \Omega) \\ \mathcal{B}_D^\circ(\phi) w = & -\mathcal{B}_D(\phi) + \mathcal{B}_D^\circ(\phi)\phi & (J \times \Gamma_D) \\ \mathcal{B}_N^\circ(\phi) w = & -\mathcal{B}_N(\phi) + \mathcal{B}_N^\circ(\phi)\phi & (J \times \Gamma_N) \\ w|_{t=0} = & u_0 & (\Omega). \end{array}\right. \tag{38}$$

Given $\rho > 0$, let

$$\Sigma(\rho, T, \phi) = \{v \in Z^T : \; v|_{t=0} = u_0, \; \partial_t v|_{t=0} = u_1 \text{ (if } \alpha > 1/p), \; |v - w|_{Z^T} \le \rho\},$$

which is a closed subset of Z^T. Since $Z^T \hookrightarrow C(J; C^1(\overline{\Omega}))$, we may further put

$$\mu_w(T) = \max\{|w(t,x) - u_0(x)| + |\nabla w(t,x) - \nabla u_0(x)| : t \in J, \, x \in \overline{\Omega}\}.$$

Apparently, $\mu_w(T) \to 0$ as $T \to 0$, due to $w|_{t=0} = u_0$.

Suppose that $u \in \Sigma(\rho, T, \phi)$. Then for $t \in J$ and $x \in \overline{\Omega}$, we estimate

$$\begin{aligned} |u(t,x) - u_0(x)| + |\nabla u(t,x) - \nabla u_0(x)| &\le |u - w|_{C(J;\, C^1(\overline{\Omega}))} + \mu_w(T) \\ &\le M|u - w|_{Z^T} + \mu_w(T) \le M\rho + \mu_w(T). \end{aligned} \tag{39}$$

Here the constant $M > 0$ does not depend on u and $T \in (0, T_1]$, the latter being true because $u - w \in {}_0Z^T$. So (39) shows that $u(t,x) \in U_0$ as well as $\nabla u(t,x) \in U_1$ for all $u \in \Sigma(\rho, T, \phi)$, $t \in J$ and $x \in \overline{\Omega}$, provided that T and ρ are sufficiently small, let us say $T \le T_2 \le T_1$ and $\rho \le \rho_1$. Henceforth we will assume that these smallness conditions hold.

The last assumption allows us to define the mapping $\Upsilon : \Sigma(\rho, T, \phi) \to Z^T$ which assigns to every $u \in \Sigma(\rho, T, \phi)$ the solution $v = \Upsilon(u)$ of the linear problem

$$\left\{\begin{array}{rll} \partial_t v + dk * \mathcal{A}_0 : \nabla^2 v = & F(u) + dk * G(u) & \\ & +dk * ((\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u) & (J \times \Omega) \\ \mathcal{B}_D^\circ(\phi) v = & -\mathcal{B}_D(\phi) + \mathcal{B}_D^\circ(\phi)\phi - \mathcal{R}_D^\phi(u) & (J \times \Gamma_D) \\ \mathcal{B}_N^\circ(\phi) v = & -\mathcal{B}_N(\phi) + \mathcal{B}_N^\circ(\phi)\phi - \mathcal{R}_N^\phi(u) & (J \times \Gamma_N) \\ v|_{t=0} = & u_0 & (\Omega). \end{array}\right. \tag{40}$$

Observe now that, in view of (36), every fixed point u of Υ is a solution of (30) and vice versa, at least for some small time interval $J = [0, T]$. So the idea is to apply the contraction mapping principle to Υ.

After these introductory considerations we become now more rigorous. We have not yet shown that w and Υ are really well-defined. Nor is it clear that the contraction mapping principle is applicable.

In order to succeed we need the subsequent assumptions. Here, for short, we put $\zeta = (\xi, \eta)$ and $U = U_0 \times U_1$. For a function $b(x)$ on a boundary segment Γ_K ($K = D$ or N), we denote the surface gradient by b_{x_Γ}.

(H1) (*kernel*): $k \in BV_{loc}(\mathbb{R}_+) \cap \mathcal{K}^1(1+\alpha, \theta)$, $k(0) = 0$, $\theta < \pi$, $\alpha \in [0, 1) \setminus \left\{ \frac{1}{p}, \frac{3}{2p-1} \right\}$, $\kappa := (1+\alpha)(1 - \frac{1}{2p}) \neq 1$;

(H2) (*exponents*): $n \geq 2$, $\frac{2}{1+\alpha} + n < p < \infty$;

(H3) (*smoothness of nonlinearities*):

(a) $a \in C(J_0 \times \overline{\Omega} \times U; \mathbb{R}^{n^2})$,

$$|a(t, x, \zeta) - a(t, x, \bar{\zeta})| \leq C|\zeta - \bar{\zeta}|, \ t \in J_0, \ x \in \Omega, \zeta, \bar{\zeta} \in U;$$

(b) $g(\cdot, \cdot, \zeta)$ is measurable on $J_0 \times \Omega$ for all $\zeta \in U$; $\exists C_g \in X^{T_0}$, $C_g \geq 0$, s.t.

$$|g(t, x, \zeta) - g(t, x, \bar{\zeta})| \leq C_g(t, x)|\zeta - \bar{\zeta}|, \ t \in J_0, \ x \in \Omega, \ \zeta, \bar{\zeta} \in U;$$

(c) f is as g in case $\alpha = 0$, otherwise: $f(\cdot, \cdot, \zeta)$ and $f_\zeta(\cdot, \cdot, \zeta)$ are measurable on $J_0 \times \Omega$ for all $\zeta \in U$; $f_\zeta \in L_\infty(J_0 \times \Omega \times U; \mathbb{R}^{n+1})$; $\exists C_f \in L_p(\Omega)$, $C_f \geq 0$ and $\sigma_f > \alpha$ s.t.

$$|f_\zeta(t, x, \zeta) - f_\zeta(\bar{t}, x, \bar{\zeta})| \leq C_f(x)|t - \bar{t}|^{\sigma_f} + C|\zeta - \bar{\zeta}|, \ t, \bar{t} \in J_0, \ x \in \Omega, \ \zeta, \bar{\zeta} \in U;$$

(d) $b^D \in C(J_0 \times \Gamma_D \times U_0)$, $\exists C_{b_1} \in L_p(\Gamma_D)$, $C_{b_1} \geq 0$, $\exists C_{b_2} \in L_p(J)$, $C_{b_2} \geq 0$, and $\exists \sigma_2 > 1 - \frac{1}{p}$ such that in case $\kappa < 1$: $\exists \sigma_1 > \kappa$ s.t.

$$|b^D_{x_\Gamma}(t, x, \xi) - b^D_{x_\Gamma}(t, \bar{x}, \xi)| \leq C_{b_2}(t)|x - \bar{x}|^{\sigma_2}, \tag{41}$$
$$|b^D_\xi(t, x, \xi) - b^D_\xi(\bar{t}, x, \xi)| \leq C_{b_1}(x)|t - \bar{t}|^{\sigma_1},$$
$$|b^D_{x_\Gamma \xi}(t, x, \xi) - b^D_{x_\Gamma \xi}(t, \bar{x}, \bar{\xi})| \leq C_{b_2}(t)|x - \bar{x}|^{\sigma_2} + C|\xi - \bar{\xi}|, \tag{42}$$
$$|b^D_{\xi\xi}(t, x, \xi) - b^D_{\xi\xi}(t, \bar{x}, \bar{\xi})| \leq C(|x - \bar{x}|^{\sigma_2} + |\xi - \bar{\xi}|),$$

and in case $\kappa > 1$: $\exists \sigma_1 > \kappa - 1$ s.t. (41), (42),

$$|b^D_t(t, x, \xi) - b^D_t(\bar{t}, x, \xi)| \leq C_{b_1}(x)|t - \bar{t}|^{\sigma_1},$$
$$|b^D_{t\xi}(t, x, \xi) - b^D_{t\xi}(\bar{t}, x, \bar{\xi})| \leq C_{b_1}(x)|t - \bar{t}|^{\sigma_1} + C|\xi - \bar{\xi}|,$$
$$|b^D_{\xi\xi}(t, x, \xi) - b^D_{\xi\xi}(\bar{t}, \bar{x}, \bar{\xi})| \leq C(|t - \bar{t}|^{\sigma_1} + |x - \bar{x}|^{\sigma_2} + |\xi - \bar{\xi}|),$$

all these inequalities being true for $t, \bar{t} \in J_0$, $x, \bar{x} \in \Gamma_D$, $\xi, \bar{\xi} \in U_0$; each of the derivatives of b^D occurring above is Carathéodory and essentially bounded on $J_0 \times \Gamma_D \times U_0$;

(e) $b^N \in C(J_0 \times \Gamma_N \times U)$, $b^N_\zeta \in L_\infty(J_0 \times \Gamma_N \times U; \mathbb{R}^{n+1})$ is Carathéodory, $\exists C_{b_1} \in L_p(\Gamma_N)$, $C_{b_1} \geq 0$, $\exists C_{b_2} \in L_p(J)$, $C_{b_2} \geq 0$, $\exists \sigma_1 > (1+\alpha)(\frac{1}{2} - \frac{1}{2p})$, $\exists \sigma_2 > 1 - \frac{1}{p}$, s.t.

$$|b^N(t, x, \zeta) - b^N(\bar{t}, \bar{x}, \zeta)| \leq C_{b_1}(x)|t - \bar{t}|^{\sigma_1} + C_{b_2}(t)|x - \bar{x}|^{\sigma_2},$$
$$|b^N_\zeta(t, x, \zeta) - b^N_\zeta(\bar{t}, \bar{x}, \bar{\zeta})| \leq C_{b_1}(x)|t - \bar{t}|^{\sigma_1} + C_{b_2}(t)|x - \bar{x}|^{\sigma_2} + C|\zeta - \bar{\zeta}|,$$

$t, \bar{t} \in J_0$, $x, \bar{x} \in \Gamma_N$, $\zeta, \bar{\zeta} \in U$;

(H4) (*initial data*): $u_0 \in Y_0$; $u_1 \in Y_1$, if $\alpha > \frac{1}{p}$ $(u_1(\cdot) := f(0,\cdot,u_0(\cdot),\nabla u_0(\cdot)))$;
$f(\cdot,\cdot,u_0(\cdot),\nabla u_0(\cdot)), g(\cdot,\cdot,u_0(\cdot),\nabla u_0(\cdot)) \in X^{T_0}$;

(H5) (*compatibility*): $(u_0(x), \nabla u_0(x)) \in U_0 \times U_1$, $x \in \overline{\Omega}$;

$$b^D(0,x,u_0(x)) = 0,\ x \in \Gamma_D;$$
$$b^N(0,x,u_0(x),\nabla u_0(x)) = 0,\ x \in \Gamma_N;$$
$$b_t^D(0,x,u_0(x)) + b_\xi^D(0,x,u_0(x))u_1(x) = 0,\ x \in \Gamma_D,\ \text{if } \alpha > \tfrac{3}{2p-1};$$

(H6) (*ellipticity*): $a(0,x,u_0(x),\nabla u_0(x)) \in \mathrm{Sym}\{n\}$, $x \in \overline{\Omega}$; $\exists c_0 > 0$ s.t.

$$a(0,x,u_0(x),\nabla u_0(x))\varrho\cdot\varrho \geq c_0|\varrho|^2,\ x \in \overline{\Omega},\ \varrho \in \mathbb{R}^n;$$

(H7) (*normality*):

$$b_\xi^D(0,x,u_0(x)) \neq 0,\ x \in \Gamma_D;$$
$$b_\eta^N(0,x,u_0(x),\nabla u_0(x))\cdot\nu(x) \neq 0,\ x \in \Gamma_N.$$

We have now the following result.

Theorem 4.1 *Suppose that the assumptions (H1)-(H7) are satisfied. Let $\phi \in Z^{T_0}$ be as above, and assume that $\rho \leq \rho_1$. Then there exists $T_3 \in (0,T_2]$ such that for each $T \in (0,T_3]$ the following statements hold:*

(i) (38) has a unique solution w in Z^T;

(ii) for every $u \in \Sigma(\rho,T,\phi)$, (40) has a unique solution $v = \Upsilon(u)$ in Z^T;

(iii) there exist positive constants M and $\mu(T)$ both not depending on ρ, with M being also independent of T and $\mu(T) \to 0$ as $T \to 0$, such that for all $u, v \in \Sigma(\rho,T,\phi) \cup \{\phi|_J\}$ and $K = D, N$, the subsequent inequalities are fulfilled:

$$|(\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u|_{X^T} \leq M(\mu(T)+\rho)^2, \tag{43}$$
$$|(\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u - (\mathcal{A}_0 - \mathcal{A}(v)) : \nabla^2 v|_{X^T} \leq M(\mu(T)+\rho)|u-v|_{Z^T}, \tag{44}$$
$$|F(u) - F(v)|_{X_1^T} + |G(u) - G(v)|_{X^T} \leq \mu(T)|u-v|_{Z^T}, \tag{45}$$
$$|\mathcal{R}_K^\phi(u) - \mathcal{R}_K^\phi(v)|_{Y_K^T} \leq M(\mu(T)+\rho)|u-v|_{Z^T}. \tag{46}$$

Proof. To prove (i) and (ii), we have to consider the linear problem (37). Since $\phi|_{t=0} = u_0$, it follows from (H7) and the compactness of Γ that there exist $T_3 \in (0,T_2]$ and $c > 0$ such that $|\mathcal{B}_D^\circ(\phi)(t,x)| \geq c$ as well as $|\mathcal{B}_N^\circ(\phi)(t,x)\cdot\nu(x)| \geq c$ for all $t \in [0,T_3]$ and $x \in \Gamma_D$ resp. $x \in \Gamma_N$. Hereafter, we suppose that $T \in (0,T_3]$. We may then normalize the boundary conditions in (37) by dividing by $\mathcal{B}_D^\circ(\phi)(t,x)$ resp. $\mathcal{B}_N^\circ(\phi)(t,x)\cdot\nu(x)$, and integrate the first equation in (37) w.r.t. time; observe that $1 * dk = k$. This way we can rewrite (37) as a problem of the form (28). One has now to check that Theorem 3.4 is applicable to the reformulations of (38) and (40).

As to regularity of the data, clearly the initial data u_0 and u_1 belong to the right regularity classes, by assumption (H4). Let us next look at the term which involves the function g. Suppose $u \in \Sigma(\rho,T,\phi) \cup \{\phi|_J\}$. By (H3b) and (H4), we have

$$\begin{aligned}|g(\cdot,\cdot,u,\nabla u)|_{X^T} &\leq |g(\cdot,\cdot,u,\nabla u) - g(\cdot,\cdot,u_0,\nabla u_0)|_{X^T} + |g(\cdot,\cdot,u_0,\nabla u_0)|_{X^T}\\ &\leq |C_g|_{X^T}(|u-u_0|_\infty + |\nabla u - \nabla u_0|_\infty) + |g(\cdot,\cdot,u_0,\nabla u_0)|_{X^T}.\end{aligned}$$

So $G(u) \in X^T$, that is, this term lies in the right regularity class. Furthermore, we have $(\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u \in X^T$, in view of (43), which will be shown below. Concerning the other terms, we refer to [35, Section 6.2], where the author proved that the regularity assumptions on f, b^D, and b^N, together with (H4), ensure that these terms enjoy the regularity required for the application of Theorem 3.4, i.e. that $F(u) \in X_1^T$ and $\mathcal{B}_K(\phi) - \mathcal{B}_K^\circ(\phi)\phi + \mathcal{R}_K^\phi(u) \in Y_K^T$ for all $u \in \Sigma(\rho, T, \phi) \cup \{\phi|_J\}$, $K = D, N$. In [35, Section 6.2] it was also shown that this regularity is preserved under the above normalization on the boundary.

Observe further that the compatibility conditions are satisfied for (38) and (40). This follows from (H5), the definition of $\Sigma(\rho, T, \phi)$, and from the fact that $\phi|_{t=0} = u_0$ and $\partial_t \phi|_{t=0} = u_1$ in case $\alpha > 1/p$. Normality has already been discussed above. Hence, (i) and (ii) are established for all $T \in (0, T_3]$.

Turning to (iii), we here only show (43), (44), and one half of (45). For the remaining estimates, which take more effort to be proved, we refer again to [35, Section 6.2] (see also Remark 4.1(ii) below), due to limitations of space. In the subsequent inequalities, M and $\mu(T)$ are constants, which may differ from line to line, but which are such that both do not depend on ρ, M is independent of T, too, and $\mu(T) \to 0$ as $T \to 0$.

Let $u, v \in \Sigma(\rho, T, \phi) \cup \{\phi|_J\}$. By means of (H3a) we get

$$\begin{aligned} |(\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u|_{X^T} &\leq (|\mathcal{A}_0 - \mathcal{A}(w)|_\infty + |\mathcal{A}(w) - \mathcal{A}(u)|_\infty) \\ &\quad \times (|\nabla^2 u - \nabla^2 w|_{(X^T)^{n^2}} + |\nabla^2 w|_{(X^T)^{n^2}}) \\ &\leq (\mu(T) + M\rho)(\rho + \mu(T)), \end{aligned}$$

which entails (43). Correspondingly,

$$\begin{aligned} &|(\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u - (\mathcal{A}_0 - \mathcal{A}(v)) : \nabla^2 v|_{X^T} \\ &\quad \leq |(\mathcal{A}_0 - \mathcal{A}(u)) : (\nabla^2 u - \nabla^2 v)|_{X^T} + |(\mathcal{A}(u) - \mathcal{A}(v)) : (\nabla^2 v - \nabla^2 w)|_{X^T} \\ &\qquad + |(\mathcal{A}(u) - \mathcal{A}(v)) : \nabla^2 w|_{X^T} \\ &\quad \leq (\mu(T) + M\rho)|u - v|_{Z^T} + M\rho|u - v|_{Z^T} + M\mu(T)|u - v|_{Z^T} \\ &\quad \leq M(\mu(T) + \rho)|u - v|_{Z^T}, \end{aligned}$$

showing (44). We finally estimate the term $|G(u) - G(v)|_{X^T}$. By virtue of (H3b), we obtain similarly as above

$$|g(\cdot, \cdot, u, \nabla u) - g(\cdot, \cdot, v, \nabla v)|_{X^T} \leq |C_g|_{X^T} M|u - v|_{Z^T} \leq M\mu(T)|u - v|_{Z^T}.\square$$

Remarks 4.1 (i) The above fixed point construction does not work when one takes u_0 instead of ϕ for the linearizations of the nonlinear boundary operators. The reason for this lies in the resulting lack of regularity. So e.g. the term $b^D(0, \cdot, u_0(\cdot))$ does not lie in general in $Y_D^T \subset L_p(J; B_{pp}^{2-\frac{1}{p}}(\Gamma_D))$, since $u_0 \in B_{pp}^{2-\frac{2}{p(1+\alpha)}}(\Omega)$ only yields $u_0|_{\Gamma_D} \in B_{pp}^{2-\frac{2}{p(1+\alpha)}-\frac{1}{p}}(\Gamma_D)$, and the regularity is not improved by a (nontrivial) substitution operator.

(ii) To indicate how the temporal traces of w and ϕ are being used in the proof of (46), we remark that in [35, Section 6.2] it was shown that under the assumptions of Theorem 4.1,

$$|\mathcal{R}_D^\phi(u) - \mathcal{R}_D^\phi(v)|_{Y_D^T} \leq M(\mu(T) + \rho + |b_\xi^D(\cdot, \cdot, w) - b_\xi^D(\cdot, \cdot, \phi)|_\infty)|u - v|_{Z^T}$$

for all $u, v \in \Sigma(\rho, T, \phi) \cup \{\phi|_J\}$. Since $|b_\xi^D(\cdot, \cdot, w) - b_\xi^D(\cdot, \cdot, \phi)|_\infty \to 0$ as $T \to 0$, the inequality (46) with $K = D$ follows. The case $K = N$ is similar.

Existence and uniqueness of a local strong solution of (30) can now be obtained by means of Theorem 4.1 and the contraction mapping principle.

Theorem 4.2 *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary Γ which decomposes as $\Gamma = \Gamma_D \cup \Gamma_N$ with dist$(\Gamma_D, \Gamma_N) > 0$. Let further $U_0 \subset \mathbb{R}$, $U_1 \subset \mathbb{R}^n$ be nonempty open convex sets. Suppose that the assumptions (H1)-(H7) are satisfied. Then there exists $T \in (0, T_0]$ such that (30) restricted to $J = [0, T]$ admits a unique solution in Z^T.*

Proof. Let $\rho \le \rho_1$ and $T \in (0, T_3]$, so that the reference function $w \in Z^T$ as well as $v = \Upsilon(u) \in Z^T$ are well-defined for each $u \in \Sigma(\rho, T, \phi)$, cf. Theorem 4.1. We want to show that, for sufficiently small T and ρ, Υ maps $\Sigma(\rho, T, \phi)$ into itself and is strictly contractive.

To show the first property we have to estimate $v - w$ in the Z^T-norm. By definition of w and $v = \Upsilon(u)$, we see that $v - w$ satisfies

$$\left\{\begin{array}{rll} \partial_t(v-w) + dk * \mathcal{A}_0 : \nabla^2(v-w) = & F(u) - F(\phi) + dk * (G(u) - G(\phi)) & \\ & +dk * ((\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u) & (J \times \Omega) \\ \mathcal{B}_D^\circ(\phi)(v-w) = & -\mathcal{R}_D^\phi(u) & (J \times \Gamma_D) \\ \mathcal{B}_N^\circ(\phi)(v-w) = & -\mathcal{R}_N^\phi(u) & (J \times \Gamma_N) \\ (v-w)|_{t=0} = & 0 & (\Omega). \end{array}\right.$$

The maximal regularity estimate for problem (37) thus yields

$$\begin{aligned} |v-w|_{Z^T} \le M_1 \Big(& |F(u) - F(\phi) + dk * (G(u) - G(\phi)) + dk * ((\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u)|_{X_1^T} \\ & + |\mathcal{R}_D^\phi(u)|_{Y_D^T} + |\mathcal{R}_N^\phi(u)|_{Y_N^T} \Big), \end{aligned}$$

with a constant $M_1 > 0$ *not* depending on T ($v - w \in {}_0Z^T$!). Using the estimates from Theorem 4.1, combined with $\mathcal{R}_K^\phi(\phi) = 0$, $K = D, N$, we obtain an inequality of the form

$$|v-w|_{Z^T} \le M(\rho + \mu(T))^2, \tag{47}$$

where $M > 0$ is independent of T and ρ, and $\mu(T) > 0$ vanishes as $T \to 0$. Here we employ the simple inequality $|u - \phi|_{Z^T} \le \rho + |\phi - w|_{Z^T}$, the last term of which behaves like $\mu(T)$. From (47) it is clear that Υ is a self-mapping of $\Sigma(\rho, T, \phi)$, if T and ρ are sufficiently small; choose e.g. ρ so small that $4M\rho^2 \le \rho$, and diminish T until $\mu(T) \le \rho$.

Strict contractivity of Υ can be established in similar fashion. Let $u, \bar{u} \in \Sigma(\rho, T, \phi)$ and put $v = \Upsilon(u)$ and $\bar{v} = \Upsilon(\bar{u})$. Then $\bar{v} - v$ satisfies

$$\left\{\begin{array}{rll} \partial_t(\bar{v}-v) + dk * \mathcal{A}_0 : \nabla^2(\bar{v}-v) = & F(\bar{u}) - F(u) + dk * (G(\bar{u}) - G(u)) & (J \times \Omega) \\ & + dk * (((\mathcal{A}_0 - \mathcal{A}(\bar{u})) : \nabla^2 \bar{u}) - (\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u) & \\ \mathcal{B}_D^\circ(\phi)(\bar{v}-v) = & -(\mathcal{R}_D^\phi(\bar{u}) - \mathcal{R}_D^\phi(u)) & (J \times \Gamma_D) \\ \mathcal{B}_N^\circ(\phi)(\bar{v}-v) = & -(\mathcal{R}_N^\phi(\bar{u}) - \mathcal{R}_N^\phi(u)) & (J \times \Gamma_N) \\ (\bar{v}-v)|_{t=0} = & 0 & (\Omega), \end{array}\right.$$

whence

$$\begin{aligned} |\bar{v}-v|_{Z^T} \le M_1 \Big(& |G(\bar{u}) - G(u) + (((\mathcal{A}_0 - \mathcal{A}(\bar{u})) : \nabla^2 \bar{u}) - (\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u)|_{X^T} \\ & + |F(\bar{u}) - F(u)|_{X_1^T} + |\mathcal{R}_D^\phi(\bar{u}) - \mathcal{R}_D^\phi(u)|_{Y_D^T} + |\mathcal{R}_N^\phi(\bar{u}) - \mathcal{R}_N^\phi(u)|_{Y_N^T} \Big) \\ \le {} & M(\mu(T) + \rho)|\bar{u} - u|_{Z^T}, \end{aligned}$$

by maximal regularity and the estimates from Theorem 4.1. Here the constants M and $\mu(T)$ are like those in (47). Hence Υ becomes a strict contraction, when ρ and T are selected sufficiently small.

The assertion follows now by the contraction mapping principle and the fact that fixed points of Υ correspond to solutions of (30) for small time-intervals $J = [0, T]$. $\square$

Remarks 4.2 (i) The statement of Theorem 4.2 is also true, if k is of the form $k = k_1 + dl * k_1$, where k_1 is like k in (H1) and $l \in BV_{loc}(\mathbb{R}_+)$ with $l(0) = l(0+) = 0$.

(ii) One can further replace k on the right-hand side of (30) by an arbitrary kernel $k_1 \in BV_{loc}(\mathbb{R}_+) \cap \mathcal{K}^1(1+\alpha_1, \theta_1)$ with $k(0) = 0$, $\alpha_1 \geq \alpha$, $\theta_1 < \pi$, and the theorem still holds true.

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