EXPONENTIAL AND POLYNOMIAL DICHOTOMIES OF OPERATOR SEMIGROUPS ON BANACH SPACES

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Abstract. Let A generate a C_0–semigroup $T(\cdot)$ on a Banach space X such that the resolvent $R(i\tau, A)$ exists and is uniformly bounded for $\tau \in \mathbb{R}$. We show that there exists a closed, possibly unbounded projection P on X commuting with $T(t)$. Moreover, $T(t)x$ decays exponentially as $t \to \infty$ for x in the range of P and $T(t)x$ exists and decays exponentially as $t \to -\infty$ for x in kernel of P. The domain of P depends on the Fourier type of X. If $R(i\tau, A)$ is only polynomially bounded, one obtains a similar result with polynomial decay. As an application we study a partial functional differential equation.

1. Introduction and preliminaries

Exponential stability and dichotomy are among the most basic and most important asymptotic properties of a strongly continuous operator semigroup $T(\cdot)$ on a Banach space X. One strives to characterize these notions in terms of the generator A of $T(\cdot)$ which is the given object in most applications. There is a well developed theory for this problem which nevertheless does not answer several important questions. In this paper we want to address one of these open issues, treating also polynomial dichotomies within the same approach.

To provide the background for our main theorems, let us describe the relevant known results in this area. Our notation is explained at the end of this section. We first recall the well known resolvent estimates which are necessary for exponential stability and dichotomy, namely

$$T(\cdot) \text{ is exp. stable} \quad \Longrightarrow \quad s(A) < 0 \;\text{ and }\; \|R(\lambda, A)\| \le c, \;\; \operatorname{Re}\lambda \ge 0, \tag{1.1}$$
$$T(\cdot) \text{ is exp. dichotomous} \quad \Longrightarrow \quad i\mathbb{R} \subset \rho(A) \;\text{ and }\; \|R(i\tau, A)\| \le c, \;\; \tau \in \mathbb{R}. \tag{1.2}$$

If $t \mapsto T(t)$ is continuous in operator norm for some $t > 0$, then the pure spectral criteria $s(A) < 0$, resp. $i\mathbb{R} \subset \rho(A)$, already imply the exponential stablity, resp. dichotomy, of $T(\cdot)$, see e.g. [8, Cor.IV.3.11, Thm.V.1.17]. In this way exponential stability and dichotomy of, e.g., analytic semigroups can be characterized very conveniently. Unfortunately, for general semigroups these criteria fail even on a Hilbert space X, see [4, Ex.5.8], [5, §2.1.5], [8, §IV.3.a], [18, Ex.1.2.4]. (Some of these examples arise from wave equations.) On the other hand, on a Hilbert space X Gearhart's spectral mapping theorem establishes the converse implications in (1.1) and (1.2); i.e., it holds

$$T(\cdot) \text{ is exp. stable} \quad \Longleftrightarrow \quad s(A) < 0 \;\text{ and }\; \|R(\lambda, A)\| \le c, \;\; \operatorname{Re}\lambda \ge 0, \tag{1.3}$$
$$T(\cdot) \text{ is exp. dichotomous} \quad \Longleftrightarrow \quad i\mathbb{R} \subset \rho(A) \;\text{ and }\; \|R(i\tau, A)\| \le c, \;\; \tau \in \mathbb{R}, \tag{1.4}$$

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if X is a Hilbert space, see e.g. [5, §2.1], [18, Thm.2.2.4], and [18, p.70] for further references. However, on non Hilbertian X there are semigroups violating the implications '$\Leftarrow$' in (1.3) and (1.4), see [5, §2.1.5], [8, §IV.3.a], [18], [24, §4].

Going back to the general case of a C_0–semigroup on a Banach space X without additional regularity properties, one can look for stronger assumptions on the resolvent which imply the exponential stability or dichotomy of $T(\cdot)$. Such conditions were found in the papers [5, §2.2], [10], [11], [12], [13], [17, Thm.A-III-7.10], [23]. However, these conditions are quite sophisticated and (as it seems) difficult to check in applications. Alternatively, one can ask whether the uniform boundedness of $R(\lambda, A)$ for $\lambda \in i\mathbb{R}$ or $\operatorname{Re}\lambda \geq 0$ implies interesting asymptotic properties of the semigroup which are related to stability of dichotomy. In the case of exponential stability this question was settled in a line of research culminating in the paper [24] by Weis and Wrobel, see also [18], [19], [22], [23], and the references therein. Theorem 3.2 and Remark 3.3 of [24] (or [18, Thm.4.2.4]) show that

$$\|R(\lambda, A)\| \leq c, \;\; \operatorname{Re}\lambda \geq 0 \quad \Longrightarrow \quad \|T(t)x\| \leq M e^{-\varepsilon t} \|(w-A)^{\beta} x\| \tag{1.5}$$

for $\beta = \frac{1}{p} - \frac{1}{p'}$, $x \in \mathcal{D}((w-A)^{\beta})$, and some constants $M, \varepsilon > 0$. Here w is a fixed real number larger than the growth bound of $T(\cdot)$, $p' = p/(p-1)$, and $p \in [1,2]$ is the *Fourier type* of X, i.e., the Fourier transform $\mathcal{F}$ is bounded from $L^p(\mathbb{R}, X)$ to $L^{p'}(\mathbb{R}, X)$. Clearly, each Banach space has at least Fourier type $p = 1$. Hilbert spaces have Fourier type $p = 2$ by Plancherel's theorem. In fact, only Hilbert spaces have Fourier type 2. The space $X = L^q(\Omega)$ has Fourier type $p = \min\{q, q'\}$. Moreover, uniformly convex Banach spaces have nontrivial Fourier type $p > 1$. (See [18, p.116] for references concerning these facts.) In particular, (1.5) implies (1.3) if X is a Hilbert space. By means of an example it can be shown that the exponent β in (1.5) is optimal, see [24, §4] or [18, Ex.4.2.9].

In Theorem 2.2 we establish a result on exponential dichotomy which is analogous to the Weis–Wrobel theorem. As in (1.5) we expect exponential estimates only for x in a space $X_\alpha := \mathcal{D}((w-A)^\alpha)$, correspondingly the dichotomy projection P will only be defined on a subspace $\mathcal{D}(P)$ of X containing X_α. This leads us to the following concept which is weaker than the usual exponential dichotomy (where $\alpha = 0$ and $\mathcal{D}(P) = X$).

Definition 1.1. *Let $T(\cdot)$ be strongly continuous semigroup on a Banach space X and $\alpha \geq 0$. We say that $T(\cdot)$ has an* exponential α–dichotomy *if there is a closed projection P on X and constants $N, \delta > 0$ such that*

(a) $X_\alpha \hookrightarrow \mathcal{D}(P), \quad T(t)\mathcal{D}(P) \subset \mathcal{D}(P), \quad T(t)Px = PT(t)x, \quad x \in \mathcal{D}(P),$

(b) $T(t) : \mathcal{N}(P) \to \mathcal{N}(P)$ *has the bounded inverse* $T_Q(-t)$,

(c) $\|T(t)Px\| \leq Ne^{-\delta t}\,\|(w-A)^\alpha x\|, \quad \|T_Q(-t)(I-P)x\| \leq Ne^{-\delta t}\,\|(w-A)^\alpha x\|$

for $t \geq 0$ and $x \in X_\alpha$. We call α and δ the regularity and decay exponent, respectively.

We note that unbounded splitting projections also occur in the study of bi–sectorial operators, see e.g. [16], [21]. However, this is a different situation in so far bisectorial generators already generate analytic semigroups.

Our main Theorem 2.2 shows that $T(\cdot)$ has an exponential α–dichotomy if $R(i\tau, A)$ exists and is uniformly bounded for $\tau \in \mathbb{R}$ and $\alpha > \frac{1}{p} - \frac{1}{p'} \geq 0$, where $p \in [1,2]$ is the Fourier type of X. We can take $\alpha = 1$ in the case of a non–trivial Fourier type $p > 1$ (e.g., if X is uniformly convex). In this case Definition 1.1(c) gives exponential estimates for $x \in \mathcal{D}(A)$, i.e., for classical solutions $u(t) = T(t)x$ of the Cauchy problem. Unfortunately, we do not quite obtain the exponent $\alpha = \frac{1}{p} - \frac{1}{p'}$ from (1.5). For $P = I$, our theorem corresponds to the results of the paper [19], where (1.5) was shown for $\beta > \frac{1}{p} - \frac{1}{p'}$. Then in [24] additional arguments were developed which allow to pass to the equality $\beta = \frac{1}{p} - \frac{1}{p'}$ in the regularity exponent. A different approach is contained in the proof of Theorem 4.5.2 of [18]. But it seems that these techniques do not work in the presence of an unbounded projection or if one deals with a spectral gap (as in our situation).

We are aware of only one result dealing with dichotomies in the setting of Theorem 2.2, namely Theorem 5.5 of the paper [7] by deLaubenfels and Latushkin. These authors obtain an exponential dichotomy on a Banach space Z such that $\mathcal{D}(A^2) \subset Z \subset X$ assuming that $R(\lambda, A)$ is bounded on $i\mathbb{R}$. But it seems that Z is hard to describe conveniently and that it is smaller than our X_α. The approach of [7] is based on deLaubenfels' work on regularized functional calculi, [6]. (Concerning these calculi we also refer to the recent contribution [9] and the fundamental work by McIntosh in e.g. [15] and [16].) We proceed in a different, rather direct and self–contained way: The contour integrals (2.8) and (2.2) define operators on X which turn out to be equal to $T(t)P(w - A)^{-\alpha}$ and $T_Q(-t)(I - P)(w - A)^{-\alpha}$ and to have the asserted properties. Our reasoning is inspired by methods from [11], [13], [19] and from the theory of functional calculi. But the verification of the exponential estimates in Definition 1.1(c) and the presence of an unbounded projection posed several new difficulties. We also note that we do not use the Weis–Wrobel theorem in our arguments, cf. Remark 2.3.

In Example 2.5 we study of parabolic partial differential equations in $L^q(\Omega)$ with delays in the highest spatial derivatives based on Theorem 2.2 and our work in [4]. We show that the exponential α–dichotomy follows also in this case from a resolvent type estimate (where $\alpha > |\frac{1}{q} - \frac{1}{q'}|$), and we give quite explicit sufficient conditions in a special case. Here pure spectral criteria for exponential dichotomy may fail, see [4, Ex.5.8].

In fact, our main Theorem 2.2 is stated in a somewhat more general way allowing for polynomial growth of $R(\lambda, A)$ on an open vertical strip around $i\mathbb{R}$. This extension only affects the value of the regularity exponent α. If one merely assumes that $\|R(i\tau, A)\| \leq c(1 + |\tau|^\gamma)$ for $\tau \in \mathbb{R}$ and some $\gamma > 0$, then it may happen that the spectrum approaches the imaginary axis at $\pm i\infty$. There are various examples arising from wave equations where $\sigma(A)$ belongs to the open left half plane, the semigroup is bounded (thus $s(A) = 0$), and $T(t)x$ decays polynomially, but not exponentially as $t \to \infty$ for $x \in \mathcal{D}(A)$, see e.g. [1], [3], [14], and the references therein. This situation was investigated in detail in [3]. To our knowledge there are no papers treating the case that the generator spectrum $\sigma(A)$ approaches $i\mathbb{R}$ at $\pm i\infty$ from the left and the right. We address this point in the third section. In fact, the arguments of Section 2 can be modified in order to obtain again a closed projection P with the properties from Definition 1.1 except for (c) where we now take $\alpha > \gamma + \frac{1}{p} - \frac{1}{p'}$. In particular we have $X_\alpha \hookrightarrow \mathcal{D}(P)$. Extending the methods of Section 2, we can further show the polynomial decay on appropriate subspaces, see Theorem 3.1, (3.3), and (3.4).

Notation and definitions. For Banach spaces X and Y the space of bounded linear operators is denoted by $\mathcal{B}(X, Y)$, where $\mathcal{B}(X) := \mathcal{B}(X, X)$. By $\mathcal{D}(B)$, $\mathcal{N}(B)$, $\mathcal{R}(B)$, $\sigma(B)$, $\rho(B)$, we designate the domain, kernel, range, spectrum, resolvent set of a linear operator B, respectively, and we set $R(\lambda, B) = (\lambda I - B)^{-1} = (\lambda - B)^{-1}$. The domain of a linear operator B is always endowed with the graph norm of B. For $p \geq 1$ we set $p' = p/(p - 1)$

if $p > 1$ and $p' = \infty$ if $p = 1$. The Fourier transform is defined by $\mathcal{F}f(\tau) = \int_{\mathbb{R}} e^{-i\tau t} f(t)\, dt$, $\tau \in \mathbb{R}$, for $f \in L^1(\mathbb{R}, X)$. We write $c = c(\alpha, \beta, \cdots)$ for a generic constant depending on the quantities $\alpha, \beta, \cdots$.

Let A be the generator of a C_0–semigroup $T(\cdot) = (T(t))_{t \geq 0}$ on X. The spectral bound of A is defined by $s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$, and the growth bound by $\omega_0(A) = \inf\{w \in \mathbb{R} : \exists M : \|T(t)\| \leq Me^{wt},\ t \geq 0\}$. We say that $T(\cdot)$ is *exponentially stable* if $\omega_0(A) < 0$. Recall that $s(A) \leq \omega_0(A) < \infty$. We fix a number $w > \omega_0(A)$ and define for $\alpha > 0$ the fractional power

$$(w - A)^{-\alpha} = \frac{1}{2\pi i} \int_\Gamma (w - \lambda)^{-\alpha} R(\lambda, A)\, d\lambda \tag{1.6}$$

where Γ is a piecewise smooth path in the set $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega_0(A),\ \lambda \notin [w, \infty)\}$ running from $\infty e^{-i\phi}$ to $\infty e^{i\phi}$ for some $0 < \phi < \pi/2$. We further set $(w - A)^0 = I$. The operators $(w - A)^{-\alpha}$ are injective, bounded, and satisfy the power law with respect to α. In particular, $(w - A)^{-\alpha}$ has a closed inverse denoted by $(w - A)^{\alpha}$. The domain $X_\alpha := \mathcal{D}((w - A)^\alpha)$ does not depend on the choice of $w > \omega_0(A)$. It is known that $X_n = \mathcal{D}(A^n)$ for $n \in \mathbb{N}$ (with equivalent norms) and that $X_\beta \hookrightarrow X_\alpha \hookrightarrow X$ for $\beta \geq \alpha \geq 0$, where '$\hookrightarrow$' designates a continuous embedding (which is also dense in our case). Since $(w - A)^{-\alpha} T(t) = T(t)(w - A)^{-\alpha}$ it is easy to see that the restriction $A_\alpha : X_{1+\alpha} \to X_\alpha$ of A generates the semigroup in X_α given by the restrictions of $T(t)$ to X_α. Moreover, $R(\lambda, A_\alpha)$ is the restriction of $R(\lambda, A)$ to X_α for $\lambda \in \sigma(A_\alpha) = \sigma(A)$. We refer to [2] or [8] for proofs of these facts.

2. Exponential dichotomy

Before presenting our main theorem we state a standard lemma on closed projections. The proof is given for the reader's convenience. Here a 'closed projection' P on X is a linear operator such that $P\mathcal{D}(P) \subset \mathcal{D}(P)$ and $Px = P^2x$ for $x \in \mathcal{D}(P)$. Throughout we set $Q = I - P$ with $\mathcal{D}(Q) = \mathcal{D}(P)$.

Lemma 2.1. *If P is a closed projection on a Banach space X, then the spaces $\mathcal{N}(P) = \mathcal{R}(Q)$ and $\mathcal{R}(P) = \mathcal{N}(Q)$ are closed in X and in $\mathcal{D}(P)$. Moreover, $\mathcal{D}(P) = \mathcal{N}(P) \oplus \mathcal{R}(P)$.*

Proof. We first observe that Q ist also a closed projection on X so that the kernels $\mathcal{N}(P)$ and $\mathcal{N}(Q)$ are closed in X and $\mathcal{D}(P)$. It is clear that $\mathcal{N}(Q) \subset \mathcal{R}(P)$. Conversely, a vector $y = Px$ belongs to $\mathcal{D}(P) = \mathcal{D}(Q)$ and $Qy = Px - P^2x = 0$. As a result, $\mathcal{N}(P) = \mathcal{R}(Q)$ and $\mathcal{N}(Q) = \mathcal{R}(I - Q) = \mathcal{R}(P)$. To show that $\mathcal{D}(P) = \mathcal{N}(P) \oplus \mathcal{R}(P)$, we take $x \in \mathcal{N}(P) \cap \mathcal{R}(P)$. Then $x = Px = 0$. Further, $x \in \mathcal{D}(P)$ can be decomposed into $x = Px + (I - P)x \in \mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{R}(P) + \mathcal{N}(P)$. ☐

Theorem 2.2. *Let A be the generator of a C_0–semigroup $T(\cdot)$ on a Banach space X with Fourier type $p \in [1, 2]$.*
(1) Suppose that

- (a) *either $i\mathbb{R} \subset \rho(A)$ and $\|R(i\tau, A)\| \leq C$ for $\tau \in \mathbb{R}$ and a constant $C > 0$,*
- (b) *or $\{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| \leq \delta\} \subset \rho(A)$ and $\|R(\lambda, A)\| \leq \tilde{C}(1 + |\lambda|^\gamma)$ for $|\operatorname{Re}\lambda| \leq \delta$ and some constants $\tilde{C}, \delta, \gamma > 0$.*

In case (a), we set $\gamma = 0$ and take $\delta \in (0, 1/C)$. Let $\alpha > \frac{1}{p} - \frac{1}{p'} + \gamma$. Then $T(\cdot)$ has an exponential α–dichotomy with decay exponent δ and a constant $N = N(\alpha)$.

(2) Conversely, if $T(\cdot)$ has an exponential α–dichotomy with decay exponent $\delta_0 > 0$, then (b) holds with $\gamma = \alpha$ and $\delta \in (0, \delta_0)$.

Proof. To show the second part of the theorem, assume that $T(\cdot)$ has an exponential α–dichotomy with decay exponent $\delta_0 > 0$, and let $x \in X_\alpha$. Then the operator

$$R_\lambda x = \int_0^\infty e^{-\lambda t} T(t) P x \, dt - \int_0^\infty e^{\lambda t} T_Q(-t) Q x \, dt$$

maps X_α into X and is uniformly bounded for $|\operatorname{Re}\lambda| \le \delta$ and a fixed $0 < \delta < \delta_0$. It is then straightforward to check that $R_\lambda x \in \mathcal{D}(A)$ and $(\lambda - A) R_\lambda x = x$ and that $R_\lambda(\lambda - A) y = y$ for $y \in X_{1+\alpha}$. Hence, $\lambda \in \rho(A_\alpha) = \rho(A)$, where A_α is the part of A in X_α, and $R(\lambda, A)$ is an extension of R_λ. Property (b) now follows from [13, Lem.3.2] and the uniform boundedness of $R(\lambda, A)(w - A)^{-\alpha} = R_\lambda (w - A)^{-\alpha}$ for $|\operatorname{Re}\lambda| \le \delta$.

We prove the first part of Theorem 2.2 in four steps. We first observe that assumption (a) implies (b) with $\gamma = 0$ and some $0 < \delta < 1/C$ by a standard perturbation argument. We fix numbers $\alpha > \frac{1}{p} - \frac{1}{p'} + \gamma \ge \gamma \ge 0$, $0 \le a \le \delta$, and $\max\{\delta, \omega_0(A)\} < \tilde{w} < w$. Hence, $\|R(\lambda, A)\| \le c$ for $\operatorname{Re}\lambda \ge \tilde{w}$.

Step 1. First part of the construction of $\mathbf{T_Q(-t)Q}$. We define the path $\Gamma_Q = (\tilde{w} + [-i\infty, +i\infty]) \cup (a + [+i\infty, -i\infty])$ which is oriented counter clockwise. Let $x \in \mathcal{D}(A)$ and $t \ge 0$. Since

$$R(\lambda, A)x = (w - \lambda)^{-1}(R(\lambda, A) - R(w, A))(w - A)x = \mathcal{O}(|\lambda|^{\gamma - 1}), \quad |\operatorname{Im}\lambda| \to \infty, \quad (2.1)$$

for $|\operatorname{Re}\lambda| \le \delta$ or $\operatorname{Re}\lambda = \tilde{w}$, the integral

$$G_Q(t)x = \frac{1}{2\pi i} \int_{\Gamma_Q} e^{-\lambda t} (w - \lambda)^{-\alpha} R(\lambda, A) x \, d\lambda \qquad (2.2)$$

converges absolutely. Consider the counter clockwise oriented rectangular path Γ_n with vertices $\tilde{w} \pm in$ and $a \pm in$. For $s \in [0, t]$ we then obtain

$$\begin{aligned} T(s) G_Q(t) x = G_Q(t) T(s) x &= \frac{1}{2\pi i} \int_{\Gamma_Q} e^{-\lambda(t-s)} (w - \lambda)^{-\alpha} \Big[R(\lambda, A)x - \int_0^s e^{-\lambda r} T(r) x \, dr \Big] d\lambda \\ &= G_Q(t-s)x - \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\Gamma_n} e^{-\lambda(t-s)} (w - \lambda)^{-\alpha} \int_0^s e^{-\lambda r} T(r) x \, dr \, d\lambda \\ &= G_Q(t-s)x, \end{aligned} \qquad (2.3)$$

where we have used Cauchy's theorem and

$$\int_0^s e^{-\lambda r} T(r) x \, dr = (e^{-\lambda s} T(s) - I) R(\lambda, A) x = \mathcal{O}(|\lambda|^{\gamma - 1})$$

on Γ_Q as $|\operatorname{Im}\lambda| \to \infty$ due to (2.1). We define

$$\begin{aligned} \tilde{Q}x := G_Q(0)x &= \frac{1}{2\pi i} \int_{+i\infty}^{-i\infty} (w - \lambda)^{-\alpha} R(\lambda, A) x \, d\lambda + \frac{1}{2\pi i} \int_{\tilde{w} - i\infty}^{\tilde{w} + i\infty} (w - \lambda)^{-\alpha} R(\lambda, A) x \, d\lambda \\ &= -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} (w - \lambda)^{-\alpha} R(\lambda, A) x \, d\lambda + (w - A)^{-\alpha} x \end{aligned} \qquad (2.4)$$

for $x \in \mathcal{D}(A)$. The last equality follows from a standard deformation of the path $[\tilde{w} - i\infty, \tilde{w} + i\infty]$ to the path Γ from (1.6).

Observe that the function $t \mapsto T_{\tilde{w}}(t) := e^{-\tilde{w}t}T(t)$ belongs to $L^2(\mathbb{R}_+, \mathcal{B}(X))$. Since $R(\tilde{w}+i\tau, A)x = (\mathcal{F}T_{\tilde{w}}(\cdot)x)(\tau)$ and p is the Fourier type of X, we obtain $R(\tilde{w}+i\cdot, A)x \in L^{p'}(\mathbb{R}, X)$ with p'–norm less than $c\|x\|$. The resolvent equation further yields

$$\begin{aligned}\|R(a+i\tau, A)x\| &= \|[I + (\tilde{w}-a)R(a+i\tau, A)]\, R(\tilde{w}+i\tau, A)x\| \\ &\leq c\,(1+|\tau|)^{\gamma}\, \|R(\tilde{w}+i\tau, A)x\|. \end{aligned} \tag{2.5}$$

for $\tau \in \mathbb{R}$. We set

$$f(\lambda) = (w-\lambda)^{-\alpha}R(\lambda, A)x = \begin{cases} f_1(\tau), & \lambda = a+i\tau,\ \tau \in \mathbb{R}, \\ f_2(\tau), & \lambda = \tilde{w}+i\tau,\ \tau \in \mathbb{R}. \end{cases}$$

Let $p \in (1,2]$. Hölder's inequality, estimate (2.5), and the relation $(-\alpha+\gamma)pp'(p'-p)^{-1} < -1$ then yield

$$\begin{aligned}\int_{\mathbb{R}} \|f_1(\tau)\|^p\, d\tau &\leq c\Big[\int_{\mathbb{R}} \|R(\tilde{w}+i\tau, A)x\|^{p'}\, d\tau\Big]^{\frac{p}{p'}}\ \Big[\int_{\mathbb{R}} (1+|\tau|)^{(-\alpha+\gamma)\frac{pp'}{p'-p}}\, d\tau\Big]^{\frac{p'-p}{p'}} \\ &\leq c(\alpha)\, \|x\|^p. \end{aligned} \tag{2.6}$$

The function f_2 can be estimated similarly. Thus the identity

$$2\pi\, e^{at}G_Q(t)x = \frac{1}{i}\int_{\Gamma_Q} e^{(a-\lambda)t}(w-\lambda)^{-\alpha}R(\lambda, A)x\, d\lambda = -(\mathcal{F}f_1)(t) + e^{(a-\tilde{w})t}\,(\mathcal{F}f_2)(t),$$

inequality (2.6), and the Fourier type of X imply that

$$\int_0^\infty \|e^{at}\, G_Q(t)x\|^{p'}\, dt \leq c\,(\|f_1\|_{p'}^{p'} + \|f_2\|_{p'}^{p'}) \leq c(\alpha)\, \|x\|^{p'}.$$

From this estimate and formula (2.3) we then deduce

$$\begin{aligned}\|e^{ar}G_Q(r)x\|^{p'} &\leq \int_0^1 e^{p'a(r+s)}\, \|T(s)\|^{p'}\, \|G_Q(r+s)x\|^{p'}\, ds \\ &\leq c\int_r^{r+1} \|e^{at}\, G_Q(t)x\|^{p'}\, dt \leq c(\alpha)\, \|x\|^{p'} \end{aligned}$$

for $r \geq 0$ and $x \in \mathcal{D}(A)$. Therefore we can extend $G_Q(t)$ to a bounded operator on X denoted by the same symbol and satisfying

$$\|G_Q(t)\| \leq c(\alpha)\, e^{-\delta t}, \qquad t \geq 0, \tag{2.7}$$

where we let $a = \delta$. In particular, the extension $\tilde{Q} : X \to X$ is bounded. The same results hold for $p = 1$ by an analogous, but simpler argument.

Step 2. Construction of $\mathbf{T(t)P}$. We define for $x \in \mathcal{D}(A)$ and $t \geq 0$ the operator

$$G_P(t)x = \frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} e^{\lambda t}(w-\lambda)^{-\alpha}R(\lambda, A)x\, d\lambda. \tag{2.8}$$

Again (2.1) shows that the integral converges absolutely. If $t = 0$, we set

$$\tilde{P}x := G_P(0)x = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} (w-\lambda)^{-\alpha}R(\lambda, A)x\, d\lambda \tag{2.9}$$

(after shifting a to 0). Thus (2.4) implies

$$\tilde{P}x + \tilde{Q}x = (w-A)^{-\alpha}x \tag{2.10}$$

for $x \in \mathcal{D}(A)$. There exists a bounded extension $\tilde{P}: X \to X$ due to (2.7) and (2.10). For $x \in \mathcal{D}(A^2)$ and $t \geq 0$ we further compute

$$\begin{aligned} G_P(t)x &= \frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} e^{\lambda t}(w-\lambda)^{-\alpha-1}(R(\lambda,A) - R(w,A))(w-A)x \, d\lambda \\ &= \frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} e^{\lambda t}(w-\lambda)^{-\alpha-1}R(\lambda,A)(w-A)x \, d\lambda, \end{aligned}$$

using the resolvent equation and

$$\frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} e^{\lambda t}(w-\lambda)^{-\alpha-1} d\lambda = 0 \tag{2.11}$$

(which can be shown by applying Cauchy's theorem to the triangular path with vertices $-a - in$, $-a + in$, $-n$). Because of the identity

$$\begin{aligned} &\frac{d}{dt} e^{\lambda t}(w-\lambda)^{-\alpha-1}R(\lambda,A)(w-A)x \\ &\qquad = e^{\lambda t}(w-\lambda)^{-\alpha-1}R(\lambda,A)(w-A)Ax + e^{\lambda t}(w-\lambda)^{-\alpha-1}(w-A)x, \end{aligned}$$

equation (2.11), and the closedness of A, the function $G_P(\cdot)x$ is continuously differentiable in X, $G_P(t)\mathcal{D}(A^2) \subset \mathcal{D}(A)$, and

$$\tfrac{d}{dt} G_P(t)x = G_P(t)Ax = AG_P(t)x, \qquad t \geq 0, \tag{2.12}$$

for $x \in \mathcal{D}(A^2)$. Therefore the uniqueness of the Cauchy problem corresponding to A yields

$$G_P(t)x = T(t)\tilde{P}x, \quad t \geq 0, \tag{2.13}$$

at first for $x \in \mathcal{D}(A^2)$. Since $T(t)\tilde{P}$ is bounded, we can extend $G_P(t)$ to a bounded operator (denoted by the same symbol) which satisfies (2.13) for all $x \in X$. We have $A\tilde{P}x = \tilde{P}Ax$ on $\mathcal{D}(A^2)$ by (2.12), so that

$$\begin{aligned} &\tilde{P}\mathcal{D}(A) \subset \mathcal{D}(A), \qquad A\tilde{P}x = \tilde{P}Ax \text{ for } x \in \mathcal{D}(A), \qquad \text{and hence} \\ &T(t)\tilde{P} = \tilde{P}T(t), \quad t \geq 0. \end{aligned} \tag{2.14}$$

Let $p \in (1, 2]$. As in the first step one shows that

$$\int_0^\infty \|e^{at}T(t)\tilde{P}x\|^{p'} \, dt \leq c(\alpha) \, \|x\|^{p'}$$

for $x \in \mathcal{D}(A)$. For $r \geq 1$, this estimate implies that

$$\|e^{ar}T(r)\tilde{P}x\|^{p'} \leq \int_0^1 \|e^{as}T(s)\|^{p'} \, \|e^{a(r-s)}\, T(r-s)\tilde{P}x\|^{p'} \, ds \leq c(\alpha) \, \|x\|^{p'}.$$

Taking $a = \delta$, we have shown that

$$\|T(t)\tilde{P}\| \leq c(\alpha) \, e^{-\delta t}, \qquad t \geq 0, \tag{2.15}$$

Again, this inequality still holds for $p = 1$ due to a similar argument.

Step 3. The projection P. We define the bounded operator

$$P = \tilde{P}(w - A)^{\alpha} : X_\alpha \to X. \tag{2.16}$$

Observe the first part of Definition 1.1(c) now follows from (2.15). For $x \in X_{1+\alpha}$, formula (2.9) and the closedness of the fractional power further imply that $\tilde{P}X_{1+\alpha} \subset X_\alpha$ and

$$Px = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} (w - \lambda)^{-\alpha} R(\lambda, A)(w - A)^{\alpha} x \, d\lambda = (w - A)^{\alpha} \tilde{P}x.$$

Hence, $\tilde{P}X_\alpha \subset X_\alpha$ and $P = (w - A)^{\alpha}\tilde{P}$ on X_α. The operator P thus possesses a closure (denoted by $(P, \mathcal{D}(P))$) in X. We further obtain $X_\alpha \hookrightarrow \mathcal{D}(P)$, $\tilde{P}\mathcal{D}(P) \subset X_\alpha$, $P = (w - A)^{\alpha}\tilde{P}$ on $\mathcal{D}(P)$, and therefore

$$\tilde{P} = P(w - A)^{-\alpha}, \qquad \tilde{P}x = (w - A)^{-\alpha}Px \quad \text{for} \;\; x \in \mathcal{D}(P). \tag{2.17}$$

We set $Q = I - P$. Then the identities (2.10) and (2.17) yield

$$\tilde{Q} = Q(w - A)^{-\alpha}, \qquad \tilde{Q}x = (w - A)^{-\alpha}Qx \quad \text{for} \;\; x \in \mathcal{D}(P). \tag{2.18}$$

We are going to show that P is a projection. For $x \in \mathcal{D}(A^2)$ and $a > 0$ we calculate

$$\begin{aligned}
\tilde{P}^2 x &= \frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} (w - \lambda)^{-\alpha}(w - \mu)^{-\alpha} \, R(\lambda, A)R(\mu, A)x \, d\mu \, d\lambda \qquad (2.19)\\
&= \frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} (w - \lambda)^{-\alpha} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{(w - \mu)^{-\alpha}}{\mu - \lambda} \, d\mu \; R(\lambda, A)x \, d\lambda \\
&\quad - \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} (w - \mu)^{-\alpha} \frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} \frac{(w - \lambda)^{-\alpha}}{\lambda - \mu} \, d\lambda \; R(\mu, A)x \, d\mu
\end{aligned}$$

using the resolvent equation and Fubini's theorem. The integration path of the scalar λ–integral in the last line can be closed in the left halfplane where the integrand is holomorphic. Thus the second summand on the right hand side is equal to 0. In the scalar μ–integral in first summand, we use the triangular path with vertices $-in$, in, $-n$ for large $n \in \mathbb{N}$ such that the (fixed) number λ belongs to the interior of the triangle. Then Cauchy's integral formula yields

$$\begin{aligned}
\tilde{P}^2 x &= \frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} (w - \lambda)^{-2\alpha} R(\lambda, A)x \, d\lambda \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} (w - \lambda)^{-2\alpha} R(\lambda, A)x \, d\lambda. \qquad (2.20)
\end{aligned}$$

On the other hand, by similar arguments we deduce

$$\begin{aligned}
\tilde{P}(w - A)^{-\alpha}x &= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{1}{2\pi i} \int_{\Gamma} (w - \lambda)^{-\alpha}(w - \mu)^{-\alpha} \, R(\lambda, A)R(\mu, A)x \, d\mu \, d\lambda \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} (w - \lambda)^{-\alpha} \frac{1}{2\pi i} \int_{\Gamma} \frac{(w - \mu)^{-\alpha}}{\mu - \lambda} \, d\mu \; R(\lambda, A)x \, d\lambda \\
&\quad - \frac{1}{2\pi i} \int_{\Gamma} (w - \mu)^{-\alpha} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{(w - \lambda)^{-\alpha}}{\lambda - \mu} \, d\lambda \; R(\mu, A)x \, d\mu \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} (w - \lambda)^{-2\alpha} R(\lambda, A)x \, d\lambda. \qquad (2.21)
\end{aligned}$$

In the first summand of the middle equation we have deformed a bounded part of the path Γ from (1.6) to a path in $\{\mu : \operatorname{Re}\mu < w\}$ around the (fixed) number $\lambda \in \mathbb{R}$. By approximation, the identities (2.20) and (2.21) can be extended to all $x \in X$. These two equations and (2.17) then yield

$$\tilde{P}^2 = \tilde{P}(w-A)^{-\alpha} = (w-A)^{-\alpha}\tilde{P}.$$

Hence, $\mathcal{R}(\tilde{P}^2) \subset X_\alpha$ and $\tilde{P} = (w-A)^\alpha \tilde{P}^2$. Using again formula (2.17) and the closedness of P, we conclude that $\tilde{P}X \subset \mathcal{D}(P)$ and $\tilde{P} = P\tilde{P}$. This fact leads to $P = P^2$ on X_α, due to (2.17). As a result, P is a projection. Combining (2.14) and (2.16) we further deduce $T(t)P = PT(t)$ on X_α, so that $T(t)\mathcal{D}(P) \subset \mathcal{D}(P)$ and $T(t)P = PT(t)$ on $\mathcal{D}(P)$ by approximation.

Step 4. Second part of the construction of $\mathbf{T_Q(-t)Q}$. Formula (2.2) implies that $G_Q(t)X_{1+\alpha} \subset X_\alpha$ and $G_Q(t)(w-A)^\alpha x = (w-A)^\alpha G_Q(t)x$ for $x \in X_{1+\alpha}$ and $t \geq 0$. Thus $G_Q(t)X_\alpha \subset X_\alpha$ and $G_Q(t)$ commutes with $(w-A)^\alpha$. For $x \in X_{1+\alpha}$ and $t \geq 0$, the equalities (2.3) and (2.18) then show that

$$(w-A)^\alpha G_Q(t)T(t)x = (w-A)^\alpha \tilde{Q}x = Qx \tag{2.22}$$

$$T(t)(w-A)^\alpha G_Q(t)x = (w-A)^\alpha \tilde{Q}x = Qx. \tag{2.23}$$

One can extend (2.22) to $\mathcal{D}(P)$ and (2.23) to X_α. Therefore the restriction $T_Q(t)$ of $T(t)$ to $\mathcal{N}(P) = \mathcal{R}(Q)$ is injective; its (closed) inverse is denoted by $T_Q(-t)$. Then (2.23) and an approximation argument yield

$$G_Q(t)x = (w-A)^{-\alpha}T_Q(-t)Qx$$

for $x \in \mathcal{D}(P)$. Inserting this identity into (2.22), we see that $T(t) : \mathcal{N}(P) \to \mathcal{N}(P)$ is bijective, and its inverse $T_Q(-t) : \mathcal{N}(P) \to \mathcal{N}(P)$ is bounded by the closed graph theorem. Moreover,

$$T_Q(-t)Qx = G_Q(t)(w-A)^\alpha x, \quad x \in X_\alpha\,,$$

and the second part of Definition 1.1(c) follows from (2.7). $\square$

Remark 2.3. Suppose that the assumptions of Theorem 2.2 hold for $\gamma = 0$. Let $T_s(t)$ be the restriction of $T(t)$ to $X_s := \mathcal{R}(P)$. Then the generator A_s of this semigroup is the restriction of A to X_s. Definition 1.1(c) implies that $R(\lambda, A_s)(w-A_s)^{-\alpha}$ (initially defined for $\operatorname{Re}\lambda > \omega_0(A_s)$) has a bounded extension $F(\lambda)$ to $\operatorname{Re}\lambda \geq -\delta/2$. This means that the spectral bound of the part of A_s in $\mathcal{D}((w-A_s)^\alpha)$ is less than $-\delta/2$, and hence $s(A_s) \leq -\delta/2 < 0$. The boundedness of $F(\lambda) = R(\lambda, A_s)(w-A_s)^{-\alpha}$ further yields $\|R(\lambda, A_s)\| \leq c\,(1+|\lambda|^\alpha)$ for $\operatorname{Re}\lambda \geq -\delta/2$ by [13, Lem.3.2]. On the other hand, $R(\lambda, A_s)$ is the restriction of $R(\lambda, A)$ for $\lambda \in i\mathbb{R}$, and thus uniformly bounded on $i\mathbb{R}$ by assumption (a). The Phragmen Lindelöf principle then implies that $R(\lambda, A_s)$ is uniformly bounded for $\operatorname{Re}\lambda \geq 0$. The Weis–Wrobel result (1.5) now shows that

$$\|T(t)x\| \leq Ne^{-\delta t}\|(w-A)^\beta x\|, \qquad t \geq 0,$$

for $\beta = \frac{1}{p} - \frac{1}{p'}$ and $x \in \mathcal{R}(P) \cap X_\beta$. Similar arguments work for the restriction $-A_u$ of $-A$ in $X_u = \mathcal{R}(Q)$ which generates the semigroup $T_Q(-t)$, $t \geq 0$, on X_u. Hence

$$\|T_Q(-t)x\| \leq Ne^{-\delta t}\|(w-A)^\beta x\|, \qquad t \geq 0,$$

for $x \in \mathcal{R}(Q) \cap X_\beta$. If one wants to derive an estimate for $T(t)P$ as in our Definition 1.1, then one obtains

$$\|T(t)Px\| \le Ne^{-\delta t}\|(w-A)^\beta Px\| \le ce^{-\delta t}\|(w-A)^{\beta+\alpha}x\|$$

for $x \in X_{\beta+\alpha}$ with $\alpha > \frac{1}{p} - \frac{1}{p'}$ (and analogously for $T_Q(-t)Q$). Observe that one looses $\beta = \frac{1}{p} - \frac{1}{p'}$ in the regularity exponent, and it is thus necessary to estimate the *products* $T(t)P$ and $T_Q(-t)Q$ in order to prove Theorem 2.2.

Remark 2.4. Suppose that the assumptions of Theorem 2.2 hold. Arguing as in (2.21), one can verify that

$$\tilde{P}(w-A)^{-\beta}x = (w-A)^{-\beta}\tilde{P}x = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}(w-\lambda)^{-\alpha-\beta}R(\lambda,A)x\,d\lambda$$

for $\beta > 0$ and $x \in \mathcal{D}(A)$. This identity implies in particular that the projection P in Theorem 2.2 does not depend on the choice of $\alpha > \frac{1}{p} - \frac{1}{p'} + \gamma$.

In the next example we extend Theorem 4.4(a) and Proposition 5.7 from [4] from an L^2 to an L^q–setting. Due to the lack of a Weis–Wrobel type result for dichotomies, we were forced to restrict ourselves to a Hilbert space setting in [4].

Example 2.5. Let $A = \Delta$ be the Dirichlet Laplacian with $\mathcal{D}(A) = W^{2,q}(\Omega) \cap W^{1,q}(\Omega) =: X_1$ on $X = L^q(\Omega)$ for $1 < q < \infty$ and a bounded domain Ω with a C^2 boundary. (In fact, one can replace A by more general generators of analytic semigroups, see (H) and Proposition 5.7 of [4] for the details.) Let $r > 0$ and $B : [-r, 0] \to \mathcal{B}(X_1, X)$ be of bounded variation db such that $db([-t, 0]) \to 0$ as $t \searrow 0$. We define the 'history function' $u_t(\theta) = u(t+\theta)$ for $u : [-r, \infty) \to X$, $t \ge 0$, and $\theta \in [-r, 0]$. Let $p \in [\min\{q, q'\}, \max\{q, q'\}]$. For $\varphi \in W^{1,p}([-r, 0], X_1)$ we study the retarded problem

$$u'(t) = Au(t) + \int_{-r}^{0} dB(\theta)\, u(t+\theta), \quad t \ge 0, \qquad u(t) = \varphi(t), \quad t \in [-r, 0]. \tag{2.24}$$

This problem can be solved using Theorems 7.4–7.6 of [20] which address even more general equations. In [4] we have developed a semigroup approach which gives an alternative proof of a part of these theorems. We need the real interpolation space $Y := (X, X_1)_{1-1/p,p}$. The product space $\mathcal{X} = Y \times L^p([-r, 0], Y)$ has Fourier type $\min\{q, q'\}$, see e.g. the proof of [4, Thm.4.5]. There is a unique solution $u \in C^1(\mathbb{R}_+, Y) \cap W^{1,p}_{loc}(\mathbb{R}_+, X_1)$ of (2.24). There exists a C_0–semigroup $\mathcal{T}(\cdot)$ on $\mathcal{X}$ satisfying

$$\binom{u(t)}{u_t} = \mathcal{T}(t)\binom{\varphi(0)}{\varphi}, \qquad t \ge 0,$$

which is generated by

$$\mathcal{A} = \begin{pmatrix} A & L \\ 0 & \frac{d}{d\theta} \end{pmatrix}, \quad D(\mathcal{A}) = \{\tbinom{x}{\varphi} \in X_1 \times W^{1,p}([-r, 0], X_1) : \varphi(0) = x,\ L\varphi + Ax \in Y\},$$

where $L = L_0$, $L_\lambda = \int_{-r}^0 dB(\theta)\, e^{\lambda\theta}x$ for $\lambda \in \mathbb{C}$ and $x \in X_1$, see [4, Thm.3.6]. In certain cases this semigroup violates the spectral mapping theorem, see [4, Ex.5.8]. By Proposition 4.3 of [4], a complex number λ belongs to $\rho(\mathcal{A})$ if and only if the operator $H(\lambda) = (\lambda - A - L_\lambda)^{-1}$ exists in $\mathcal{B}(X, X_1)$, and one has $\|R(\lambda, \mathcal{A})\|_{\mathcal{B}(\mathcal{X})} \le c(a)\,\|H(\lambda)\|_{\mathcal{B}(X,X_1)}$ for $\lambda \in \rho(\mathcal{A})$ with $\operatorname{Re}\lambda \ge a$. Thus Theorem 2.2 immediately implies that

$$\sup_{\tau\in\mathbb{R}} \|H(i\tau)\|_{\mathcal{B}(X,X_1)} < \infty \quad \Longrightarrow \quad \mathcal{T}(\cdot) \text{ has an exp. } \alpha\text{–dichotomy for } \alpha > \left|\tfrac{1}{q} - \tfrac{1}{q'}\right|.$$

Observe that the dichotomy of $\mathcal{T}(\cdot)$ means a splitting of both the solutions $u(t)$ and their history functions u_t with decay estimates in the norm of $\mathcal{X}$.

To obtain a more explicit condition, suppose in addition that $B(\theta) = \eta(\theta)A$, where $\eta \in BV([-r,0],\mathbb{C})$. We set

$$\widehat{d\eta}(\lambda) = \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta), \qquad \lambda \in \mathbb{C}.$$

Then $H(i\tau) : X \to X_1$ exists and is uniformly bounded for $\tau \in \mathbb{R}$ provided that

$$\overline{1 + \widehat{d\eta}(i\mathbb{R})} \;\subset\; \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| < \psi \text{ or } |\arg(-\lambda)| < \psi\}$$

for some $0 < \psi < \pi/2$, see the proof of Proposition 5.7 of [4]. In this case we thus obtain an exponential α–dichotomy.

3. Polynomial dichotomy

In this section we assume that A is a generator of a semigroup $T(\cdot)$ on a Banach space X such that $i\mathbb{R} \subset \rho(A)$ and $\|R(i\tau, A)\| \le \hat{C}\,(1 + |\tau|^{\gamma})$ for $\tau \in \mathbb{R}$ and some $\gamma > 0$. We take again $\alpha > \gamma + \frac{1}{p} - \frac{1}{p'}$ where $p \in [1,2]$ is the Fourier type of X. We want to repeat the reasoning in steps 1–4 of the proof of Theorem 2.2 with $a = \delta = 0$. This can be done literally in the same way with the exception of (2.19). Here we have to replace the path $-a + i\mathbb{R}$ by the path Γ' given by

$$\lambda = i\tau - [2\hat{C}\,(1 + |\tau|^{\gamma})]^{-1}, \qquad \tau \in \mathbb{R}.$$

Standard perturbation arguments show that $\Gamma' \subset \rho(A)$ and

$$\|R(\lambda, A)\| \le 2\hat{C}\,(1 + |\operatorname{Im}\lambda|^{\gamma}) \le c\,(1 + |\lambda|^{\gamma}), \quad \lambda \in \Gamma'.$$

The modification of (2.19) can be justified using Cauchy's integral theorem. Then one verifies (2.20) as before. Thus we have constructed a closed projection P on X such that $X_\alpha \hookrightarrow \mathcal{D}(P)$, $T(t)\mathcal{D}(P) \subset \mathcal{D}(P)$, $T(t)Px = PT(t)x$ for $x \in \mathcal{D}(P)$, $T(t) : \mathcal{N}(P) \to \mathcal{N}(P)$ has the bounded inverse $T_Q(-t)$. Moreover, $T(t)P(w-A)^{-\alpha}x$ and $T_Q(-t)Q(w-A)^{-\alpha}x$, $t \ge 0$, are given by (2.8) and (2.2) with $a = 0$ if $x \in \mathcal{D}(A)$, and these operators are uniformly bounded for $t \ge 0$. We use these formulas in order to derive the desired polynomial decay estimates, starting with the stable case. First observe that we can replace α by $\alpha + \gamma$ in the above reasoning. Let $x \in \mathcal{D}(A)$ and $t \ge 1$. Then we obtain

$$\begin{aligned} tT(t)P(w-A)^{-\alpha-\gamma}x &= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} (\tfrac{d}{d\lambda}\, e^{\lambda t})\,(w-\lambda)^{-\alpha-\gamma}R(\lambda, A)x\, d\lambda. \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{\lambda t}(w-\lambda)^{-\alpha-\gamma}R(\lambda, A)^2 x\, d\lambda \\ &\qquad - \frac{\alpha+\gamma}{2\pi i}\int_{-i\infty}^{+i\infty} e^{\lambda t}(w-\lambda)^{-\alpha-\gamma-1}R(\lambda, A)x\, d\lambda =: J_1 + J_2, \end{aligned}$$

integrating by parts in the second equality. It is clear that $\|J_2\|$ is less than $c\,\|x\|$. Due to (2.5), we can estimate the integrand in J_1 by

$$c\,(1 + |\lambda|)^{2\gamma-\alpha-\gamma}\|R(\tilde{w} + i\operatorname{Im}\lambda, A)\|^2\,\|x\|.$$

We know that $\tau \mapsto \|R(\tilde{w}+i\tau, A)\|$ belongs to $L^{p'}(\mathbb{R}, \mathcal{B}(X))$. Thus $\|J_1\|$ is also bounded by $c\,\|x\|$ if $p=2$ because of $\gamma-\alpha<0$. In this case we could have replaced $\alpha+\gamma$ by 2γ. If $1 \le p < 2$, we use Hölder's inequality with exponents $p'/2$ and $p/(2-p)$. Since

$$\frac{p(\gamma-\alpha)}{2-p} < \frac{p(p-p')}{(2-p)pp'} = -1,$$

we also obtain $\|J_1\| \le c\,\|x\|$. The same arguments work for $T_Q(-t)Q$.

Theorem 3.1. *Let A be the generator of a C_0–semigroup $T(\cdot)$ on a Banach space X with Fourier type $p \in [1,2]$. Suppose that $i\mathbb{R} \subset \rho(A)$ and $\|R(i\tau, A)\| \le C(1+|\tau|^\gamma)$ for $\tau \in \mathbb{R}$ and constants $C, \gamma > 0$. Let $\alpha > \frac{1}{p} - \frac{1}{p'} + \gamma$. Then there is a closed projection P on X such that $X_\alpha \hookrightarrow \mathcal{D}(P)$, $T(t)\mathcal{D}(P) \subset \mathcal{D}(P)$, $T(t)Px = PT(t)x$ for $x \in \mathcal{D}(P)$, $T(t) : \mathcal{N}(P) \to \mathcal{N}(P)$ has the bounded inverse $T_Q(-t)$, and*

$$\|T(t)Px\| \le \frac{N(\alpha)}{t}\,\|(w-A)^{\alpha+\gamma}x\|\,, \quad \|T_Q(-t)Qx\| \le \frac{N(\alpha)}{t}\,\|(w-A)^{\alpha+\gamma}x\| \tag{3.1}$$

for $t \ge 1$, $x \in X_{\alpha+\gamma}$, and a constant $N(\alpha) \ge 0$. If $p=2$ we may take $\alpha=\gamma$ in (3.1). We further have

$$\|T(t)Px\| \le N'(\alpha)\,\|(w-A)^{\alpha}x\|\,, \quad \|T_Q(-t)Qx\| \le N'(\alpha)\,\|(w-A)^{\alpha}x\| \tag{3.2}$$

for $t \ge 0$, $x \in X_\alpha$, and a constant $N'(\alpha) \ge 0$.

If $T(\cdot)$ is a *bounded* semigroup, we obtained a similiar result in Theorem 3.5 of [3] with $P = I$ and an arbitrary $\alpha > 0$ in (3.1). The difference in the regularity exponent is caused by the unboundedness of $T(t)P$ in Theorem 3.1.

Under the assumptions of Theorem 3.1, we easily obtain decay estimates on the spaces X_β for each $\beta > 0$. First observe that $T(t)P$ and $T_Q(-t)Q$ commute with fractional powers (say, on X_n for a sufficiently large $n \in \mathbb{N}$). Using (3.1), (3.2), and the moment inequality, see e.g. [8, Thm.II.5.34], we then deduce for $0<\theta<1$ and $t \ge 1$ that

$$\|T(t)P(w-A)^{-(\alpha+\theta\gamma)}\| \le c(\alpha)\,t^{-\theta}\,, \qquad \|T_Q(-t)Q(w-A)^{-(\alpha+\theta\gamma)}\| \le c(\alpha)\,t^{-\theta}. \tag{3.3}$$

Second, since $T(t)Px = T(\frac{t}{n})P \cdots T(\frac{t}{n})Px$ and $T_Q(-t)Qx = T_Q(-\frac{t}{n})Q \cdots T_Q(-\frac{t}{n})Qx$, the estimates (3.1) yield

$$\|T(t)Px\| \le c(\alpha,n)t^{-n}\,\|(w-A)^{(\alpha+\gamma)n}x\|\,, \quad \|T_Q(-t)Qx\| \le c(\alpha,n)t^{-n}\,\|(w-A)^{(\alpha+\gamma)n}x\|$$

for $n \in \mathbb{N}$, $t \ge 1$, and $x \in X_{n(\alpha+\theta)}$. Interpolating once more, we arrive at

$$\|T(t)P(w-A)^{-\theta(\alpha+\gamma)}\| \le c(\alpha,\theta)\,t^{-\theta}, \qquad \|T_Q(-t)Q(w-A)^{-\theta(\alpha+\gamma)}\| \le c(\alpha,\theta)\,t^{-\theta} \tag{3.4}$$

for $\theta > 1$, $t \ge 1$, and a constant $c(\alpha,\theta) \ge 0$.

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