Semicontinuity of Convex-valued Multifunctions

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May 2, 2005

Abstract

We introduce semicontinuity concepts for functions f with values in the space $\mathcal{C}(Y)$ of closed convex subsets of a finite dimensional normed vector space Y by appropriate notions of upper and lower limits. We characterize the upper semicontinuity of $f : X \to \mathcal{C}(Y)$ by the upper semicontinuity of the scalarizations $\sigma_{f(\,\cdot\,)}(y^*) : X \to \overline{\mathbb{R}}$ by the support function. Furthermore, we compare our semicontinuity concepts with well-known concepts.

1 Introduction

Working in the framework of convex-valued multifunctions we expect that an appropriate notion of an upper semicontinuous hull produces a convex-valued multifunction being upper semicontinuous. This cannot be ensured by the classical concept of outer semicontinuity [4], as the following examples show. We denote by $\mathrm{LIM\,SUP}_{x' \to x} f(x')$ the outer limit of f at x and by $(\operatorname{osc} f)(x) = \mathrm{LIM\,SUP}_{x' \to x} f(x')$ the corresponding outer semicontinuous hull, see Section 2 for the exact definitions.

Example 1.1 Let $f : \mathbb{R} \rightrightarrows \mathbb{R}$, $f(x) := \{x \,/\, |x|\}$ if $x \neq 0$ and $f(0) := \{0\}$. Then the outer semicontinuous hull of f, namely $(\operatorname{osc} f) : \mathbb{R} \rightrightarrows \mathbb{R}$, $(\operatorname{osc} f)(x) = f(x)$ if $x \neq 0$ and $(\operatorname{osc} f)(0) = \{-1, 0, 1\}$, is not convex-valued.

This might suggest to redefine the outer semicontinuous hull as follows:

$$(\widetilde{\operatorname{osc}}\, f)(x) := \operatorname{cl}\operatorname{conv} \operatorname*{LIM\,SUP}_{x' \to x} f(x').$$

However, $(\widetilde{\operatorname{osc}}\, f)$ is not necessarily outer semicontinuous as the following example shows.

Example 1.2 Let $f : \mathbb{R} \rightrightarrows \mathbb{R}$,

$$f(x) := \begin{cases} \left\{\frac{1}{x}\right\} & \text{if} \quad \exists n \in \mathbb{N} : x \in \left[2^{-2n}, 2^{-2n+1}\right) \\ \left\{-\frac{1}{x}\right\} & \text{if} \quad \exists n \in \mathbb{N} : x \in \left[2^{-2n+1}, 2^{-2n+2}\right) \\ \emptyset & \text{else.} \end{cases}$$

Then the modified outer semicontinuous hull $(\widetilde{\operatorname{osc}}\, f)$ of f is obtained as

$$(\widetilde{\operatorname{osc}}\, f)(x) = \begin{cases} \left\{\frac{1}{x}\right\} & \text{if} \quad \exists n \in \mathbb{N} : x \in \left(2^{-2n}, 2^{-2n+1}\right) \\ \left\{-\frac{1}{x}\right\} & \text{if} \quad \exists n \in \mathbb{N} : x \in \left(2^{-2n+1}, 2^{-2n+2}\right) \\ \left[-\frac{1}{x}, \frac{1}{x}\right] & \text{if} \quad \exists n \in \mathbb{N} : x = 2^{-n} \\ \emptyset & \text{else.} \end{cases}$$

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It is easily seen that $\mathrm{gr}\,(\widetilde{\mathrm{osc}}\, f)$ is not closed. Indeed, the sequence $(2^{-n}, 0)_{n\in\mathbb{N}}$ belongs to the graph of $(\widetilde{\mathrm{osc}}\, f)$, but its limit $(0,0)$ does not. Hence $(\widetilde{\mathrm{osc}}\, f)$ is not outer semicontinuous.

Let us illuminate another aspect. An important idea of Convex Analysis is the relationship between a convex set $A \subset \mathbb{R}^p$ and its support function $\sigma_A : \mathbb{R}^p \to \overline{\mathbb{R}}$. In particular, for closed convex sets $A, B \subset \mathbb{R}^p$ and $\alpha \in \mathbb{R}_+$ we have the following relationships (in particular, we set $-\infty + \infty = -\infty$, $0 \cdot \emptyset = \{0\}$):

$$\Big(A \subset B \Leftrightarrow \sigma_A \le \sigma_B\Big), \qquad \sigma_A + \sigma_B = \sigma_{A+B}, \qquad \alpha\sigma_A = \sigma_{\alpha A}.$$

This yields, for instance, that a set-valued map $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is concave (i.e. graph-convex) if and only if the functions $\sigma_{f(\,\cdot\,)}(y^*) : \mathbb{R}^n \to \overline{\mathbb{R}}$ have the same property for all $y^* \in \mathbb{R}^p$. For some reasons it could be useful to have a corresponding relationship for continuity properties, too. However, the usual outer and inner semicontinuity is not appropriate for this, as the following example shows.

Example 1.3 Let $f : \mathbb{R} \rightrightarrows \mathbb{R}$, $f(x) := \left\{\frac{1}{x}\right\}$ if $x \neq 0$ and $f(0) := \{0\}$. Then f is outer semicontinuous (in particular at $x = 0$), but $\sigma_{f(\,\cdot\,)}(y^*)$ is not upper semicontinuous at $x = 0$ whenever $y^* \neq 0$.

Motivated by these examples we introduce semicontinuity concepts such that the corresponding upper semicontinuous hull operation yields a convex-valued upper semicontinuous function and such that upper semicontinuity can be described by upper semicontinuity of the functions $\sigma_{f(\,\cdot\,)}(y^*) : \mathbb{R}^n \to \overline{\mathbb{R}}$. Our investigations are based on some results on $\mathcal{C}$-convergence, which were recently obtained by C. Zălinescu and the author [2], [1].

This paper is organized as follows. In Section 2 we shortly summarize some facts on outer and inner semicontinuity in the sense of Painlevé and Kuratowski (e.g., see [4]). In the third section, we recall the definition of upper and lower limits for sequences of convex sets, as introduced in [2], and we propose our main tools. In Section 4 we extend these concepts, which leads to our semicontinuity concepts. We show that $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, having closed convex values, is upper semicontinuous at $\bar{x}$ if and only if $\sigma_{f(\,\cdot\,)}(y^*) : X \to \overline{\mathbb{R}}$ is upper semicontinuous at $\bar{x}$ for all y^* belonging to the set $\mathrm{ri}\,(0^+ f(\bar{x})^\circ)$. Section 5 is devoted to a comparison of our semicontinuity concepts with the classical outer and inner semicontinuity. Finally, in Section 6, we discuss the special case of concave (i.e. graph-convex) maps.

2 Preliminaries

Throughout the paper we denote by $\mathcal{F} := \mathcal{F}(Y)$ the space of closed subsets of a finite dimensional normed vector space Y with dimension $p \ge 1$.

We start with some basic concepts with respect to Painlevé–Kuratowski convergence (shortly PK-convergence), see also [4]. We frequently use the following notation of [4]:

$$\mathcal{N}_\infty := \{N \subset \mathbb{N} |\ \mathbb{N} \setminus N \text{ finite}\} \quad \text{ and } \quad \mathcal{N}_\infty^\# := \{N \subset \mathbb{N} |\ N \text{ infinite}\}.$$

For a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ the *outer limit* is the set

$$\operatorname*{LIM\,SUP}_{n\to\infty} A_n := \left\{ y \in Y |\ \exists N \in \mathcal{N}_\infty^\#,\ \forall n \in N,\ \exists y_n \in A_n : y_n \xrightarrow{N} y \right\}.$$

and the *inner limit* is the set

$$\operatorname*{LIM\,INF}_{n\to\infty} A_n := \left\{y \in Y \mid \exists N \in \mathcal{N}_\infty,\ \forall n \in N,\ \exists y_n \in A_n : y_n \xrightarrow{N} y\right\}.$$

A sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ is *PK-convergent* to some $A \in \mathcal{F}$ if $A = \operatorname{LIM\,SUP}_{n\to\infty} A_n = \operatorname{LIM\,INF}_{n\to\infty} A_n$. Then we write $A = \operatorname{LIM}_{n\to\infty} A_n$ or $A_n \xrightarrow{PK} A$. In contrast to [4], we use capital letters in the notation of the (outer and inner) limit, because the notation with small letters is reserved for the (upper and lower) limit in the space $\mathcal{C}$ to be defined later on. The following characterization of outer and inner limits (see [4, Exercise 4.2.(b)]) is very important for the considerations in the next section. For a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ we have

$$\operatorname*{LIM\,SUP}_{n\to\infty} A_n = \bigcap_{N\in\mathcal{N}_\infty} \operatorname{cl} \bigcup_{n\in N} A_n, \qquad \operatorname*{LIM\,INF}_{n\to\infty} A_n = \bigcap_{N\in\mathcal{N}_\infty^\#} \operatorname{cl} \bigcup_{n\in N} A_n.$$

Note that $(\mathcal{F}, \subset)$ provides a complete lattice, i.e. every nonempty subset of $\mathcal{F}$ has a supremum and an infimum (denoted by $\operatorname{SUP}\mathcal{A}$ and $\operatorname{INF}\mathcal{A}$). Of course, for a nonempty subset $\mathcal{A} \subset \mathcal{F}$ we have $\operatorname{SUP}\mathcal{A} = \operatorname{cl}\bigcup\{A \mid A \in \mathcal{A}\}$ and $\operatorname{INF}\mathcal{A} = \bigcap\{A \mid A \in \mathcal{A}\}$. Further, we set $\operatorname{INF}\emptyset = \operatorname{SUP}\mathcal{F}$ and $\operatorname{SUP}\emptyset = \operatorname{INF}\mathcal{F}$. Hence we can write

$$\operatorname*{LIM\,SUP}_{n\to\infty} A_n = \operatorname*{INF}_{N\in\mathcal{N}_\infty} \operatorname*{SUP}_{n\in N} A_n, \qquad \operatorname*{LIM\,INF}_{n\to\infty} A_n = \operatorname*{INF}_{N\in\mathcal{N}_\infty^\#} \operatorname*{SUP}_{n\in N} A_n.$$

Throughout the paper let $X = \mathbb{R}^n$, although many assertions are also valid in a more general context. The notations $\bigcup_{x_n\to\bar{x}}$ and $\bigcap_{x_n\to\bar{x}}$ stand for the union and intersection over all sequences converging to $\bar{x}$, respectively. In the following let $f : X \to \mathcal{F}$.

The *outer* and *inner limits* of f at $\bar{x} \in X$ are defined, respectively, by

$$\operatorname*{LIM\,SUP}_{x\to\bar{x}} f(x) := \bigcup_{x_n\to\bar{x}} \operatorname*{LIM\,SUP}_{n\to\infty} f(x_n) \quad \text{and} \quad \operatorname*{LIM\,INF}_{x\to\bar{x}} f(x) := \bigcap_{x_n\to\bar{x}} \operatorname*{LIM\,INF}_{n\to\infty} f(x_n).$$

The limit of f at $\bar{x}$ exists if the outer and inner limits coincide. Then we write

$$\operatorname*{LIM}_{x\to\bar{x}} f(x) = \operatorname*{LIM\,SUP}_{x\to\bar{x}} f(x) = \operatorname*{LIM\,INF}_{x\to\bar{x}} f(x).$$

Note that the outer limit (and obviously also the inner limit) is always a closed subset of Y, see [4, Proposition 4.4]. Hence we can write

$$\operatorname*{LIM\,SUP}_{x'\to x} f(x') = \operatorname*{SUP}_{x_n\to x} \operatorname*{INF}_{N\in\mathcal{N}_\infty} \operatorname*{SUP}_{n\in N} f(x_n), \qquad \operatorname*{LIM\,INF}_{x'\to x} f(x') = \operatorname*{INF}_{x_n\to x} \operatorname*{INF}_{N\in\mathcal{N}_\infty^\#} \operatorname*{SUP}_{n\in N} f(x_n).$$

The function f is said to be *outer semicontinuous (osc)*, *inner semicontinuous (isc)*, *continuous* at $\bar{x} \in X$ if $f(\bar{x}) \supset \operatorname{LIM\,SUP}_{x\to\bar{x}} f(x)$, $f(\bar{x}) \subset \operatorname{LIM\,INF}_{x\to\bar{x}} f(x)$, $f(\bar{x}) = \operatorname{LIM}_{x\to\bar{x}} f(x)$, respectively. If f is osc, isc, continuous at every $\bar{x} \in X$ we just say f is osc, isc, continuous, respectively. The *epigraph* and the *hypograph* of $f : X \to \mathcal{F}$ are defined, respectively, by

$$\operatorname{epi} f := \{(x, A) \in X \times \mathcal{F} \mid A \supset f(x)\}, \qquad \operatorname{hyp} f := \{(x, A) \in X \times \mathcal{F} \mid A \subset f(x)\}.$$

Note that, for all $x \in X$, we have $(x, \emptyset) \in \operatorname{hyp} f$ and $(x, Y) \in \operatorname{epi} f$. For a characterization of semicontinuity we need to know what is meant by closedness of the epigraph and hypograph. A subset $\mathcal{A} \subset X \times \mathcal{F}$ is said to be closed if for every sequence $(x_n, A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ with

$x_n \to \bar{x} \in X$ and $A_n \xrightarrow{PK} \bar{A} \in \mathcal{F}$ it is true that $(\bar{x}, \bar{A}) \in \mathcal{A}$. The *closure* of a set $\mathcal{A} \subset X \times \mathcal{F}$, denoted by $\operatorname{cl} \mathcal{A}$, is the set of all limits $(\bar{x}, \bar{A}) \in X \times \mathcal{F}$ of sequences $(x_n, A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$.

From [4, Exercise 5.6 (c)] and [4, Theorem 5.7 (a)] we obtain the following characterization of outer semicontinuity

$$\operatorname{hyp} f \text{ is closed} \quad \Leftrightarrow \quad f \text{ is osc} \quad \Leftrightarrow \quad \operatorname{gr} f \subset X \times Y \text{ is closed.}$$

Likewise, by [4, Exercise 5.6 (d)], inner semicontinuity of f is equivalent to the closedness of the epigraph. Note that the description by the graph fails in this case, i.e. a function $f : X \to \mathcal{F}$ that is isc has not necessarily a closed graph, see [4, Fig. 5–3. (b)].

Let us collect some basic properties of the *outer semicontinuous hull* of f, defined by $(\operatorname{osc} f) : X \to \mathcal{F}$, $(\operatorname{osc} f)(x) := \operatorname{LIM\,SUP}_{x'\to x} f(x')$.

Proposition 2.1 *Let $f : X \to \mathcal{F}$. Then it holds*

(i) $\operatorname{gr}\,(\operatorname{osc} f) = \operatorname{cl}\,(\operatorname{gr} f)$,

(ii) $\operatorname{hyp}\,(\operatorname{osc} f) \supset \operatorname{cl}\,(\operatorname{hyp} f)$,

(iii) $(\operatorname{osc} f)$ *is osc,*

(iv) $\forall x \in X : \quad (\operatorname{osc} f)(x) \supset f(x)$,

(v) f *is osc at* $\bar{x} \in X \quad \Leftrightarrow \quad (\operatorname{osc} f)(\bar{x}) = f(\bar{x})$,

(vi) $\operatorname{gr} f$ *convex* $\quad \Rightarrow \quad \operatorname{gr}\,(\operatorname{osc} f)$ *convex.*

Proof. (i) See [4, page 154, 5(2) and 5(3)]. (ii) Let $(\bar{x}, \bar{A}) \in \operatorname{cl}(\operatorname{hyp} f)$. Then, there exist $(x_n)_{n\in\mathbb{N}} \subset X$ and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $\bar{x} = \lim_{n\to\infty} x_n$, $\bar{A} = \operatorname{LIM}_{n\to\infty} A_n$ and $A_n \subset f(x_n)$ for all $n \in \mathbb{N}$. Hence, $(\operatorname{osc} f)(\bar{x}) = \operatorname{LIM\,SUP}_{x\to\bar{x}} f(x) \supset \operatorname{LIM\,SUP}_{n\to\infty} f(x_n) \supset \operatorname{LIM\,SUP}_{n\to\infty} A_n = \operatorname{LIM}_{n\to\infty} A_n = \bar{A}$, i.e. $(\bar{x}, \bar{A}) \in \operatorname{hyp}(\operatorname{osc} f)$. (iii) By (i), $\operatorname{gr}(\operatorname{osc} f)$ is closed. Hence, $(\operatorname{osc} f)$ is osc. (iv) Choosing the special sequence $x_n \equiv x$, we obtain $(\operatorname{osc} f)(x) = \operatorname{LIM\,SUP}_{x'\to x} f(x') \supset \operatorname{LIM\,SUP}_{n\to\infty} f(x_n) = \operatorname{LIM\,SUP}_{n\to\infty} f(x) = f(x)$. (v) By definition, f is osc at $\bar{x}$ if and only if $f(\bar{x}) \supset (\operatorname{osc} f)(\bar{x})$. By (iv), this equivalent to $f(\bar{x}) = (\operatorname{osc} f)(\bar{x})$. (vi) Since $\operatorname{gr} f$ is convex, $\operatorname{cl}(\operatorname{gr} f)$ is convex, too. Hence, the convexity of $\operatorname{gr}(\operatorname{osc} f)$ follows from (i). □

The next example shows that the opposite inclusion in assertion (ii) of the previous proposition does not hold true, in general.

Example 2.2 Let $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$, $f(x) := \{x\,/\,|x|\}$ if $x \neq 0$, $f(0) := \emptyset$. Then, $(0, \{-1, 1\})$ belongs to $\operatorname{hyp}(\operatorname{osc} f)$ but it does not belong to $\operatorname{cl}(\operatorname{hyp} f)$.

Remark 2.3 As noticed in [4], an analogous definition of the inner semicontinuous hull, namely by $(\operatorname{isc} f)(x) := \operatorname{LIM\,INF}_{x'\to x} f(x')$, is not constructive in the sense that $(\operatorname{isc} f)$ is not necessarily isc. In the framework of $\mathcal{C}$-valued functions we will have similar problems. An example is given there.

3 Upper and lower limits for sequences

In this section we recall some concepts and results of [2]. In the following we denote by $\mathcal{C}$ the space of all closed convex subsets of a finite dimensional normed vector space Y with dimension $p \geq 1$. Of course, $(\mathcal{C}, \subset)$ is a complete lattice and the supremum and infimum of a nonempty set $\mathcal{A} \subset \mathcal{C}$ are given, respectively, by

$$\sup \mathcal{A} := \operatorname{cl}\operatorname{conv} \bigcup_{A\in\mathcal{A}} A \qquad \text{and} \qquad \inf \mathcal{A} := \bigcap_{A\in\mathcal{A}} A.$$

The following initial result is an immediate consequence of [3, Cor. 16.5.1].

Proposition 3.1 *Let $\mathcal{A} \subset \mathcal{C}$. Then $\sigma_{\inf \mathcal{A}} \leq \inf_{A \in \mathcal{A}} \sigma_A$ and $\sigma_{\sup \mathcal{A}} = \sup_{A \in \mathcal{A}} \sigma_A$.*

The *upper* and *lower limits* [2] of a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ are defined, respectively, by

$$\limsup_{n\to\infty} A_n := \inf_{N\in\mathcal{N}_\infty} \sup_{n\in N} A_n \quad \text{and} \quad \liminf_{n\to\infty} A_n := \inf_{N\in\mathcal{N}_\infty^\#} \sup_{n\in N} A_n.$$

We say that a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ is *$\mathcal{C}$-convergent* to some $A \in \mathcal{C}$ if $A = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$. Then we write $A = \lim_{n\to\infty} A_n$ or $A_n \xrightarrow{\mathcal{C}} A$.

The following characterization of the upper limit is useful to show further properties of the upper and lower limits. For simplicity of notation we denote the set $\{m, m+1, \dots, k\} \subset \mathbb{N}$ ($m, k \in \mathbb{N}$, $m \leq k$) by $\overline{m,k}$. Further we set $\Delta_p := \left\{\lambda \in [0,1]^p \mid \sum_{i\in\overline{0,p-1}} \lambda_n^i = 1\right\}$.

Proposition 3.2 ([2]) *Consider a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$. Then $y \in \limsup_{n\in\mathbb{N}} A_n$ if and only if the following assertion holds:*

$$\begin{aligned} &\exists\, (\lambda_n)_{n\in\mathbb{N}} \subset \Delta_{p+1},\ \exists\, (k_n)_{n\in\mathbb{N}} \subset \mathbb{N}^{p+1},\ \exists\, (z_n)_{n\in\mathbb{N}} \subset Y^{p+1}, \\ &\qquad y = \lim_{n\in\mathbb{N}} \sum_{i\in\overline{0,p}} \lambda_n^i z_n^i,\ \forall n \in \mathbb{N},\ \forall j \in \overline{0,p},\ k_n^j \geq n,\ z_n^j \in A_{k_n^j}. \end{aligned}$$

The next two theorems give us sufficient conditions for the coincidence of PK-convergence and $\mathcal{C}$-convergence. Let $K \subset Y$ be a nonempty closed convex cone. By $\mathcal{C}_K$ we denote the family of all members A of $\mathcal{C} \setminus \{\emptyset\}$ satisfying $0^+A = K$.

Theorem 3.3 ([2]) *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}_K$ be a sequence such that $\sup_{n\in\mathbb{N}} A_n \in \mathcal{C}_K$. Then, $\limsup_{n\to\infty} A_n = \operatorname{cl}\operatorname{conv} \operatorname{LIM\,SUP}_{n\to\infty} A_n$.*

Theorem 3.4 ([2]) *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ be a sequence such that for all $\bar{N} \in \mathcal{N}_\infty^\#$ there exists some $\widetilde{N} \in \mathcal{N}_\infty^\#$ with $\widetilde{N} \subset \bar{N}$ and some nonempty closed convex cone $K \subset Y$ such that $A_n \in \mathcal{C}_K$ for all $n \in \widetilde{N}$ and $\sup_{n\in\widetilde{N}} A_n \in \mathcal{C}_K$. Then it holds $\liminf_{n\to\infty} A_n = \operatorname{LIM\,INF}_{n\to\infty} A_n$.*

The following lemmas provide the main tools in our investigations.

Lemma 3.5 ([2]) *Let $A, B \subset Y$ be nonempty closed and convex. Then,*

$$A \subset B \quad \Leftrightarrow \quad \forall y^* \in \operatorname{ri}(0^+B)^\circ,\ \sigma_A(y^*) \leq \sigma_B(y^*).$$

Lemma 3.6 ([2]) *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ such that $A := \limsup_{n\to\infty} A_n \neq \emptyset$. Then,*

$$\forall y^* \in \operatorname{ri}(0^+A)^\circ, \quad \limsup_{n\to\infty} \sigma_{A_n}(y^*) = \sigma_A(y^*).$$

4 Semicontinuity

Based on the considerations in Section 3 we introduce upper and lower limits for functions with values in $\mathcal{C}$. The *upper* and *lower limits* of a function $f : X \to \mathcal{C}$ at $\bar{x} \in X$ are defined, respectively by

$$\limsup_{x\to\bar{x}} f(x) := \sup_{x_n\to\bar{x}} \limsup_{n\to\infty} f(x_n) \quad \text{and} \quad \liminf_{x\to\bar{x}} f(x) := \inf_{x_n\to\bar{x}} \liminf_{n\to\infty} f(x_n).$$

The limit of f at $\bar{x}$ exists if the upper and lower limits coincide. Then we write

$$\lim_{x\to\bar{x}} f(x) = \limsup_{x\to\bar{x}} f(x) = \liminf_{x\to\bar{x}} f(x).$$

As in the case of $\mathcal{F}$-valued functions, the upper and lower limits can be expressed by the supremum and infimum with respect to the corresponding complete lattice, i.e. we have

$$\limsup_{x\to\bar{x}} f(x) = \sup_{x_n\to\bar{x}} \inf_{N\in\mathcal{N}_\infty} \sup_{n\in N} f(x_n), \qquad \liminf_{x\to\bar{x}} f(x) = \inf_{x_n\to\bar{x}} \inf_{N\in\mathcal{N}_\infty^\#} \sup_{n\in N} f(x_n).$$

In case of outer limits for $\mathcal{F}$-valued functions we see that the set $\bigcup_{x_n\to\bar{x}} \mathrm{LIM\,SUP}_{n\to\infty} f(x_n)$ is always closed, i.e. the closure operation, which is implicitly contained in the supremum, is superfluous. An analogous result is valid for $\mathcal{C}$-valued functions.

Proposition 4.1 *Let $f : X \to \mathcal{C}$ and $\bar{x} \in X$. Then it holds*

$$\limsup_{x\to\bar{x}} f(x) = \bigcup_{x_n\to\bar{x}} \limsup_{n\to\infty} f(x_n).$$

Proof. We have to show that $A := \bigcup_{x_n\to\bar{x}} \limsup_{n\to\infty} f(x_n)$ is convex and closed.

(i) *Convexity.* Let $y_1, y_2 \in A$ and let $\lambda \in [0,1]$ be given. Hence there exist sequences $(x_n^{(i)})_{n\in\mathbb{N}} \subset X$, $(i = 1,2)$ with $x_n^{(i)} \to \bar{x}$ such that $y_i \in \limsup_{n\to\infty} f(x_n^{(i)})$. We define a sequence $(x_n^{(3)})_{n\in\mathbb{N}} \subset X$ by $(x_n^{(3)})_{n\in\mathbb{N}} := (x_1^{(1)}, x_1^{(2)}, x_2^{(1)}, x_2^{(2)}, x_3^{(1)}, x_3^{(2)}, ...)$. Since $(x_n^{(i)})_{n\in\mathbb{N}}$, $(i = 1,2)$ are subsequences of $(x_n^{(3)})_{n\in\mathbb{N}}$, we deduce that $\limsup_{n\to\infty} f(x_n^{(i)}) \subset \limsup_{n\to\infty} f(x_n^{(3)})$, $(i = 1,2)$. Hence we obtain $\lambda y_1 + (1-\lambda) y_2 \in \limsup_{n\to\infty} f(x_n^{(3)})$. From $x_n^{(3)} \to \bar{x}$ it follows that $\lambda y_1 + (1-\lambda) y_2 \in A$.

(ii) *Closedness.* Let $(y_m)_{m\in\mathbb{N}} \subset A$ with $y_n \to \bar{y} \in Y$. For all $m \in \mathbb{N}$ there exists a sequence $(x_n^{(m)})_{n\in\mathbb{N}} \subset X$ such that $\bar{x} = \lim_{n\to\infty} x_n^{(m)}$ and $y_m \in \limsup_{n\to\infty} f(x_n^{(m)})$. Thus we can construct a strictly increasing function $n_0 : \mathbb{N} \to \mathbb{N}$ by

$$\forall m \in \mathbb{N},\ \exists n_0(m) \in \mathbb{N},\ \forall n \ge n_0(m),\ \forall k \in \{1, ..., m\}\,,\ \left\| x_n^{(k)} - \bar{x} \right\| < \frac{1}{m}.$$

Consider the (not necessarily strictly) increasing function $m_0 : \mathbb{N} \to \mathbb{N} \cup \{-\infty\}$ being defined by $m_0(n) := \sup \{m \in \mathbb{N} |\ n \ge n_0(m)\}$. Of course, we have $m_0(n) \to \infty$ for $n \to \infty$. Define a sequence $(\bar{x}_n)_{n\in\mathbb{N}} \subset X$ by

$$(\bar{x}_n)_{n\in\mathbb{N}} := (x_1^{(1)}, x_1^{(2)}, ..., x_1^{(m_0(1))}, x_2^{(1)}, x_2^{(2)}, ..., x_2^{(m_0(2))}, \ldots, x_n^{(1)}, x_n^{(2)}, ..., x_n^{(m_0(n))}, \ldots).$$

where, without loss of generality, it can be assumed that $m_0(n) \neq -\infty$ for all $n \in \mathbb{N}$. Clearly, the sequence $(\bar{x}_n)_{n\in\mathbb{N}}$ converges to $\bar{x}$ and we have $(x_n^{(m)})_{n\ge n_0(m)} \subset (\bar{x}_n)_{n\in\mathbb{N}}$ for all $m \in \mathbb{N}$. It follows that $\limsup_{n\to\infty} f(x_n^{(m)}) \subset \limsup_{n\to\infty} f(\bar{x}_n)$ for all $m \in \mathbb{N}$, whence $(y_m)_{m\in\mathbb{N}} \subset \limsup_{n\to\infty} f(\bar{x}_n)$. Since $\limsup_{n\to\infty} f(\bar{x}_n)$ is closed, we get $\bar{y} \in \limsup_{n\to\infty} f(\bar{x}_n) \subset A$. $\square$

The following relationship between the outer and inner limits in $\mathcal{F}$ and the upper and lower limits in $\mathcal{C}$ is an easy consequence of the definition.

$$\liminf_{x\to\bar{x}} f(x) \supset \operatorname*{LIM\,INF}_{x\to\bar{x}} f(x)\,, \qquad \limsup_{x\to\bar{x}} f(x) \supset \operatorname*{LIM\,SUP}_{x\to\bar{x}} f(x).$$

A function $f : X \to \mathcal{C}$ is said to be *lower semicontinuous (lsc)*, *upper semicontinuous (usc)*, *continuous* at $\bar{x} \in X$ if $f(\bar{x}) \subset \liminf_{x\to\bar{x}} f(x)$, $f(\bar{x}) \supset \limsup_{x\to\bar{x}} f(x)$, $f(\bar{x}) = \lim_{x\to\bar{x}} f(x)$, respectively. If f is lsc, usc, continuous at every $\bar{x} \in X$ we just say f is lsc, usc, continuous, respectively. It is easy to see that upper (inner) semicontinuity implies outer (lower) semicontinuity.

With the aid of Lemma 3.5 and 3.6 we obtain our main result.

Theorem 4.2 *Let $f : X \to \mathcal{C}$ and $\bar{x} \in \operatorname{dom} f$. Then the following statements are equivalent:*

(i) *f is usc at $\bar{x}$,*

(ii) *For all $y^* \in \operatorname{ri}\left(0^+ f(\bar{x})\right)^\circ$ the function $\sigma_{f(\,\cdot\,)}(y^*) : X \to \overline{\mathbb{R}}$ is usc at $\bar{x}$.*

Proof. Let be given an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to \bar{x}$.

(i) $\Rightarrow$ (ii). Let the sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ be defined by $\tilde{x}_{2n} := x_n$ and $\tilde{x}_{2n+1} := \bar{x}$. From (i) we deduce that $f(\bar{x}) = \limsup_{n\to\infty} f(\tilde{x}_n)$. Lemma 3.6 implies that

$$\forall y^* \in \operatorname{ri}\left(0^+ f(\bar{x})\right)^\circ,\quad \sigma_{f(\bar{x})}(y^*) = \limsup_{n\to\infty} \sigma_{f(\tilde{x}_n)}(y^*) \le \limsup_{n\to\infty} \sigma_{f(x_n)}(y^*).$$

(ii) $\Rightarrow$ (i). Without loss of generality we can assume that $A := \limsup_{n\to\infty} f(x_n) \neq \emptyset$. By Lemma 3.6 we obtain

$$\forall y^* \in \operatorname{ri}\left(0^+ f(\bar{x})\right)^\circ,\quad \sigma_{f(\bar{x})}(y^*) \ge \limsup_{n\to\infty} \sigma_{f(x_n)}(y^*) = \sigma_A(y^*).$$

From Lemma 3.5 we deduce that $f(\bar{x}) \supset A$. □

The next assertion about nested upper limits is essential for an expedient definition of the *upper semicontinuous hull* of a $\mathcal{C}$-valued function. An analogous assertion for the lower limit is not true, see Example 4.6 below.

Proposition 4.3 *Let $f : X \to \mathcal{C}$ and $\bar{x} \in X$. Then it holds*

$$\limsup_{x\to\bar{x}} f(x) = \limsup_{x\to\bar{x}}\ \limsup_{w\to x} f(w).$$

Proof. Clearly, we have $f(x) \subset \limsup_{w\to x} f(w)$ for all $x \in X$, which implies the inclusion "$\subset$". It remains to show that $A := \limsup_{n\to\infty} \limsup_{w\to x_n} f(w) \subset \limsup_{n\to\infty} f(x_n) =: B$ for an arbitrarily given sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to \bar{x}$. For all $y^* \in \operatorname{ri}\left(0^+ B\right)^\circ$ it holds

$$\sigma_A(y^*) \overset{\text{Pr. 3.1}}{\le} \limsup_{n\to\infty}\ \limsup_{w\to x_n} \sigma_{f(w)}(y^*) = \limsup_{n\to\infty} \sigma_{f(x_n)}(y^*) \overset{\text{Lem. 3.6}}{=} \sigma_B(y^*).$$

Lemma 3.5 yields that $A \subset B$. □

The *upper semicontinuous hull* of a function $f : X \to \mathcal{C}$ is defined by

$$(\operatorname{usc} f) : X \to \mathcal{C},\quad (\operatorname{usc} f)(x) := \limsup_{x'\to x} f(x').$$

The *hypograph* of a function $f : X \to \mathcal{C}$ is the set $\operatorname{hyp} f := \{(x, A) \in X \times \mathcal{C} |\ A \subset f(x)\}$. A subset $\mathcal{A} \subset X \times \mathcal{C}$ is said to be closed if for every sequence $(x_n, A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ with $x_n \to \bar{x} \in X$ and $A_n \xrightarrow{\mathcal{C}} \bar{A} \in \mathcal{C}$ it is true that $(\bar{x}, \bar{A}) \in \mathcal{A}$. The *closure* of a set $\mathcal{A} \subset X \times \mathcal{C}$, denoted by $\operatorname{cl} \mathcal{A}$, is the set of all limits $(\bar{x}, \bar{A}) \in X \times \mathcal{C}$ of sequences $(x_n, A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$.

Let us collect some properties of the upper semicontinuous hull of a $\mathcal{C}$-valued function.

Proposition 4.4 *For $f : X \to \mathcal{C}$ the following statements hold true:*

(i) $\mathrm{gr}\,(\mathrm{usc}\, f) \supset \mathrm{cl}\,(\mathrm{gr}\, f)$,

(ii) $\mathrm{hyp}\,(\mathrm{usc}\, f) \supset \mathrm{cl}\,(\mathrm{hyp}\, f)$,

(iii) $(\mathrm{usc}\, f)$ *is usc,*

(iv) $\forall x \in X,\ (\mathrm{usc}\, f)(x) \supset f(x)$,

(v) f *is usc at* $\bar{x} \in X \quad \Leftrightarrow \quad (\mathrm{usc}\, f)(\bar{x}) = f(\bar{x})$,

(vi) $\mathrm{gr}\,(\mathrm{usc}\, f)$ *is closed,*

(vii) $\mathrm{hyp}\,(\mathrm{usc}\, f)$ *is closed.*

Proof. (i) Let $(\bar{x}, \bar{y}) \in \mathrm{cl}\,(\mathrm{gr}\, f)$. Then there exists a sequence $(x_n, y_n)_{n\in\mathbb{N}} \subset \mathrm{gr}\, f$ converging to $(\bar{x}, \bar{y})$. For all $n \in \mathbb{N}$, we have $\{y_n\} \subset f(x_n)$. Hence

$$\{\bar{y}\} = \lim_{n\to\infty} \{y_n\} = \limsup_{n\to\infty} \{y_n\} \subset \limsup_{n\to\infty} f(x_n) \subset \limsup_{x\to\bar{x}} f(x) = (\mathrm{usc}\, f)(\bar{x}),$$

i.e. $(\bar{x}, \bar{y}) \in \mathrm{gr}\,(\mathrm{usc}\, f)$. The proof of (ii) is similar. Statement (iii) follows from Proposition 4.3. The proofs of (iv) and (v) are analogous to those of Proposition 2.1 (iv) and (v). (vi) Let $(x_n, y_n)_{n\in\mathbb{N}} \subset \mathrm{gr}\,(\mathrm{usc}\, f)$ with $(x_n, y_n) \to (\bar{x}, \bar{y}) \in X \times Y$ be given. Proceeding as in (i), but replacing f by $(\mathrm{usc}\, f)$, we obtain $\{\bar{y}\} \subset (\mathrm{usc}\,(\mathrm{usc}\, f))(\bar{x})$. From (iii) we conclude that $(\mathrm{usc}\,(\mathrm{usc}\, f))(\bar{x}) = (\mathrm{usc}\, f)(\bar{x})$. Hence $(\bar{x}, \bar{y}) \in \mathrm{gr}\,(\mathrm{usc}\, f)$. The proof of (vii) is similar to that of (iv). □

The next example shows that neither the closedness of $\mathrm{hyp}\, f$ nor the closedness of $\mathrm{gr}\, f$ implies that f is usc.

Example 4.5 Let $f : \mathbb{R} \to \mathcal{C}(\mathbb{R})$ be defined by $f(x) := \{1/x\}$ if $x \neq 0$ and $f(0) := \emptyset$. Then, it easily follows that $\mathrm{gr}\, f \subset \mathbb{R} \times \mathbb{R}$ as well as $\mathrm{hyp}\, f \subset \mathbb{R} \times \mathcal{C}(\mathbb{R})$ are closed, but f is not usc.

In Remark 2.3 (due to [4]) we noticed that an inner semicontinuous hull of a $\mathcal{F}$-valued function that is defined analogously to the outer semicontinuous hull is not necessarily inner semicontinuous. There are analogous problems with the lower semicontinuous hull of a $\mathcal{C}$-valued function. This is due to the fact that there is no analogous assertion to Proposition 4.3 for lower limits, as the following example shows.

Example 4.6 For functions $f : X \to \mathcal{C}$, in general, we have

$$\liminf_{x\to\bar{x}} f(x) \neq \liminf_{x\to\bar{x}}\ \liminf_{w\to x} f(w).$$

Indeed, consider the function $f : \mathbb{R}^2 \to \mathcal{C}(\mathbb{R})$, defined by

$$f(x) := \begin{cases} \{\|x\|\} & \text{if} \quad x_1 \geq 0 \\ \{-\|x\|\} & \text{if} \quad x_1 < 0. \end{cases}$$

Then it holds

$$\liminf_{w\to x} f(w) := \begin{cases} \{\|x\|\} & \text{if} \quad x_1 > 0 \text{ or } x_2 = 0 \\ \{-\|x\|\} & \text{if} \quad x_1 < 0 \\ \emptyset & \text{if} \quad x_1 = 0 \text{ and } x_2 \neq 0. \end{cases}$$

Hence we obtain $\{0\} = \liminf_{x\to 0} f(x) \neq \liminf_{x\to 0} \liminf_{w\to x} f(w) = \emptyset$.

5 Locally bounded functions

The concept of local boundedness of a set-valued map plays an important role in Variational Analysis, see [4]. As an easy consequence of the definition ([4, Definition 5.14]), local boundedness of a map $f : X \rightrightarrows Y$ at $\bar{x}$ implies that $f(\bar{x})$ is a bounded subset of Y. This means, local boundedness is (at least locally) adapted to set-valued maps with bounded values. Therefore we introduce a slightly generalized concept, adapted to the framework of $\mathcal{C}$-valued functions. It turns out that this concept provides a sufficient condition for the coincidence of upper (lower) semicontinuity with outer (inner) semicontinuity.

A function $f : X \to \mathcal{C}$ is said to be *locally bounded* at $\bar{x} \in \operatorname{dom} f$ if there exists a neighborhood $V \in \mathcal{N}(\bar{x})$ such that the following conditions are satisfied:

(i) $0^+ \sup_{x \in V} f(x) \subset 0^+ f(\bar{x})$,

(ii) $\forall x \in V \cap \operatorname{dom} f,\ 0^+ f(x) \supset 0^+ f(\bar{x})$.

Note that, if $f : X \to \mathcal{C}$ is locally bounded at $\bar{x} \in \operatorname{dom} f$, (i) and (ii) of the previous definition are always satisfied with equality. Moreover, if $f(\bar{x}) \subset Y$ is bounded, our concept coincides with the classical one.

Theorem 5.1 *Let $f : X \to \mathcal{C}$ be locally bounded at $\bar{x} \in \operatorname{dom} f$. Then,*

$$\limsup_{x \to \bar{x}} f(x) = \operatorname{cl} \operatorname{conv} \operatorname*{LIM\,SUP}_{x \to \bar{x}} f(x).$$

Proof. Clearly, we have $\limsup_{x \to \bar{x}} f(x) \supset \operatorname{cl} \operatorname{conv} \operatorname{LIM\,SUP}_{x \to \bar{x}} f(x)$. To show the opposite inclusion let $y \in \limsup_{x \to \bar{x}} f(x)$ be given. Then there exists a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \to \bar{x}$ such that $y \in \limsup_{n \to \infty} f(x_n)$. Assuming that there exists some $n_0 \in \mathbb{N}$ such that $f(x_n) = \emptyset$ for all $n \geq n_0$, we obtain $\limsup_{n \to \infty} f(x_n) = \emptyset$, which contradicts $y \in \limsup_{x \to \bar{x}} f(x)$. Hence, by $(x_{n_k})_{k \in \mathbb{N}} := (x_n)_{n \in \mathbb{N}} \cap \operatorname{dom} f$, we obtain a subsequence of $(x_n)_{n \in \mathbb{N}}$. Of course, we have $\limsup_{n \to \infty} f(x_n) = \limsup_{k \to \infty} f(x_{n_k})$. By the local boundedness, we find $k_0 \in \mathbb{N}$ such that, $f(x_{n_k}) \in \mathcal{C}_K$ for all $k \geq k_0$ and $\sup_{k \geq k_0} f(x_{n_k}) \in \mathcal{C}_K$, where $K := 0^+ f(\bar{x})$. Theorem 3.3 yields $y \in \operatorname{cl} \operatorname{conv} \operatorname{LIM\,SUP}_{k \to \infty} f(x_{n_k}) \subset \operatorname{LIM\,SUP}_{x \to \bar{x}} f(x)$. □

In the next example we show that the assertion of the preceeding theorem can fail if one of the conditions in the definition of the local boundedness concept is not satisfied.

Example 5.2 Let $f : \mathbb{R} \to \mathcal{C}(\mathbb{R})$ be defined by $f(x) := \left\{ \frac{1}{x} \right\}$ if $x \neq 0$ and $f(0) := \{0\}$, i.e. (ii) is satisfied, but (i) is not. Then, $\mathbb{R} = \limsup_{x \to 0} f(x) \neq \operatorname{cl} \operatorname{conv} \operatorname{LIM\,SUP}_{x \to 0} f(x) = \{0\}$.

Example 5.3 Let $f : \mathbb{R} \to \mathcal{C}(\mathbb{R}^2)$ be defined by $f(x) = \left\{ y \in \mathbb{R}^2 |\ y_2 = 1, y_1 = 1/x \right\}$ if $x \neq 0$ and $f(0) := \left\{ y \in \mathbb{R}^2 |\ y_2 = 0 \right\}$, i.e. (i) is satisfied, but (ii) is not. An easy calculation shows that $\left\{ y \in \mathbb{R}^2 |\ 0 \leq y_2 \leq 1 \right\} = \limsup_{x \to 0} f(x) \neq \operatorname{cl} \operatorname{conv} \operatorname{LIM\,SUP}_{x \to 0} f(x) = \left\{ y \in \mathbb{R}^2 |\ y_2 = 0 \right\}$.

Local boundedness of a function $f : X \to \mathcal{C}$ at a point $\bar{x} \in \operatorname{dom} f$ also implies that $\liminf_{x \to \bar{x}} f(x) = \operatorname{LIM\,INF}_{x \to \bar{x}} f(x)$ (see Corollary 5.6 below). Moreover, as shown in the next theorem, a weaker assumption is already sufficient.

Theorem 5.4 *Let $f : X \to \mathcal{C}$ and $\bar{x} \in \operatorname{dom} f$ such that for all sequences $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to \bar{x}$ there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and a nonempty closed convex cone $K \subset Y$ with $f(x_{n_k}) \in \mathcal{C}_K$ for all $k \in \mathbb{N}$ and $\sup_{k\in\mathbb{N}} f(x_{n_k}) \in \mathcal{C}_K$. Then it holds $\liminf_{x\to\bar{x}} f(x) = \operatorname{LIM\,INF}_{x\to\bar{x}} f(x)$.*

Proof. Of course, $\liminf_{x\to\bar{x}} f(x) \supset \operatorname{LIM\,INF}_{x\to\bar{x}} f(x)$. In order to show the opposite inclusion let $y \in Y \setminus \operatorname{LIM\,INF}_{x\to\bar{x}} f(x)$ be given (the case $\operatorname{LIM\,INF}_{x\to\bar{x}} f(x) = Y$ is obvious). Hence there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to \bar{x}$ such that $y \notin \operatorname{LIM\,INF}_{n\to\infty} f(x_n)$. Every subsequence of $(x_n)_{n\in\mathbb{N}}$ is again a sequence converging to $\bar{x}$, hence our assumption ensures that Theorem 3.4 is applicable. It follows that $y \notin \liminf_{n\to\infty} f(x_n)$. □

The next example shows that the assertion of the previous theorem can fail if the assumption is not satisfied.

Example 5.5 Let $f : \mathbb{R} \to \mathcal{C}(\mathbb{R}^2)$ be defined by $f(x) := \operatorname{conv}\left\{\left(-1, -\frac{1}{x}\right), \left(1, \frac{1}{x}\right)\right\}$ if $x > 0$ and $f(x) := \mathbb{R}^2$ if $x \le 0$, i.e. the condition in the previous theorem is not satisfied. Then we have $\left\{y \in \mathbb{R}^2 \mid -1 \le y_2 \le 1\right\} = \liminf_{x\to 0} f(x) \neq \operatorname{LIM\,INF}_{x\to 0} f(x) = \left\{y \in \mathbb{R}^2 \mid y_2 = 0\right\}$.

Corollary 5.6 *Let $f : X \to \mathcal{C}$ be locally bounded at $\bar{x} \in \operatorname{dom} f$. Then,*

$$\liminf_{x\to\bar{x}} f(x) = \operatorname*{LIM\,INF}_{x\to\bar{x}} f(x).$$

Proof. By the local boundedness of f at $\bar{x}$, for every sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to \bar{x}$ there exists some $n_0 \in \mathbb{N}$ such that $f(x_n) \in \mathcal{C}_K$ for all $n \ge n_0$ and $\sup_{n\ge n_0} f(x_n) \in \mathcal{C}_K$, where $K := 0^+ f(\bar{x})$. Hence, Theorem 5.4 yields the desired assertion. □

Corollary 5.7 *Let $f : X \to \mathcal{C}$ be locally bounded on $\operatorname{dom} f$. Then the following statements are equivalent:*

(i) $\operatorname{hyp} f \subset X \times \mathcal{C}$ *is closed,*

(ii) f *is usc,*

(iii) $\operatorname{gr} f \subset X \times Y$ *is closed.*

Proof. (i) ⇒ (iii). Elementary (see also [2]).

(iii) ⇒ (ii). [4, Theorem 5.7 (a)] yields that f is osc. By Theorem 5.1, f is usc.

(ii) ⇒ (i). Follows from Proposition 4.4 (v), (vii). □

6 Concave functions

This section is devoted to the special case of concave $\mathcal{C}$-valued functions. We show that semicontinuity in the sense of Section 4 coincides with the classical concepts of outer and inner semicontinuity in this case.

A function $f : X \to \mathcal{C}$ is said to be *concave* if

$$\forall \lambda \in [0,1],\ \forall x_1, x_2 \in X,\ f(\lambda \cdot x_1 + (1-\lambda)\cdot x_2) \supset \lambda\, f(x_1) + (1-\lambda)\, f(x_2).$$

It is easy to see that a function $f : X \to \mathcal{C}$ is concave if and only if $\operatorname{hyp} f \subset X \times \mathcal{C}$ is convex. Of course, concavity (which is often called convexity) of a set-valued map is equivalent to the convexity of its graph. The following proposition shows that the values of a concave $\mathcal{C}$-valued function essentially have the same recession cone.

Proposition 6.1 *Let $f : X \to \mathcal{C}$ be concave. If $\bar{x} \in \mathrm{ri}\,(\mathrm{dom}\, f)$, then $0^+ f(x) \subset 0^+ f(\bar{x})$ for all $x \in \mathrm{dom}\, f$ and $0^+ f(x) = 0^+ f(\bar{x})$ for all $x \in \mathrm{ri}\,(\mathrm{dom}\, f)$.*

Proof. Note that $\mathrm{dom}\, f$ is convex. Let $\bar{x} \in \mathrm{ri}\,(\mathrm{dom}\, f)$ and $x \in \mathrm{dom}\, f$. By [3, Theorem 6.4], there exists $\mu > 1$ such that $\hat{x} := \mu\bar{x} + (1-\mu)x \in \mathrm{dom}\, f$. Set $\lambda := 1/\mu \in (0,1)$. The concavity of f yields $f(\bar{x}) \supset \lambda f(\hat{x}) \oplus (1-\lambda) f(x)$. Since $\hat{x} \in \mathrm{dom}\, f$ we can choose some $\hat{y} \in f(\hat{x})$, hence $f(\bar{x}) \supset \lambda\{\hat{y}\} + (1-\lambda) f(x) := C_x$. It follows that $0^+ C_x \subset 0^+ f(\bar{x})$. With the aid of [3, Theorem 8.1] we conclude that $0^+ C_x = 0^+ f(x)$, hence $0^+ f(x) \subset 0^+ f(\bar{x})$. Assume there is some $\tilde{x} \in \mathrm{ri}\,(\mathrm{dom}\, f)$ with $0^+ f(\tilde{x}) \subsetneq 0^+ f(\bar{x})$, then the first part yields $0^+ f(x) \subset 0^+ f(\tilde{x})$ for all $x \in \mathrm{dom}\, f$, whence the contradiction $0^+ f(\bar{x}) \subsetneq 0^+ f(\bar{x})$. $\square$

Theorem 6.2 *Let $f : X \to \mathcal{C}$ be concave. Then, for all $\bar{x} \in X$ it holds*

$$\limsup_{x \to \bar{x}} f(x) = \mathop{\mathrm{LIM\,SUP}}_{x \to \bar{x}} f(x).$$

Proof. Of course, we always have $\limsup_{x\to\bar{x}} f(x) \supset \mathrm{LIM\,SUP}_{x\to\bar{x}} f(x)$. To show the opposite inclusion let $y \in \limsup_{x\to\bar{x}} f(x)$ be given. Hence there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to \bar{x}$ such that $y \in \limsup_{n\to\infty} f(x_n)$. By Proposition 3.2 this can be written as

$$\exists\, (\lambda_n)_{n\in\mathbb{N}} \subset \Delta_{p+1},\ \exists\, (k_n)_{n\in\mathbb{N}} \subset \mathbb{N}^{p+1},\ \exists\, (z_n)_{n\in\mathbb{N}} \subset Y^{p+1},$$
$$y = \lim_{n\in\mathbb{N}} \sum_{i\in\overline{0,p}} \lambda_n^i z_n^i,\ \forall n \in \mathbb{N},\ \forall j \in \overline{0,p},\ k_n^j \geq n,\ z_n^j \in A_{k_n^j}.$$

We define two sequences $(y_m)_{m\in\mathbb{N}} \subset Y$ and $(\tilde{x}_m)_{m\in\mathbb{N}} \subset X$ by

$$y_m := \sum_{j\in\overline{0,p}} \lambda_j^{(m)} z_j^{(m)}, \qquad \tilde{x}_m := \sum_{j\in\overline{0,p}} \lambda_j^{(m)} x_{k_j^{(m)}}.$$

Then we have $y_m \to y$, $\tilde{x}_m \to \bar{x}$ and the concavity of f yields that

$$y_m = \sum_{j\in\overline{0,p}} \lambda_j^{(m)} z_j^{(m)} \in \sum_{j\in\overline{0,p}} \lambda_j^{(m)} f\left(x_{k_j^{(m)}}\right) \subset f\left(\sum_{j\in\overline{0,p}} \lambda_j^{(m)} x_{k_j^{(m)}}\right) = f\left(\tilde{x}_m\right)$$

for all $m \in \mathbb{N}$. By [4, 5(1)], this means $y \in \mathrm{LIM\,SUP}_{x\to\bar{x}} f(x)$. $\square$

Corollary 6.3 *Let $f : X \to \mathcal{C}$ be concave. Then the following statements hold true:*

(i) $(\mathrm{usc}\, f) = (\mathrm{osc}\, f)$,

(ii) $(\mathrm{usc}\, f)$ *is concave,*

(iii) $(\mathrm{usc}\, f) : X \to \mathcal{C}_K \cup \{\emptyset\}$ *for some nonempty closed convex cone $K \subset Y$.*

Proof. (i) Follows from Theorem 6.2.

(ii) f concave $\Leftrightarrow$ $\mathrm{gr}\, f$ convex $\Rightarrow$ $\mathrm{cl}\,(\mathrm{gr}\, f) = \mathrm{gr}\,(\mathrm{osc}\, f)$ convex $\Leftrightarrow$ $\mathrm{osc}\, f = \mathrm{usc}\, f$ convex.

(iii) Since $(\mathrm{usc}\, f)$ is osc and concave, its graph is closed and convex. If $\mathrm{dom}\,(\mathrm{usc}\, f) = \emptyset$ there is nothing to prove, otherwise, there exists some $\bar{x} \in \mathrm{ri}\,\mathrm{dom}\,(\mathrm{usc}\, f)$. From Proposition 6.1 we deduce that $0^+(\mathrm{usc}\, f)(x) \subset 0^+(\mathrm{usc}\, f)(\bar{x}) =: K$ for all $x \in \mathrm{dom}\,(\mathrm{usc}\, f)$. It remains to prove the opposite inclusion for all $x \in \mathrm{dom}\,(\mathrm{usc}\, f)$. Indeed, let $\hat{y} \in 0^+(\mathrm{usc}\, f)(\bar{x})$ and

$\bar{y} \in (\operatorname{usc} f)(\bar{x})$ be arbitrarily chosen. By [3, Theorem 8.3] we have $\bar{y} + \lambda\hat{y} \in (\operatorname{usc} f)(\bar{x})$ for all $\lambda \geq 0$ and equivalently $(0, \hat{y}) \in 0^+\operatorname{gr}(\operatorname{usc} f)$. Given some $x \in \operatorname{dom}(\operatorname{usc} f)$ we can choose $y \in (\operatorname{usc} f)(x)$. Since $(0, \hat{y}) \in 0^+\operatorname{gr}(\operatorname{usc} f)$, [3, Theorem 8.3] yields that $y + \lambda\hat{y} \in (\operatorname{usc} f)(x)$ for all $\lambda \geq 0$ and equivalently $\hat{y} \in 0^+(\operatorname{usc} f)(x)$. □

Corollary 6.4 *Let $f : X \to \mathcal{C}_K \cup \{\emptyset\}$. Then the following statements are equivalent:*

(i) *f is concave and usc,*

(ii) *$\operatorname{gr} f \subset X \times Y$ is convex and closed,*

(iii) *$\operatorname{hyp} f \subset X \times \mathcal{C}$ is convex and closed,*

(iv) *For all $y^* \in \operatorname{ri} K^\circ$ the function $\sigma_{f(\,\cdot\,)}(y^*) : X \to \overline{\mathbb{R}}$ is concave and usc.*

Proof. The equivalence of the convexity/concavity assertions is immediate.

(i) ⇔ (ii) ⇔ (iii). The equivalence of the upper semicontinuity and closedness assertions follows similarly to the proof of Corollary 5.7 (using Corollary 6.3 (i) instead of Theorem 5.1).

(i) ⇔ (iv). From Theorem 4.2 taking into account Corollary 6.3 (iii). □

Theorem 6.5 *Let $f : X \to \mathcal{C}$ be concave. Then the following assertions hold true:*

(i) *f is usc at every $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$,*

(ii) *f is continuous at every $\bar{x} \in \operatorname{int}(\operatorname{dom} f)$.*

Proof. (i) Let $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$ be given and let $K := 0^+f(\bar{x})$. By Theorem 4.2, it remains to show that, for all $y^* \in \operatorname{ri} K^\circ$, $\sigma_{f(\,\cdot\,)}(y^*)$ is usc at $\bar{x}$. From Proposition 6.1 we deduce that $0^+f(x) = K$ for all $x \in \operatorname{ri}(\operatorname{dom} f)$. Hence, for all $y^* \in \operatorname{ri} K^\circ$ it is true that $\bar{x} \in \operatorname{ri}\left(\operatorname{dom} \sigma_{f(\,\cdot\,)}(y^*)\right)$, whence, by [3, Theorem 7.4], $\sigma_{f(\,\cdot\,)}(y^*)$ is usc at $\bar{x}$.

(ii) By [4, Theorem 5.9 (b)], f is isc at $\bar{x} \in \operatorname{int}(\operatorname{dom} f)$. Hence f is lsc at $\bar{x}$. Now the assertion follows from (i). □

We close this paper with some assertions with respect to local boundedness of concave functions.

Theorem 6.6 *Let $f : X \to \mathcal{C}$ be concave and usc. Then, f is locally bounded on* $\operatorname{dom} f$.

Proof. Let $\bar{x} \in \operatorname{dom} f$, $V := \{x \in X |\; \|x - \bar{x}\| \leq 1\}$ and $K := 0^+f(\bar{x})$. By Proposition 4.4 (v) and Corollary 6.3 (iii) we have $0^+f(x) = K$ for all $x \in \operatorname{dom} f$. Hence, condition (ii) in the definition of the local boundedness is satisfied. It remains to show $0^+ \sup_{x\in V} f(x) \subset K$.

Since V and f are convex, the set $\bigcup_{x\in V} f(x)$ is convex. Since V is compact and $\operatorname{gr} f$ is closed, we deduce that $\bigcup_{x\in V} f(x)$ is closed. Hence $0^+ \sup_{x\in V} f(x) = 0^+ \bigcup_{x\in V} f(x)$. Let $k \in 0^+ \bigcup_{x\in V} f(x)$ be given. By [3, Theorem 8.2], k is the limit of a sequence $(\lambda_n y_n)_{n\in\mathbb{N}}$ where $\lambda_n \downarrow 0$ and $y_n \in \bigcup_{x\in V} f(x)$. Clearly, for all $n \in \mathbb{N}$ there exists $x_n \in V$ such that $y_n \in f(x_n)$. Since V is bounded, we have $(\lambda_n x_n, \lambda_n y_n) \to (0, k)$. Applying [3, Theorem 8.2] to the closed convex set $\operatorname{gr} f \subset X \times Y$, we obtain $(0, k) \in 0^+\operatorname{gr} f$. With the aid of [3, Theorem 8.3] we deduce that $\bar{y} + \lambda k \in f(\bar{x} + \lambda \cdot 0) = f(\bar{x})$ for all $\lambda \geq 0$ and arbitrary $\bar{y} \in f(\bar{x})$, which is equivalent to $k \in 0^+f(\bar{x}) = K$. □

Corollary 6.7 *If $f : X \to \mathcal{C}$ is concave, then f is locally bounded at every $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$.*

Proof. Theorem 6.6 yields that $\operatorname{usc} f$ is locally bounded at every $x \in \operatorname{dom}(\operatorname{usc} f)$. By Theorem 6.5 (i), we know that $f(\bar{x}) = (\operatorname{usc} f)(\bar{x})$ for all $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$. $\square$

Corollary 6.8 *Let $f : X \to \mathcal{C}$ be concave. Then, for all $\bar{x} \in X$ it holds*

$$\liminf_{x \to \bar{x}} f(x) = \operatorname*{LIM\,INF}_{x \to \bar{x}} f(x).$$

Proof. If $\bar{x} \in \operatorname{int}(\operatorname{dom} f)$, this follows from Corollary 6.7 and Corollary 5.6. Otherwise, we have $\liminf_{x \to \bar{x}} f(x) = \operatorname{LIM\,INF}_{x \to \bar{x}} f(x) = \emptyset$. $\square$

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