MAXIMAL L_p - REGULARITY FOR THE CAHN-HILLIARD EQUATION WITH NONCONSTANT TEMPERATURE AND DYNAMIC BOUNDARY CONDITIONS

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Dedicated to Philippe Clément

Abstract. In this paper we investigate the nonlinear Cahn-Hilliard equation with nonconstant temperature and dynamic boundary conditions. We show maximal L_p - regularity for this problem with inhomogeneous boundary data. Furthermore we show global existence and use the Lojasiewicz-Simon inequality to show that each solution converges to a steady state as time tends to infinity, as soon as the potential Φ and the latent heat λ satisfy certain growth conditions.

1. Introduction and the model

During the last years there has been much interest in the so called phase-field models arising from the theory of phase transitions. Basically there are four models, namely the Cahn-Allen equation, the Cahn-Hilliard equation, the Penrose-Fife system and the Caginalp system. There is a quite large literature concerning these phase-field models, especially for the Cahn-Hilliard equation

$$\partial_t\psi = \Delta\mu, \quad \mu = -\Delta\psi + \Phi'(\psi), \quad t > 0,\ x \in \Omega, \tag{1.1}$$

with constant temperature. Here ψ is the so called order-parameter for the investigated physical system, e.g. the mass density or the magnetic flux etc. The functions μ and Φ are the chemical and physical potentials, respectively.
Consider the free energy functional

$$F(\psi,\vartheta) = \int_\Omega \left(\frac{1}{2}|\nabla\psi|^2 + \Phi(\psi) - \lambda(\psi)\vartheta - \frac{1}{2}\vartheta^2\right)\,dx,$$

where we assume that the temperature ϑ varies in time and space and Φ, λ are given functions. One obtains the chemical potential μ and the internal energy e by differentiating F with respect to ψ and ϑ, respectively. We then have

$$\mu = \frac{\partial F}{\partial\psi}(\psi,\vartheta) = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta$$

and

$$e = -\frac{\partial F}{\partial\vartheta}(\psi,\vartheta) = \lambda(\psi) + \vartheta.$$

To obtain kinetic equations we assume that the order parameter ψ and the internal energy e are conserved quantities. The according conservation laws read as follows

$$\partial_t\psi + \operatorname{div}\mathcal{F} = 0, \quad \partial_t e + \operatorname{div} q = 0.$$

Here q is the heat flux of the internal energy e, which in this paper is assumed to be given by Fourier's law $q = -\nabla\vartheta$ and $\mathcal{F}$ denotes the phase flux of the order parameter ψ which is assumed to be of the form

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$\mathcal{F} = -\nabla\mu$. Thus the kinetic equations read as follows

$$\partial_t\psi - \Delta\mu = 0, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, \quad t \in J,\ x \in \Omega, \tag{1.2}$$

$$\partial_t\vartheta + \lambda'(\psi)\partial_t\psi - \Delta\vartheta = 0, \quad t \in J,\ x \in \Omega, \tag{1.3}$$

where $J = [0,T]$ or $J = \mathbb{R}_+$, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\Gamma = \partial\Omega$ of class C^4. To ensure that ψ is a conserved quantity we take Neumann boundary conditions for μ, i.e. $\partial_\nu\mu = 0$. Concerning ϑ we will use Robin boundary conditions, namely $\alpha\vartheta + \partial_\nu\vartheta = 0$, where $\alpha \geq 0$ is a constant. Since (1.2) is an equation of fourth order, we need another boundary condition for ψ. Usually one uses further classical boundary conditions, i.e. $\partial_\nu\psi = 0$ etc. (cf. also [1]). Recently, to account for boundary effects, the authors in [7] proposed a dynamic boundary condition of the form

$$\partial_t\psi - \sigma_s\Delta\psi + \gamma\partial_\nu\psi + \kappa\psi = 0, \tag{1.4}$$

where $\sigma_s, \gamma > 0$, $\kappa \geq 0$. Well-posedness and maximal L_p - regularity of the classical Cahn-Hilliard equation (1.1) with dynamic boundary conditions have been shown in [8]. Concerning the asymptotic behavior of solutions of (1.1) subject to dynamic boundary conditions we refer to the references [2] and [11]. There the authors made use of the Lojasiewicz-Simon inequality to show that every solution converges to a steady state as time tends to infinity. So far there seems to be no result on well-posedness and asymptotic behavior of (1.2), (1.3) with dynamic boundary conditions. This is the subject of this paper. For results concerning well-posedness and asymptotic behaviour of solutions of (1.2) and (1.3) with classical boundary conditions, we refer to [1], [5], [6], [13] and the references cited therein. Here the system we investigate is

$$\begin{aligned}
\partial_t\psi - \Delta\mu = f_1, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, &\quad t \in J,\ x \in \Omega,\\
\partial_t\vartheta + \lambda'(\psi)\partial_t\psi - \Delta\vartheta = f_2, &\quad t \in J,\ x \in \Omega,\\
\partial_\nu\mu = g_1, &\quad t \in J,\ x \in \Gamma,\\
\psi_t - \sigma_s\Delta_\Gamma\psi + \gamma\partial_\nu\psi + \kappa\psi = g_2, &\quad t \in J,\ x \in \Gamma,\\
\alpha\vartheta + \partial_\nu\vartheta = g_3, &\quad t \in J,\ x \in \Gamma,\\
\psi(0) = \psi_0, &\quad t = 0,\ x \in \Omega,\\
\vartheta(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega,
\end{aligned} \tag{1.5}$$

where f_i,g_j are given functions in appropriate function spaces to be defined later. Furthermore the initial functions ψ_0, ϑ_0 are given and $\sigma_s, \gamma > 0$, $\alpha, \kappa \geq 0$. Typically one has $\lambda(\psi) = \lambda_0(\psi - \psi_c)$ and $\Phi(\psi) = \Phi_0(\frac{1}{4}\psi^4 - \frac{1}{2}\psi^2)$, with $\lambda_0, \Phi_0 > 0$. In this case Φ is called a *double-well* potential and λ is called *latent heat*; the two distinct minima of Φ characterize the two stable phases during the phase transition.
We are interested in solutions

$$\psi \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^4_p(\Omega)) =: Z^1,$$
$$\vartheta \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega)) =: Z^2,$$

with

$$\psi|_\Gamma \in H^1_p(J; W^{2-1/p}_p(\Gamma)) \cap L_p(J; W^{4-1/p}_p(\Gamma)) =: Z^1_\Gamma.$$

In the following section we will solve the corresponding linear problem to (1.5) to obtain maximal L_p - regularity. Then, in section 3, the property of maximal L_p - regularity and the contraction mapping principle are applied to overcome the nonlinearities and to obtain local well-posedness of (1.5). Here we employ the technique introduced in the paper of Clément and Li [3]. Section 4 is dedicated to global well-posedness of (1.5). We will make use of some energy estimates and the Gagliardo-Nirenberg inequality to establish global existence again with the help of maximal L_p-regularity. For the sake of readability, the technical parts of the proofs of theorems 3.3 and 4.2 are given in the appendix. Finally, we investigate the asymptotic behavior of the solutions to (1.5) in section 5. Here we show relative compactness of the orbits $\psi(\mathbb{R}_+)$ and $\vartheta(\mathbb{R}_+)$ in a suitable energy space and we make use of some differentiability properties of the underlying energy-functionals to (1.5). Lastly the Lojasiewicz-Simon inequality is the key, which leads to convergence of solutions to steady states as time tends to infinity.

2. The linear problem

Before we deal with the linearized version of (1.5), we need some preliminaries. Firstly, we want to set $f_2, g_3, \vartheta_0 = 0$. For this we consider the system

$$\begin{aligned}
\partial_t v - \Delta v = f_2, &\quad t \in J,\ x \in \Omega,\\
\alpha v + \partial_\nu v = g_3, &\quad t \in J,\ x \in \Gamma,\\
v(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega.
\end{aligned} \tag{2.1}$$

By [4, Theorem 2.1] there is a unique solution $\vartheta_1 \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega))$ of (2.1), provided $f_2 \in L_p(J \times \Omega) =: X$, $\vartheta_0 \in W^{2-2/p}_p(\Omega) =: X_p$,

$$g_3 \in W^{1/2-1/2p}_p(J; L_p(\Gamma)) \cap L_p(J; W^{1-1/p}_p(\Gamma)) =: Y_3,$$

and the compatibility condition $\alpha\vartheta_0 + \partial_\nu \vartheta_0 = g_3|_{t=0}$ is satisfied, whenever $p > 3$. Taking the latter for granted, we may set $f_2, g_3, \vartheta_0 = 0$ in (1.5).
Secondly we want to replace ϑ in $(1.5)_1$ by a term only depending on ψ and some given data. Therefore we will solve the inhomogeneous heat equation

$$\partial_t \vartheta - \Delta\vartheta = -\partial_t \lambda(\psi), \quad \vartheta(0) = 0 \tag{2.2}$$

with homogeneous Robin- or Neumann-boundary conditions. Suppose we already know a solution $(\psi, \vartheta) \in (Z^1 \cap Z^1_\Gamma) \times Z^2$ of (1.5). Assuming that λ' is bounded we have $\partial_t\lambda(\psi) \in L_p(J \times \Omega)$. Let $A_K = -\Delta_K$, $K = R, N$, where R and N stand for Robin- and Neumann-boundary conditions, respectively. By $e^{-A_K t}$ we denote the bounded analytic semigroup, generated by $-A_K$ in $L_p(\Omega)$. The solution ϑ to (2.2) may then be represented by the variation of parameters formula

$$\vartheta(t) = -\int_0^t e^{-A_K(t-s)} \partial_t \lambda(\psi(s))\, ds.$$

Our aim is to split the derivative ∂_t into $\partial_t^{1/2}\partial_t^{1/2}$. But this is only possible if one applies ∂_t to a function with vanishing trace at $t = 0$. Since in general $\lambda(\psi_0) \neq 0$, we insert a function, say $w_0 \in Z^1$, with $w_0(0) = \lambda(\psi_0)$. The existence of such a function w_0 may be seen by solving the problem

$$\begin{aligned} \partial_t w + \Delta^2 w &= 0, \quad t \in J,\ x \in \Omega,\\ \partial_\nu \Delta w &= e^{-\Delta_\Gamma^2 t}\partial_\nu\Delta\lambda(\psi_0), \quad t \in J,\ x \in \Gamma,\\ \partial_\nu w &= e^{-\Delta_\Gamma^2 t}\partial_\nu\lambda(\psi_0), \quad t \in J,\ x \in \Gamma,\\ w(0) &= \lambda(\psi_0), \quad t = 0,\ x \in \Omega, \end{aligned}$$

under the assumption $\psi_0 \in W^{4-4/p}_p(\Omega)$ which is necessary for $\psi \in Z^1$. Then we may write

$$\begin{aligned} \vartheta(t) &= -\int_0^t e^{-A_K(t-s)}\partial_t w_0(s)\, ds - \int_0^t e^{-A_K(t-s)}\partial_t(\lambda(\psi(s)) - w_0(s))\, ds\\ &= \vartheta_2(t) - \partial_t^{1/2}(\partial_t + A_K)^{-1}\partial_t^{1/2}(\lambda(\psi(t)) - w_0(t)), \end{aligned} \tag{2.3}$$

with $\vartheta_2(t) := -\int_0^t e^{-A_K(t-s)}\partial_t w_0(s)\, ds$. As we will see in section 3, the splitting of the time derivative ∂_t yields a lower order term, compared to $(\partial_t + \Delta^2)\psi$. Thus it remains to solve the problem

$$\begin{aligned} \partial_t\psi + \Delta^2\psi &= \Delta\Phi'(\psi) + \Delta(\lambda'(\psi)F(\psi)) - \Delta(\lambda'(\psi)\vartheta^*) + f, \quad t \in J,\ x \in \Omega,\\ \partial_\nu\Delta\psi &= \partial_\nu\Phi'(\psi) + \partial_\nu(\lambda'(\psi)F(\psi)) - \partial_\nu(\lambda'(\psi)\vartheta^*) + g, \quad t \in J,\ x \in \Gamma,\\ \partial_t\psi - \sigma_s\Delta_\Gamma\psi + \gamma\partial_\nu\psi + \kappa\psi &= h, \quad t \in J,\ x \in \Gamma,\\ \psi(0) &= \psi_0, \quad t = 0,\ x \in \Omega, \end{aligned} \tag{2.4}$$

with $F(\psi) := \partial_t^{1/2}(\partial_t + A_K)^{-1}\partial_t^{1/2}(\lambda(\psi) - w_0)$ and $\vartheta^* = \vartheta_1 + \vartheta_2$. The corresponding linear system to (2.4) reads as follows

$$\begin{aligned} \partial_t u + \Delta^2 u &= f, \quad t \in J,\ x \in \Omega,\\ \partial_\nu \Delta u &= g, \quad t \in J,\ x \in \Gamma,\\ \partial_t u - \sigma_s\Delta_\Gamma u + \gamma\partial_\nu u + \kappa u &= h, \quad t \in J,\ x \in \Gamma,\\ u(0) &= u_0, \quad t = 0,\ x \in \Omega. \end{aligned} \tag{2.5}$$

Here is the main result on maximal L_p - regularity of (2.5).

Theorem 2.1. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $p \neq 3, 5$ and let $\sigma_s, \gamma > 0$ and $\kappa \geq 0$ be constants. Suppose $\Omega \subset \mathbb{R}^n$ is bounded open with compact boundary $\Gamma = \partial\Omega \in C^4$ and let $J_0 = [0, T_0]$. Then there is a unique solution u of (2.5) such that*

$$u \in H^1_p(J_0; L_p(\Omega)) \cap L_p(J_0; H^4_p(\Omega)) = Z^1$$

with

$$u|_\Gamma \in H^1_p(J_0; W^{2-1/p}_p(\Gamma)) \cap L_p(J_0; W^{4-1/p}_p(\Gamma)) = Z^1_\Gamma,$$

if and only if the data are subject to the following conditions.

(i) $f \in L_p(J_0 \times \Omega) = X$,
(ii) $g \in W^{1/4-1/4p}_p(J_0; L_p(\Gamma)) \cap L_p(J_0; W^{1-1/p}_p(\Gamma)) =: Y_1$,

(iii) $h \in L_p(J_0; W_p^{2-1/p}(\Gamma)) =: Y_2$,
(iv) $u_0 \in \{w \in W_p^{4-4/p}(\Omega) : w|_\Gamma \in W_p^{4-3/p}(\Gamma)\} =: X_{p,\Gamma}$,
(v) $\partial_\nu \Delta u_0 = g|_{t=0}$, *if* $p > 5$.

Proof. This theorem is a special case of [8, Theorem 2.1]. □

3. Local Well-Posedness

In this section we will apply the contraction mapping principle to overcome the nonlinearities in

$$\begin{aligned} \partial_t u + \Delta^2 u &= \Delta G(u) + f, \quad t \in J,\ x \in \Omega,\\ \partial_\nu \Delta u &= \partial_\nu G(u) + g, \quad t \in J,\ x \in \Gamma,\\ \partial_t u - \sigma_s \Delta_\Gamma u + \gamma \partial_\nu u + \kappa u &= h, \quad t \in J,\ x \in \Gamma,\\ u(0) &= \psi_0, \quad t = 0,\ x \in \Omega, \end{aligned} \tag{3.1}$$

where

$$G(u) := \Phi'(u) + \lambda'(u) F(u) - \lambda'(u)\vartheta^*.$$

For that purpose let $\psi_0 \in X_{p,\Gamma}$, $f \in X$, $g \in Y_1$ and $h \in Y_2$, as well as $\vartheta_0 \in X_p$ be given, such that the compatibility condition

$$\partial_\nu \Delta \psi_0 = \Phi'(\psi_0) - \lambda'(\psi_0)\vartheta_0 + g|_{t=0}, \quad \text{if } p > 5,$$

is satisfied. We furthermore assume that $\lambda, \Phi \in C^{4-}(\mathbb{R})$ and we will use the embeddings

$$H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^4(\Omega)) \hookrightarrow C(J; W_p^{4-4/p}(\Omega)) \hookrightarrow C(J \times \overline{\Omega}), \tag{3.2}$$

valid for $p > n/4 + 1$. For $[0,T] \subset [0, T_0]$ we define

$$\mathbb{E}_1 := \{w \in Z^1(T) : w|_\Gamma \in Z_\Gamma^1(T)\}, \quad \text{and} \quad {}_0\mathbb{E}_1 := \{w \in \mathbb{E}_1 : w|_{t=0} = 0\}$$

and

$$\mathbb{E}_0 := X(T) \times Y_1(T) \times Y_2(T), \quad \text{and} \quad {}_0\mathbb{E}_0 := \{(f,g,h) \in \mathbb{E}_0 : g|_{t=0} = 0\}.$$

The spaces $\mathbb{E}_1$ and $\mathbb{E}_0$ are endowed with canonical norms $|\cdot|_1$ and $|\cdot|_0$, respectively. Let furthermore $A := -\Delta_\Gamma$. By Theorem 2.1 there is a unique solution $u^* \in \mathbb{E}_1$ of the linear system

$$\begin{aligned} \partial_t u + \Delta^2 u &= f, \quad t \in J,\ x \in \Omega,\\ \partial_\nu \Delta u &= g - e^{-A^2 t} g_0, \quad t \in J,\ x \in \Gamma,\\ \partial_t u - \sigma \Delta_\Gamma u + \gamma \partial_\nu u + \kappa u &= h, \quad t \in J,\ x \in \Gamma,\\ u(0) &= \psi_0, \quad t = 0,\ x \in \Omega, \end{aligned} \tag{3.3}$$

where $g_0 = 0$, if $p < 5$ and $g_0 = g|_{t=0} - \partial_\nu \Delta \psi_0$, if $p > 5$. We define a linear operator $\mathbb{L} : \mathbb{E}_1 \to \mathbb{E}_0$ by

$$\mathbb{L}w = \begin{bmatrix} \partial_t w + \Delta^2 w \\ \partial_\nu \Delta w \\ \partial_t w - \sigma_s \Delta_\Gamma w + \mu \partial_\nu w + \kappa w \end{bmatrix}.$$

Consider $\mathbb{L}$ as an operator from ${}_0\mathbb{E}_1$ to ${}_0\mathbb{E}_0$. Then by Theorem 2.1, $\mathbb{L}$ is bounded and bijective, i.e. an isomorphism. Hence, by the open-mapping theorem, $\mathbb{L}$ is invertible with bounded inverse.
Next define a nonlinear mapping $\tilde{G} : \mathbb{E}_1 \times {}_0\mathbb{E}_0 \to {}_0\mathbb{E}_0$ by means of

$$\tilde{G}(u^*, w) = \begin{bmatrix} \Delta G(u^* + w) \\ \partial_\nu G(u^* + w) - g_1 \\ 0 \end{bmatrix},$$

where $g_1 = 0$, if $p < 5$ and $g_1 = e^{-A^2 t} \partial_\nu G(u^*)|_{t=0}$ if $p > 5$. We will show in a subsequent proposition, that the range of $\tilde{G}$ is indeed a subset of ${}_0\mathbb{E}_0$. For the moment, assume that this result is already at our disposal. It is then obvious that $u := u^* + w$ is a solution of (3.1) if and only if $\mathbb{L}w = \tilde{G}(u^*, w)$ or equivalently $w = \mathbb{L}^{-1}\tilde{G}(u^*, w)$. In addition we define a ball $\mathbb{B}_R \subset {}_0\mathbb{E}_1$ by

$$\mathbb{B}_R := \mathbb{B}_R(0) := \{w \in_0 \mathbb{E}_1 \ : \ |w|_1 \le R\}, \ R \in (0,1],$$

and an operator $S : \mathbb{B}_R \to_0 E_1$ by $Sw = \mathbb{L}^{-1}\tilde{G}(u^*, w)$. In order to apply the contraction mapping principle we have to ensure that S is a self-mapping, i.e. $S\mathbb{B}_R \subset \mathbb{B}_R$ and that S defines a strict contraction on $\mathbb{B}_R$, i.e. there exists a number $\beta < 1$ with

$$|Sw - S\bar{w}|_1 \le \beta |w - \bar{w}|_1, \quad w, \bar{w} \in \mathbb{B}_R.$$

We need some preliminaries to prove these properties. First we observe that all functions belonging to $\mathbb{B}_R$ are uniformly bounded on $J\times\Omega$. Indeed, by (3.2) it holds that

$$|w|_\infty \le M|w|_{Z^1} \le M|w|_1 \le MR \le M,$$

with a constant $M>0$, independent of T, since $w|_{t=0}=0$ for all $w\in\mathbb{B}_R$.
For all forthcoming considerations we define the shifted ball $\mathbb{B}_R(u^*)\subset\mathbb{E}_1$ by means of

$$\mathbb{B}_R(u^*) := \{w\in\mathbb{E}_1 : w=\tilde w+u^*,\ \tilde w\in\mathbb{B}_R\}.$$

Note that all functions $w\in\mathbb{B}_R(u^*)$ are uniformly bounded, too. In the sequel we will also make use of the well known estimate

$$|fg|_{H_p^s(L_p)} \le C(|f|_{L_{\sigma_1' p}(L_{r_1' p})}|g|_{H^s_{\sigma_1 p}(L_{r_1 p})} + |g|_{L_{\sigma_2' p}(L_{r_2' p})}|f|_{H^s_{\sigma_2 p}(L_{r_2 p})}), \tag{3.4}$$

where $1/\sigma_i+1/\sigma_i' = 1/r_i+1/r_i'=1,\ i=1,2,\ s\in[0,1]$. Indeed, this is a consequence of the definition of the spaces H_p^s and Hölders inequality (cf. also [10]). Next, by [12, Lemma 6.2.3] there is a function $\mu(T)>0$, with $\mu(T)\to0$ as $T\to0$, such that

$$|f(u)-f(v)|_{H_p^s(L_p)} \le \mu(T)(|u-v|_{H_p^{s_0}(L_p)} + |u-v|_\infty),\quad 0<s<s_0<1, \tag{3.5}$$

for every $f\in C^{1-}(\mathbb{R})$ and all $u,v\in\mathbb{B}_R(u^*)$. Now we show that $F(w),\ w\in\mathbb{B}_R(u^*)$, represents a lower order term in (3.1). By the mixed-derivative theorem we obtain

$$w_0\in Z^1 \hookrightarrow H_p^{3/4}(J;H_p^1(\Omega)) \hookrightarrow H_p^s(J;H_p^1(\Omega)),$$

for every $s\in(0,3/4)$. By (3.4) we see that $\lambda(w)\in H_p^s(J;H_p^1(\Omega)),\ s\in(0,3/4)$, too. Thus $(\lambda(w)-w_0)\in {}_0H_p^s(J;H_p^1(\Omega))$, and for $1/2\le s<3/4$ and every $\eta\in {}_0H_p^s(J;H_p^1(\Omega))$ we obtain

$$(\partial_t+A_K)^{-1}\partial_t^{1/2}\eta\in {}_0H_p^{s+1/2}(J;H_p^1(\Omega))\cap {}_0H_p^{s-1/2}(J;H_p^3(\Omega)) \hookrightarrow {}_0H_p^{s+\theta-1/2}(J;H_p^{3-2\theta}(\Omega)),$$

for each $\theta\in[\frac12,1]$, where the latter embedding is due to the mixed-derivative theorem. Finally it holds

$$\partial_t^{1/2}(\partial_t+A_K)^{-1}\partial_t^{1/2} :\ {}_0H_p^s(J;H_p^1(\Omega)) \to {}_0H_p^{s+\theta-1}(J;H_p^{3-2\theta}(\Omega)),\quad \theta\in[\frac{1}{2},1]. \tag{3.6}$$

The following proposition shows that the lipschitz property of λ carries over to F.

Proposition 3.1. *Let $p>n/4+1$, $\lambda\in C^{2-}(\mathbb{R})$ and $J=[0,T]\subset[0,T_0]$. Then there exists a function $\mu(T)>0$, with $\mu(T)\to0$ as $T\to0$, such that for every $s\in[\frac12,\frac34)$, and all $u,v\in\mathbb{B}_R(u^*)$ the estimate*

$$\begin{aligned}|F(u)-F(v)|_{H_p^{s-1/2}(H_p^2)} &+ |\nabla F(u)-\nabla F(v)|_{H_p^{s-1/2}(H_p^1)}\\ &+ |\Delta F(u)-\Delta F(v)|_{H_p^{s-1/2}(L_p)} \le \mu(T)|u-v|_1\end{aligned}$$

is valid.

Proof. By (3.6) it suffices to show that

$$|\lambda(u)-\lambda(v)|_{H_p^s(H_p^1)} \le \mu(T)|u-v|_1.$$

Obviously $|\lambda(u)-\lambda(v)|_{H_p^s(H_p^1)} \le C(|\lambda(u)-\lambda(v)|_{H_p^s(L_p)} + |\nabla u\lambda'(u)-\nabla v\lambda'(v)|_{H_p^s(L_p)})$ and

$$|\nabla u\lambda'(u)-\nabla v\lambda'(v)|_{H_p^s(L_p)}) \le C(|\nabla u(\lambda'(u)-\lambda'(v))|_{H_p^s(L_p)} + |\lambda'(v)(\nabla u-\nabla v)|_{H_p^s(L_p)}).$$

Now (3.4) yields

$$\begin{aligned}|\nabla u(\lambda'(u)-\lambda'(v))|_{H_p^s(L_p)} \le C(T_0)\Big(&|\nabla u|_{L_{r'p}(L_{r'p})}|\lambda'(u)-\lambda'(v)|_{H^s_{rp}(L_{rp})}\\ &+ T^{1/\sigma'}|\lambda'(u)-\lambda'(v)|_\infty|\nabla u|_{H^s_{\sigma p}(L_{\sigma p})}\Big),\end{aligned}$$

as well as

$$\begin{aligned}&|\lambda'(v)(\nabla u-\nabla v)|_{H_p^s(L_p)}\\ &\qquad\le C(T_0)\Big(|\nabla u-\nabla v|_{L_{r'p}(L_{r'p})}|\lambda'(v)|_{H^s_{rp}(L_{rp})} + T^{1/\sigma'}|\nabla u-\nabla v|_{H^s_{\sigma p}(L_{\sigma p})}\Big).\end{aligned}$$

Again by (3.4) we see that $|\lambda'(v)|_{H^s_{rp}(L_{rp})} \le T^{1/\rho'}|\lambda'(v)|_{H^s_{r\rho p}(L_{r\rho p})}$. Observe that the embedding $Z^1\hookrightarrow H^s_{rp}(L_{rp})$ holds, whenever $s\le 1/r$. To meet this requirement we set $r=4/3$, i.e. $r'=4$. Hence for sufficiently small $\rho>1$ and by (3.5) we obtain the desired estimate.

□

The next proposition collects all the facts we need, to show the desired properties of the operator S defined above.

Proposition 3.2. *Let $p > n/4+1$, $\lambda, \Phi \in C^{4-}(\mathbb{R})$ and $J = [0,T] \subset [0,T_0]$. Then there exists a constant $C > 0$, independent of T, and functions $\mu_j = \mu_j(T)$ with $\mu_j(T) \to 0$ as $T \to 0$, $j = 1, \dots, 4$, such that for all $u, v \in \mathbb{B}_R(u^*)$ the following statements hold.*

(i) $|\Delta\Phi'(u) - \Delta\Phi'(v)|_p \le \mu_1(T)|u-v|_1$,

(ii) $|(\Delta(\lambda'(u)F(u)) - \Delta(\lambda'(v)F(v))|_p \le \mu_2(T)|u-v|_1$,

(iii) $|\partial_\nu\Phi'(u) - \partial_\nu\Phi'(v)|_{Y_1(T)} \le \mu_3(T)|u-v|_1$,

(iv) $|\partial_\nu(\lambda'(u)F(u)) - \partial_\nu(\lambda'(v)F(v))|_{Y_1(T)} \le \mu_4(T)|u-v|_1$.

For each $\eta \in H^1_p(J;L_p(\Omega)) \cap L_p(J;H^2_p(\Omega))$ we have

(v) $|(\Delta(\lambda'(u)\eta) - \Delta(\lambda'(v)\eta)|_p \le C|\eta|_{Z^2}|u-v|_1$,

(vi) $|(\partial_\nu(\lambda'(u)\eta) - \partial_\nu(\lambda'(v)\eta)|_{Y_1(T)} \le C|\eta|_{Z^2}|u-v|_1$.

The proof is given in the Appendix.
Note that we have also shown that $\Delta G(w) \in X(T)$ and $\partial_\nu G(w) \in Y_1(T)$ for each $w \in \mathbb{B}_R(u^*)$, where G was defined in (3.1). Thus the operator $S : \mathbb{B}_R \to {}_0\mathbb{E}_1$ is well defined. With our previous considerations, it is now easy to verify the self-mapping property as well as the strict contraction property of S. Let $w \in \mathbb{B}_R$. Then we obtain

$$\begin{aligned}|Sw|_1 &= |\mathbb{L}^{-1}\tilde{G}(u^*,w)|_1 \le |\mathbb{L}^{-1}||\tilde{G}(u^*,w)|_0\\ &\le C(|\tilde{G}(u^*,w) - \tilde{G}(u^*,0)|_0 + |\tilde{G}(u^*,0)|_0)\\ &\le C(|\Delta G(u^*+w) - \Delta G(u^*)|_{X(T)} + |\partial_\nu(G(u^*+w) - G(u^*))|_{Y_1(T)}\\ &\qquad + |\Delta G(u^*)|_{X(T)} + |\partial_\nu G(u^*)|_{Y_1(T)} + |g_1|_{Y_1(T)}.\end{aligned}$$

By Proposition 3.2 there exists a function $\mu(T)$, with $\mu(T) \to 0$ as $T \to 0$, such that

$$|\Delta G(u^*+w) - \Delta G(u^*)|_{X(T)} + |\partial_\nu(G(u^*+w) - G(u^*))|_{Y_1(T)} \le \mu(T)|w|_1 \le \mu(T)R,$$

since $w + u^* \in \mathbb{B}_R(u^*)$. Thus we see that $|Sw|_1 \le R$, if $T > 0$ is sufficiently small. We remark that g_1 and $G(u^*)$ are fixed functions, hence $|g_1|_{Y_1(T)}, |\Delta G(u^*)|_{X(T)}, |\partial_\nu G(u^*)|_{Y_1(T)} \to 0$ as $T \to 0$. This shows that $S\mathbb{B}_R \subset \mathbb{B}_R$. Furthermore for all $w \in \mathbb{B}_R$ we have

$$\begin{aligned}|Sw - S\bar{w}|_1 &= |\mathbb{L}^{-1}(\tilde{G}(u^*,w) - \tilde{G}(u^*,\bar{w}))|_1 \le |\mathbb{L}^{-1}||\tilde{G}(u^*,w) - \tilde{G}(u^*,\bar{w})|_0\\ &\le C(|\Delta G(u^*+w) - \Delta G(u^*+\bar{w})|_{X(T)} + |\partial_\nu G(u^*+w) - \partial_\nu G(u^*+\bar{w})|_{Y_1(T)}).\end{aligned}$$

It is a consequence of Proposition 3.2 that

$$|\Delta G(u^*+w) - \Delta G(u^*+\bar{w})|_{X(T)} + |\partial_\nu G(u^*+w) - \partial_\nu G(u^*+\bar{w})|_{Y_1(T)} \le \mu(T)|w-\bar{w}|_1,$$

hence $S : \mathbb{B}_R \to \mathbb{B}_R$ is a strict contraction, if $T > 0$ is sufficiently small. Thus the contraction-mapping principle yields a unique fixed-point w^* of S, i.e. a solution $\psi \in Z^1(T) \cap Z^1_\Gamma(T)$ of (2.4), which depends continously on the given data $f \in X$, $g \in Y_1$, $h \in Y_2$ and $\psi_0 \in X_{p,\Gamma}$. Since $\partial_t\lambda(\psi) = \partial_t\psi\lambda'(\psi) \in L_p(J\times\Omega)$, there is a unique solution $\vartheta \in Z^2(T)$ of

$$\begin{aligned}\partial_t v - \Delta v &= -\partial_t\lambda(\psi) + f_2, && t \in J,\ x \in \Omega,\\ \alpha v + \partial_\nu v &= g_3, && t \in J,\ x \in \Gamma,\\ v(0) &= \vartheta_0, && t = 0,\ x \in \Omega.\end{aligned}$$

Finally we see that $(\psi, \vartheta) \in (Z^1(T) \cap Z^1_\Gamma(T)) \times Z^2(T)$ is the unique solution of (1.5) on the interval $[0,T]$. We summarize these considerations in

Theorem 3.3. *Let $1 < p < \infty$, $p > n/4+1$, $p \ne 3, 5$ and let $\sigma_s, \gamma > 0$, $\alpha, \kappa \ge 0$ be constants. Assume furthermore that $\lambda, \Phi \in C^{4-}(\mathbb{R})$. Then there exists is an interval $J = [0,T] \subset [0,T_0]$ and a unique solution (ψ, ϑ) of (1.5) on J, with*

$$\psi \in H^1_p(J;L_p(\Omega)) \cap L_p(J;H^4_p(\Omega)) = Z^1(T), \quad \vartheta \in H^1_p(J;L_p(\Omega)) \cap L_p(J;H^2_p(\Omega)) = Z^2(T),$$

and

$$\psi|_\Gamma \in H^1_p(J;W^{2-1/p}_p(\Gamma)) \cap L_p(J;W^{4-1/p}_p(\Gamma)) = Z^1_\Gamma(T),$$

provided the data are subject to the following conditions.

(i) $f_1, f_2 \in L_p(J\times\Omega) = X$,

(ii) $g_1 \in W^{1/4-1/4p}_p(J;L_p(\Gamma)) \cap L_p(J;W^{1-1/p}_p(\Gamma)) = Y_1$,

(iii) $g_2 \in L_p(J;W^{2-1/p}_p(\Gamma)) = Y_2$,

(iv) $g_3 \in W_p^{1/2-1/2p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{1-1/p}(\Gamma)) = Y_3$,
(v) $\psi_0 \in \{u \in W_p^{4-4/p}(\Omega) : u|_\Gamma \in W_p^{4-3/p}(\Gamma)\} = X_{p,\Gamma}$,
(vi) $\vartheta_0 \in W_p^{2-2/p}(\Omega) = X_p$,
(vii) $\partial_\nu \Delta\psi_0 = -\Phi'(\psi_0) + \lambda'(\psi_0)\vartheta_0 + g_1|_{t=0}$, *if* $p > 5$,
(viii) $\alpha\vartheta_0 + \partial_\nu\vartheta_0 = g_3|_{t=0}$, *if* $p > 3$.

The solution depends continously on the given data and if the data are independent of t*, the map* $(\psi_0, \vartheta_0) \mapsto (\psi, \vartheta)$ *defines a local semiflow on the natural phase manifold* $\mathcal{M} \subset X_{p,\Gamma} \times X_p$*, defined by (vii) and (viii).*

4. Global Well-Posedness

Throughout this section we assume that $n \leq 3$ and that the potential Φ satisfies the growth conditions

$$\Phi(s) \geq -\frac{\mu}{2}s^2 - c_0, \quad c_0 > 0,\ s \in \mathbb{R}, \tag{4.1}$$

where $\mu < \lambda_1$, with λ_1 being the smallest nontrivial eigenvalue of the negative Laplacian on Ω with Neumann boundary conditions,

$$|\Phi'(s)| \leq (c_1\Phi(s) + c_2 s^2 + c_3)^\theta, \quad c_i > 0,\ \theta \in (0,1),\ s \in \mathbb{R}, \tag{4.2}$$

and

$$|\Phi'''(s)| \leq C(1 + |s|^\beta), \quad s \in \mathbb{R}, \tag{4.3}$$

with $\beta < 3$ in case $n = 3$. Furthermore let $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$.

Remark: (i) The conditions (4.1)-(4.3) are certainly fulfilled, if Φ is a polynomial of degree $2m$, $m \in \mathbb{N}$, $m < 3$. Then we may set $\theta = 1 - 1/2m$.
(ii) As we will see we may omit (4.2) if $f_1 = g_1 = 0$.

A successive application of Theorem 3.3 yields a maximal time interval $J_{max} = [0, T_{max}) \subset [0, T_0]$ for the solution $\psi \in Z^1 \cap Z^1_\Gamma$ of (3.1). If $T_{max} < T_0$, this interval is characterized by the following two equivalent conditions

$$\lim_{t \to T_{max}} \psi(t) \quad \text{does not exist in } X_{p,\Gamma}$$

and

$$|\psi|_{Z^1(T_{max})} + |\psi_{|\Gamma}|_{Z^1_\Gamma(T_{max})} = \infty.$$

First of all, we need some a priori estimates for ψ. We multiply $(1.5)_1$ by μ and $(1.5)_2$ by ϑ. Integration by parts and the boundary conditions $(1.5)_{3,4,5}$ lead to the energy-equation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(|\nabla\psi|_2^2 + |\vartheta|_2^2 + \frac{\sigma_s}{\gamma}|\nabla_\Gamma\psi|_{2,\Gamma}^2 + \frac{\kappa}{\gamma}|\psi|_{2,\Gamma}^2 + 2\int_\Omega \Phi(\psi)\Big) + |\nabla\mu|_2^2 + |\partial_t\psi|_{2,\Gamma}^2 + |\nabla\vartheta|_2^2 + \alpha|\vartheta|_{2,\Gamma}^2 \\ = \int_\Omega (f_1\mu + f_2\vartheta) + \int_\Gamma (g_1\mu + g_2\partial_t\psi + g_3\vartheta).\end{aligned} \tag{4.4}$$

The Poincaré-Wirtinger inequality as well as the embedding $H_2^1(\Omega) \hookrightarrow L_2(\Gamma)$ imply

$$\int_\Omega f_1\mu \leq C|f_1|_2(|\nabla\mu|_2 + |\int_\Omega \mu|) \quad \text{and} \quad \int_\Gamma g_1\mu \leq C|g_1|_{2,\Gamma}(|\nabla\mu|_2 + |\int_\Omega \mu|). \tag{4.5}$$

By (4.2) and since

$$\int_\Omega \mu = \int_\Omega (\Phi'(\psi) - \lambda'(\psi)\vartheta) + \frac{1}{\gamma}\int_\Gamma (\partial_t\psi + \kappa\psi - g_2),$$

we obtain using the Cauchy-Schwartz inequality

$$|\int_\Omega \mu| \leq C(|\vartheta|_2 + |\partial_t\psi|_{2,\Gamma} + |\psi|_{2,\Gamma} + |g_2|_{2,\Gamma}) + \int_\Omega (c_1\Phi(\psi) + c_2|\psi|^2 + c_3)^\theta.$$

For simplicity, we set

$$E(\psi,\vartheta) := \frac{1}{2}|\nabla\psi|_2^2 + \frac{1}{2}|\vartheta|_2^2 + \frac{1}{2}\frac{\sigma_s}{\gamma}|\nabla_\Gamma\psi|_{2,\Gamma}^2 + \frac{1}{2}\frac{\kappa}{\gamma}|\psi|_{2,\Gamma}^2 + \int_\Omega \Phi(\psi).$$

From (4.4), (4.5) and by Hölder's inequality, Young's inequality as well as by the Poincaré-Wirtinger inequality we obtain the estimate

$$\begin{aligned}\frac{d}{dt}\ E(\psi,\vartheta) + C(|\nabla\mu|_2^2 + &|\partial_t\psi|_{2,\Gamma}^2 + |\nabla\vartheta|_2^2 + \alpha|\vartheta|_{2,\Gamma}^2) \\ &\leq C_1E(\psi,\vartheta) + C_2(|f_1|_2^q + |f_2|_2^2 + |g_1|_{2,\Gamma}^q + |g_2|_{2,\Gamma}^2 + |g_3|_{2,\Gamma}^2 + 1),\end{aligned}$$

where $q := \max\{2, \frac{1}{1-\theta}\}$. Observe that the functional E is bounded from below. Indeed, by (4.1) we obtain

$$E(\psi(t),\vartheta(t)) \geq \frac{1}{2}(|\nabla\psi|_2^2 - \mu|\psi|_2^2) - c_0|\Omega| \geq \frac{\lambda_1-\mu}{2\lambda_1}|\nabla\psi|_2^2 - c \geq -c, \quad c>0, \tag{4.6}$$

where we used again the Poincaré inequality, since

$$|\int_\Omega \psi(t)| \leq \int_\Omega |\psi_0| + |f_1|_{L_1} + |g_1|_{L_1}.$$

Then Gronwall's lemma yields the estimate

$$E(\psi,\vartheta) \leq C\left(E(\psi_0,\vartheta_0) + \int_0^{T_{max}} (|f_1|_2^q + |f_2|_2^2 + |g_1|_{2,\Gamma}^q + |g_2|_{2,\Gamma}^2 + |g_3|_{2,\Gamma}^2 + 1)\right),$$

and by (4.1) we obtain among other things the a priori estimate $\psi \in L_\infty([0,T_{max}]; H_2^1(\Omega))$. The following lemma is the key to obtain global existence.

Lemma 4.1. *Suppose $p \geq 2$, $n \leq 3$ and let $\psi \in Z^1(T)$ be the solution of* (3.1). *Then there exist constants $C>0$ and $\delta \in (0,1)$, independent of $T>0$, such that*

$$|\Delta G(\psi)|_{X(T)} + |\partial_\nu G(\psi)|_{Y_1(T)} \leq C(1 + |\psi|_{Z^1(T)}^\delta |\psi|_{L_\infty(J;H_2^1(\Omega))}^{1-\delta}).$$

The proof is given in the Appendix.
Observe that by maximal L_p-regularity the estimate

$$|\psi|_{Z^1(T)} + |\psi|_{Z_\Gamma^1(T)} \leq M(|\Delta G(\psi)|_{X(T)} + |\partial_\nu G(\psi)|_{Y_1(T)} + |f|_X + |g|_{Y_1} + |h|_{Y_2} + |\psi_0|_{X_{p,\Gamma}}),$$

for the solution ψ of (3.1) is valid, with a constant $M = M(T_0) > 0$. Then it follows from Lemma 4.1 that

$$|\psi|_{Z^1(T)} + |\psi_{|\Gamma}|_{Z_\Gamma^1(T)} \leq M(1 + |\psi|_{Z^1(T)}^\delta),$$

hence $|\psi|_{Z^1(T)}$ is bounded. This in turn yields the boundedness of $|\psi_{|\Gamma}|_{Z_\Gamma^1(T)}$ and therefore global existence of the solution (ψ,ϑ) of (1.5), since ϑ solves the heat-equation $\partial_t\vartheta - \Delta\vartheta = -\partial_t\lambda(\psi)$, subject to homogeneous boundary and initial conditions.
We summarize our previous considerations in

Theorem 4.2. *Let $1<p<\infty$, $p \geq 2$, $p \neq 3,5$, $\theta \in (0,1)$, $q = \max\{2, \frac{1}{1-\theta}\}$ and $J_0 = [0,T_0]$. Assume furthermore that $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$ and let Φ satisfy* (4.1)-(4.3). *Then there exists a unique global solution (ψ,ϑ) of* (1.5) *on J_0, with*

$$\psi \in H_p^1(J_0; L_p(\Omega)) \cap L_p(J_0; H_p^4(\Omega)), \quad \vartheta \in H_p^1(J_0; L_p(\Omega)) \cap L_p(J_0; H_p^2(\Omega)),$$

and

$$\psi|_\Gamma \in H_p^1(J_0; W_p^{2-1/p}(\Gamma)) \cap L_p(J_0; W_p^{4-1/p}(\Gamma)),$$

if the data are subject to the following conditions.

(i) $f_1, f_2 \in L_p(J_0 \times \Omega)$, $f_1 \in L_q(J_0; L_2(\times\Omega))$,
(ii) $g_1 \in W_p^{1/4-1/4p}(J_0; L_p(\Gamma)) \cap L_p(J_0; W_p^{1-1/p}(\Gamma)) \cap L_q(J_0; L_2(\Gamma))$,
(iii) $g_2 \in L_p(J_0; W_p^{2-1/p}(\Gamma))$,
(iv) $g_3 \in W_p^{1/2-1/2p}(J_0; L_p(\Gamma)) \cap L_p(J_0; W_p^{1-1/p}(\Gamma))$,
(v) $\psi_0 \in \{u \in W_p^{4-4/p}(\Omega) : u|_\Gamma \in W_p^{4-3/p}(\Gamma)\} = X_{p,\Gamma}$,
(vi) $\vartheta_0 \in W_p^{2-2/p}(\Omega) = X_p$,
(vii) $\partial_\nu\Delta\psi_0 = -\Phi'(\psi_0) + \lambda'(\psi_0)\vartheta_0 + g_1|_{t=0}$, *if* $p>5$,
(viii) $\alpha\vartheta_0 + \partial_\nu\vartheta_0 = g_3|_{t=0}$, *if* $p>3$.

The solution depends continously on the given data and if the data are independent of t, the map $(\psi_0,\vartheta_0) \mapsto (\psi,\vartheta)$ defines a global semiflow on the natural phase manifold $\mathcal{M} \subset X_{p,\Gamma} \times X_p$, defined by (vii) and (viii).

5. Asymptotic Behaviour

In this section we study the asymptotic behavior of a solution of the system

$$
\begin{aligned}
\partial_t\psi - \Delta\mu = 0, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, &\quad t \in J,\ x \in \Omega,\\
\partial_t\vartheta + \lambda'(\psi)\partial_t\psi - \Delta\vartheta = 0, &\quad t \in J,\ x \in \Omega,\\
\partial_\nu\mu = 0, &\quad t \in J,\ x \in \Gamma,\\
\alpha\vartheta + \partial_\nu\vartheta = 0, &\quad t \in J,\ x \in \Gamma,\\
\partial_t\psi - \sigma_s\Delta_\Gamma\psi + \gamma\partial_\nu\psi + \kappa(\psi - h) = 0, &\quad t \in J,\ x \in \Gamma,\\
\psi(0) = \psi_0, &\quad t = 0,\ x \in \Omega,\\
\vartheta(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega.
\end{aligned}
\tag{5.1}
$$

For the forthcoming considerations, we need the following assumptions. Let $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$ and let Φ satisfy (4.1) as well as (4.3).
The main tool will be the *Lojasiewicz-Simon inequality* (see Proposition 5.3), which leads to the convergence result. We define two energy-functionals $E_K(u, v)$ by means of

$$E_N(u,v) = \frac{1}{2}\left(\int_\Omega |\nabla u|^2 + \int_\Omega |v + \overline{\lambda(u)}|^2 + \frac{\sigma_s}{\gamma}\int_\Gamma |\nabla_\Gamma u|^2 + \frac{\kappa}{\gamma}\int_\Gamma |u|^2\right) + \int_\Omega \Phi(u) + \frac{|\Omega|}{2}\,(\overline{\lambda(u)})^2,$$

and

$$E_R(u,v) = \frac{1}{2}\left(\int_\Omega |\nabla u|^2 + \int_\Omega |v|^2 + \frac{\sigma_s}{\gamma}\int_\Gamma |\nabla_\Gamma u|^2 + \frac{\kappa}{\gamma}\int_\Gamma |u|^2\right) + \int_\Omega \Phi(u),$$

in case of Neumann- or Robin-boundary conditions, respectively. Here we use the abbreviation

$$\overline{w} = \frac{1}{|\Omega|}\int_\Omega w(x)dx$$

for the mean value of a function $w \in L_1(\Omega)$. A suitable energy space both for E_N and E_R will be

$$W := \{(u, v) \in H^1_2(\Omega) \times L_2(\Omega) : u|_\Gamma \in H^1_2(\Gamma),\ \int_\Omega u = 0\}.$$

Note that the condition $\int_\Omega u = 0$ is compatible with our system. This might be seen by integrating $(5.1)_1$ and invoking the boundary condition $(5.1)_3$. We obtain $\int_\Omega \psi = \int_\Omega \psi_0$. If we replace the solution ψ by $\tilde\psi = \psi - c$, with $c = \frac{1}{|\Omega|}\int_\Omega \psi_0$ and set $h = c$ in (5.1), we see, that $\tilde\psi$ satisfies the corresponding homogeneous system to (5.1) (i.e. $h = 0$), if $\Phi(s)$ and $\lambda(s)$ are replaced by $\Phi_1(s) := \Phi(s + c)$ and $\lambda_1(s) := \lambda(s + c)$, respectively. In a similar way we can achieve that in case $\alpha = 0$ we have $\int_\Omega(\vartheta + \lambda(\psi)) = 0$. Indeed this follows by a shift of λ, i.e. $\tilde\lambda(s) := \lambda(s) - d$, where $d = \frac{1}{|\Omega|}\int_\Omega(\vartheta_0 + \lambda(\psi_0))$.
With the same methods as in [2, Lemma 6.2], we can compute the Fréchet-derivatives of E_N and E_R which are given by the formulas

$$
\begin{aligned}
\langle E_N'(u,v),(h,k)\rangle_{W^*,W} = \int_\Omega \nabla u\nabla h + \int_\Omega (v+\overline{\lambda(u)})(k + \frac{1}{|\Omega|}(\lambda'(u)|h)_2) + \int_\Omega \Phi'(u)h\\
+ \frac{\sigma_s}{\gamma}\int_\Gamma \nabla_\Gamma u\nabla_\Gamma h + \frac{\kappa}{\gamma}\int_\Gamma uh + \overline{\lambda(u)}(\lambda'(u)|h)_2,
\end{aligned}
\tag{5.2}
$$

and

$$\langle E_R'(u,v),(h,k)\rangle_{W^*,W} = \int_\Omega \nabla u\nabla h + \int_\Omega vk + \int_\Omega \Phi'(u)h + \frac{\sigma_s}{\gamma}\int_\Omega \nabla_\Gamma u\nabla_\Gamma h + \frac{\kappa}{\gamma}\int_\Gamma uh, \tag{5.3}$$

respectively, for all $(u, v), (h, k) \in W$.
Next we compute the derivative of $E_K(\psi(\cdot), \vartheta(\cdot))$ with respect to time to obtain

$$\frac{d}{dt}\,E_K(\psi(t),\vartheta(t)) = -|\nabla\mu|_2^2 - |\nabla\vartheta|_2^2 - |\partial_t\psi|_{2,\Gamma}^2 - \alpha|\vartheta|_{2,\Gamma}^2. \tag{5.4}$$

Note that in case of Robin boundary conditions, this is an easy consequence of (4.4), whereas in case of Neumann boundary conditions this follows from the fact that $\int_\Omega(\vartheta + \overline{\lambda(\psi)}) = 0$. Furthermore we observe that by (4.6) the functionals E_K are bounded from below. Then (5.4) and the boundedness of E_K yield

$$\psi \in C_b(\mathbb{R}_+; H^1_2(\Omega)) \cap C_b(\mathbb{R}_+; H^1_2(\Gamma)) \quad \text{and} \quad \vartheta \in C_b(\mathbb{R}_+; L_2(\Omega)). \tag{5.5}$$

Proposition 5.1. *Let $(\psi, \vartheta) \in (Z^1 \cap Z^1_\Gamma) \times Z^2$ be a global solution of* (5.1)*. Then $(\psi(t), \vartheta(t))$ has relatively compact range in $(H^1_2(\Omega) \cap H^1_2(\Gamma)) \times L_2(\Omega)$.*

Proof. We already know that a global solution is bounded in $(H_2^1(\Omega)\cap H_2^1(\Gamma))\times L_2(\Omega)$. To prove the relative compactness of the orbit $\psi(\mathbb{R}_+)$ we will proceed in two steps. First we consider the operator $A_p:=\Delta^2$ in $L_p(\Omega)$, with domain

$$D(A_p)=\{w\in H_p^4(\Omega):\Delta w=0 \text{ and } \partial_\nu w=0 \text{ on } \Gamma\}.$$

By [2, Proof of Proposition 5.2 (b)] we have $L_p(\Omega)=N(A_p)\oplus R(A_p)$ and the semigroup, generated by A_p is exponentially stable on $R(A_p)$. Let P be the corresponding projection onto $N(A_p)$ and set $Q=I-P$. Consider the evolution equation

$$\partial_t\psi_1+A_p\psi_1=Q(\Phi'(\psi)-\lambda'(\psi)\vartheta),\ t>0,\quad \psi_1(0)=\psi_{10},$$

where ψ_{10} denotes the the solution of the elliptic problem

$$\begin{cases}\Delta\psi_{10}=\psi_0,\ x\in\Omega,\\ \alpha\psi_{10}+\partial_\nu\psi_{10}=0,\ x\in\Gamma.\end{cases}$$

Since ψ_0 has mean 0 the compatibility condition is fulfilled. Choosing $p=6/(\beta+2)$ and using the fact that $\lambda'\in L_\infty(\mathbb{R})$, we obtain $Q(\Phi'(\psi)+\lambda'(\psi)\vartheta)\in C_b(\mathbb{R}_+;R(A_p))$. Thus by semigroup-theory this yields $\psi_1\in C_b(\mathbb{R}_+;H_p^r(\Omega))$, for each $r<4$ and by compact embedding the orbit $\Delta\psi_1(\mathbb{R}_+)$ is relatively compact in $H_2^1(\Omega)$.

Next we split ψ by means of $\psi=\Delta\psi_1+\psi_2$. From [2, Proof of Proposition 5.2] it follows immediately that the orbit $\psi_2(\mathbb{R}_+)$ is relatively compact in $H_2^1(\Omega)\cap H_2^1(\Gamma)$. Therefore the orbit of ψ is relatively compact in $H_2^1(\Omega)\cap H_2^1(\Gamma)$, too.

Now let $e=\vartheta+\lambda(\psi)$. Then e solves the following system

$$\begin{aligned}\partial_t e-\Delta e&=-\Delta\lambda(\psi),\quad t>0,\ x\in\Omega,\\ \alpha e+\partial_\nu e&=\alpha\lambda(\psi)+\partial_\nu\lambda(\psi),\quad t>0,\ x\in\Gamma,\\ e(0)&=e_0,\quad t=0,\ x\in\Omega,\end{aligned}$$

where $e_0:=\vartheta_0+\lambda(\psi_0)$. By (5.5) we see that $-\Delta\lambda(\psi)\in C_b(\mathbb{R}_+;H_2^1(\Omega)^*)$. The Laplacian generates an exponentially stable analytic semigroup in $H_2^1(\Omega)^*$, if $\alpha>0$. In case $\alpha=0$ the semigroup is exponentially stable on $\hat{H}_2^1(\Omega)^*$, where

$$\hat{H}_2^1(\Omega):=\{w\in H_2^1(\Omega):\int_\Omega w=0\}.$$

Therefore semigroup-theory implies that $e\in C_b(\mathbb{R}_+;H_2^r(\Omega))$, for every $r\in(0,1)$. One more time we use (5.5) to obtain

$$\vartheta=e-\lambda(\psi)\in C_b(\mathbb{R}_+;H_2^r(\Omega)),$$

hence by compact embedding the orbit $\vartheta(\mathbb{R}_+)$ is relatively compact in $L_2(\Omega)$. □

We remark, that the corresponding stationary system to (5.1), with $h=0$ is given by

$$\begin{cases}-\Delta u+\Phi'(u)-\lambda'(u)c^*=const,\quad x\in\Omega,\\ -\sigma_s\Delta_\Gamma u+\gamma\partial_\nu u+\kappa u=0,\quad x\in\Gamma,\end{cases}\tag{5.6}$$

where $c^*=const$, if $\alpha=0$ and $c^*=0$, if $\alpha>0$. Before we consider the next proposition, which justifies forthcoming calculations, we need some preliminaries. First we rewrite E_N in the following way

$$E_N(u,v)=\frac{1}{2}a((u,v),(u,v))+\int_\Omega\Phi(u)+\overline{\lambda(u)}\int_\Omega v+|\Omega|(\overline{\lambda(u)})^2,\tag{5.7}$$

with

$$a((u_1,v_1),(u_2,v_2))=\int_\Omega\nabla u_1\nabla u_2+\frac{\sigma_s}{\gamma}\int_\Gamma\nabla_\Gamma u_1\nabla_\Gamma u_2+\frac{\kappa}{\gamma}\int_\Gamma u_1u_2+\int_\Omega v_1v_2.$$

The form a is bounded, symmetric, positive and elliptic. The latter property follows from the trace map $H_2^1(\Omega)$ to $L_2(\Gamma)$. By the Lemma of Lax-Milgram there exists a bounded operator $A:W\to W^*$, with $\rho(A)\neq\varnothing$ and we have

$$a((u_1,v_1),(u_2,v_2))=\langle A(u_1,v_1),(u_2,v_2)\rangle_{W^*,W}.$$

Furthermore we denote by A_p the part of the operator A in the space

$$V_p=\{(u,v)\in L_p(\Omega)\times L_p(\Omega):\int_\Omega u=0\}$$

and we set $H=V_2$.

Proposition 5.2. *Let $K\in\{N,R\}$. For every $(u,v)\in W$ the following assertions are equivalent:*

(i) *(u,v) is a critical point of E_K, i.e. $E_K'(u,v)=0$, with the additional constraint $\int_\Omega(v+\lambda(u))=0$ in case $K=N$.*

(ii) *$(u,v)\in D(A_2)$ and (u,v) satisfy (5.6), where $v=c^*=const$, $c^*=0$ if $\alpha>0$.*

Proof. The proof for E_R is nearly the same as in [2, Proposition 6.4]. Therefore we will only consider the case $K=N$. Let (u,v) be a critical point of E_N with the constraint $\int_\Omega(v+\lambda(u))=0$. By [2, Lemma 6.3] it holds that

$$D(A_p)=\{(u,v)\in V_p : u\in H_p^2(\Omega)\cap W_p^{3-1/p}(\Gamma),\ -\sigma_s\Delta_\Gamma u+\gamma\partial_\nu u+\kappa u=0\}.$$

Next by (5.7) we obtain

$$A(u,v)=\left((h,k)\mapsto\int_\Omega\left(\Phi'(u)+\lambda'(u)(2\overline{\lambda(u)}+\bar v)\right)h+\int_\Omega\overline{\lambda(u)}k\right),$$

hence $A(u,v)\in V_p$, where p can be chosen strictly larger than $6/5$, due to (4.3). Then a bootstrap argument and the characterization of $D(A_p)$ lead to $A(u,v)\in H$. Finally we use integration by parts to obtain

$$\begin{aligned}\langle E_N'(u,v),(h,k)\rangle_{W^*,W}=&\int_\Omega(-\Delta u+\Phi'(u)-\lambda'(u)v)h\\ &+\frac{1}{\gamma}\int_\Gamma(-\sigma_s\Delta_\Gamma+\gamma\partial_\nu u+\kappa u)h+\int_\Omega(v+\overline{\lambda(u)})(k+\lambda'(u)h),\end{aligned}\tag{5.8}$$

for all $(h,k)\in W$. This yields (ii). The implication (ii)$\Rightarrow$ (i) is a simple consequence of (5.8). □

Remark. In other words, the statement of Proposition 5.2 means that every weak solution of the stationary system (5.6) is already a strong solution and vice versa.

Assuming in addition that Φ is real analytic and that in case of Neumann boundary conditions λ is real analytic too, we obtain the following result.

Proposition 5.3 (Lojasiewicz-Simon inequality)**.** *Let $K\in\{N,R\}$ and let $(\varphi,\theta)\in W$ be a critical point of the functional E_K. Assume that Φ is real analytic and in case $K=N$, λ is real analytic too, and let (4.3) as well $\lambda',\lambda'',\lambda'''\in L_\infty(\mathbb{R})$ hold. Then there exist constants $s\in(0,\frac12],C,\delta>0$ such that*

$$|E_K(u,v)-E_K(\varphi,\theta)|^{1-s}\le C|E_K'(u,v)|_{W^*},$$

whenever $|(u,v)-(\varphi,\psi)|_W\le\delta$.

Proof. The arguments used in [2, Proposition 6.6] for a proof of this assertion carries over to our case, since $\lambda',\lambda'',\lambda'''\in L_\infty(\mathbb{R})$. We will not repeat them here. □

The next proposition collects some properties of the functionals $E_K : W\to\mathbb{R}$.

Proposition 5.4. *Let (ψ,ϑ) be a global solution of (5.1) with $h=\frac{1}{|\Omega|}\int\psi_0$ and suppose that Φ satisfies (4.1) as well as (4.3). Let further $K\in\{N,R\}$. Then the following statements hold.*

(i) *The functions $E_K(\psi(\cdot),\vartheta(\cdot))$ are nonincreasing and the limits*
$$\lim_{t\to\infty}E_K(\psi(t),\vartheta(t))=:E_K^\infty$$
exists.

(ii) *The ω-limit set*
$$\omega(\psi,\vartheta):=\{(\varphi,\theta)\in W:\exists\ (t_n)_{n\in\mathbb{N}}\nearrow\infty,\ s.t.\ (\psi(t_n),\vartheta(t_n))\to(\varphi,\theta)\ in\ W\}$$
is nonempty, compact, connected and E_K is constant on $\omega(\psi,\vartheta)$.

(iii) *For every $(\varphi,\theta)\in\omega(\psi,\vartheta)$ it holds that $\theta=c^*$ and (φ,θ) is a solution of the stationary problem (5.6).*

(iv) *Every $(\varphi,\theta)\in\omega(\psi,\vartheta)$ is a critical point of E_K, i.e. $E_K'(\varphi,\theta)=0$.*

Proof. By (5.4) the functions $E_N(\psi(\cdot),\vartheta(\cdot))$ and $E_R(\psi(\cdot),\vartheta(\cdot))$ are nonincreasing, hence the limits $\lim_{t\to\infty}E_K(\psi(t),\vartheta(t))$ exist, since $E_K(\psi(\cdot),\vartheta(\cdot))$ are bounded from below. This yields (i). By Proposition 5.1, the solution (ψ,ϑ) has relatively compact range in W. Therefore, by well-known results, the ω-limit set is nonempty, compact and connected. The fact that E_K is constant on the ω-limit set, follows easily from continuity of E_K on W and (i).

Now let $(\varphi,\theta)\in\omega(\psi,\vartheta)$ and let $(t_n)_{n\in\mathbb{N}}\nearrow\infty$, such that $(\psi(t_n),\vartheta(t_n))\to(\varphi,\theta)$ in W, as $n\to\infty$. From (5.4) and the boundedness of $E_K(\psi(\cdot),\vartheta(\cdot))$ we obtain

$$\nabla\mu\in L_2(\mathbb{R}_+;L_2(\Omega)),\ \partial_t\psi|_\Gamma\in L_2(\mathbb{R}_+;L_2(\Gamma))\quad\text{and}\quad\nabla\vartheta\in L_2(\mathbb{R}_+;L_2(\Omega)).$$

This and again (5.4) imply $\partial_t\psi,\partial_t\vartheta\in L_2(\mathbb{R}_+;H_2^1(\Omega)^*)$ and for every $s\in[0,1]$,

$$\lim_{n\to\infty}\psi(t_n+s)=\varphi\text{ in }H_2^1(\Omega)^*,$$

$$\lim_{n\to\infty}\psi(t_n+s)=\varphi\text{ in }L_2(\Gamma)\text{ and}$$

$$\lim_{n\to\infty}\vartheta(t_n+s)=\theta\text{ in }H_2^1(\Omega)^*.$$

Since (ψ,ϑ) has relatively compact range in W, the above three limits exist even in $H_2^1(\Omega)$, $H_2^1(\Gamma)$ and $L_2(\Omega)$, respectively. Integrating the equations $(5.1)_1-(5.1)_5$ we finally obtain that $\theta=c^*=const$ and (φ,c^*) solves (5.6), hence the assertion (iii).

Now we turn to (iv). Suppose that $(u,v)=(\varphi,\theta)\in\omega(\psi,\vartheta)$. Then by (iii), $\theta=c^*$ and (φ,c^*) solves (5.6). Proposition 5.2, (5.3) and integration by parts lead to

$$\langle E_N'(\varphi,c^*),(h,k)\rangle_{W^*,W}=(c^*+\overline{\lambda(\varphi)})\int_\Omega(k+\lambda'(\varphi)h)=0,$$

and

$$\langle E_R'(\varphi,c^*),(h,k)\rangle_{W^*,W}=c^*\int_\Omega(k+\lambda'(\varphi)h)=0,$$

since in case of Robin boundary conditions we have $c^*=0$ and in the other case we have $\int_\Omega(\vartheta+\overline{\lambda(\psi)})=\int_\Omega(\vartheta+\lambda(\psi))=0$ and for some sequence $t_n\to\infty$ it holds $(\psi(t_n),\vartheta(t_n))\to(\varphi,c^*)$ as $n\to\infty$ in W, hence $\int_\Omega(c^*+\overline{\lambda(\varphi)})=0$, and therefore $c^*+\overline{\lambda(\varphi)}=0$. The proposition is proved. □

Now we are in a position to state our main result concerning the asymptotic behaviour of solutions of the Cahn-Hilliard equation.

Theorem 5.5. *Let (ψ,ϑ) be a global solution of the Cahn-Hilliard equation* (5.1) *with $h=\frac{1}{|\Omega|}\int_\Omega\psi_0$ and suppose that Φ satisfies the conditions* (4.1) *as well as* (4.3) *and let $\lambda',\lambda'',\lambda'''\in L_\infty(\mathbb{R})$. Assume that Φ is real analytic, and that λ is real analytic, if $\alpha=0$. Then the limits*

$$\lim_{t\to\infty}\psi(t)=:\varphi,\quad\text{and}\quad\lim_{t\to\infty}\vartheta(t)=:c^*=const$$

exist in $H_2^1(\Omega)\cap H_2^1(\Gamma)$ and $L_2(\Omega)$, respectively, and (φ,c^*) is a solution of the stationary problem (5.6).

Proof. Proposition 5.3 yields, that for every $(\varphi,\theta)\in\omega(\psi,\vartheta)$ there exist constants $s\in(0,\frac12]$, $C>0$ and $\delta>0$ such that

$$|E_K(u,v)-E_K(\varphi,\theta)|^{1-s}\le C|E_K'(u,v)|_{W^*},$$

whenever $|(u,v)-(\varphi,\theta)|_W\le\delta$. By Proposition 5.4 (iii) the ω-limit set $\omega(\psi,\vartheta)$ is compact, hence we may cover it by a union of *finitely* many balls with center (φ_i,θ_i) and radius δ_i, $i=1,\dots,N$. Since $E_K(u,v)\equiv E_K^\infty$ on $\omega(\psi,\vartheta)$, there are *uniform* constants $s\in(0,\frac12]$, $C>0$ and an open set $U\supset\omega(\psi,\vartheta)$, with

$$|E_K(u,v)-E_K^\infty|^{1-s}\le C|E_K'(u,v)|_{W^*},\tag{5.9}$$

for all $(u,v)\in U$. After these preliminaries, we define the function $H:\mathbb{R}_+\to\mathbb{R}_+$ by

$$H(t):=(E_K(\psi(t),\vartheta(t))-E_K^\infty)^s.$$

By Proposition 5.4 the function H is nonincreasing and $\lim_{t\to\infty}H(t)=0$. A well known result in the theory of dynamical systems implies further that $\lim_{t\to\infty}\operatorname{dist}((\psi(t),\vartheta(t)),\omega(\psi,\vartheta))=0$, i.e. there exists $t^*\ge0$, such that $(\psi(t),\vartheta(t))\in U$, whenever $t\ge t^*$. Next, we compute and estimate the time derivative of H. By (5.4) and (5.9) we obtain

$$\begin{aligned}-\frac{d}{dt}\,H(t)&=s\left(-\frac{d}{dt}\,E_K(\psi(t),\vartheta(t))\right)|E_K(\psi(t),\vartheta(t))-E_K^\infty|^{s-1}\\&\ge C\frac{|\nabla\mu|_2^2+|\nabla\vartheta|_2^2+|\partial_t\psi|_{2,\Gamma}^2+\alpha|\vartheta|_{2,\Gamma}^2}{|E_K'(\psi(t),\vartheta(t))|_{W^*}}.\end{aligned}\tag{5.10}$$

Together with the Cauchy-Schwartz inequality, integration by parts, as well as by (5.2) and by (5.3), we obtain

$$
\begin{aligned}
|\langle E_N'(\psi,\vartheta),(h,k)\rangle_{W^*,W}| &= |\int_\Omega (\mu-\overline{\mu})h + \int_\Omega (\vartheta+\overline{\lambda(\psi)})(k+\lambda'(\psi)h) + \int_\Gamma \partial_t\psi h| \\
&\le C(|\nabla\mu|_2|h|_2 + (|h|_2+|k|_2)|\nabla\vartheta|_2 + |\partial_t\psi|_{2,\Gamma}|h|_{H_2^1(\Omega)}),
\end{aligned} \tag{5.11}
$$

and

$$
\begin{aligned}
|\langle E_R'(\psi,\vartheta),(h,k)\rangle_{W^*,W}| &= |\int_\Omega (\mu-\overline{\mu})h + \int_\Omega \vartheta(k+\lambda'(\psi)h) + \int_\Gamma \partial_t\psi h| \\
&\le C(|\nabla\mu|_2|h|_2 + (|h|_2+|k|_2)(|\nabla\vartheta|_2 + \alpha^{1/2}|\vartheta|_{2,\Gamma}) + |\partial_t\psi|_{2,\Gamma}|h|_{H_2^1(\Omega)}),
\end{aligned} \tag{5.12}
$$

respectively, where $\bar{\mu} = \frac{1}{|\Omega|}\int_\Omega \mu$. Here we applied the Poincaré inequality to the functions $\mu - \bar{\mu}$ and $\vartheta + \overline{\lambda(\psi)}$, since their means are 0. Concerning ϑ in E_R', we made use of a version of the Poincaré inequality, namely

$$|w|_2 \le C(|\nabla w|_2 + |w|_{2,\Gamma}), \quad w \in H_2^1(\Omega).$$

If we take the supremum in (5.11) and (5.12) over all $(h,k)\in W$ with norm less than 1 this results in

$$|E_K'(\psi(t),\vartheta(t))|_{W^*} \le C(|\nabla\mu(t)|_2 + |\nabla\vartheta(t)|_2 + \alpha^{1/2}|\vartheta(t)|_{2,\Gamma} + |\partial_t\psi(t)|_{2,\Gamma}).$$

Hence from (5.10) it follows that

$$-\frac{d}{dt}\,H(t) \ge C(|\nabla\mu(t)|_2 + |\nabla\vartheta(t)|_2 + \alpha^{1/2}|\vartheta(t)|_{2,\Gamma} + |\partial_t\psi(t)|_{2,\Gamma})$$

and this in turn implies that $\nabla\mu,\nabla\vartheta \in L_1([t^*,\infty),L_2(\Omega))$ and $\vartheta,\partial_t\psi \in L_1([t^*,\infty);L_2(\Gamma))$. Furthermore this yields $\partial_t\psi,\partial_t\vartheta \in L_1([t^*,\infty),H_2^1(\Omega)^*)$, hence the limits for $t\to\infty$ exist in $L_2(\Gamma)$ and $H_2^1(\Omega)^*$, respectively. Therefore the claim follows from relative compactness of $(\psi(t),\vartheta(t))$ in $(H_2^1(\Omega)\cap H_2^1(\Gamma))\times L_2(\Omega)$ and Proposition 5.4 (iii). □

6. Appendix

(a) Proof of Proposition 3.2.

(i) By Hölders inequality it holds that

$$
\begin{aligned}
|\Delta\Phi'(u)-\Delta\Phi'(v)|_p &\le |\Delta u\Phi''(u) - \Delta v\Phi''(v)|_p + ||\nabla u|^2\Phi'''(u) - |\nabla v|^2\Phi'''(v)|_p \\
&\le |\Delta u|_{rp}|\Phi''(u)-\Phi''(v)|_{r'p} + |\Delta u - \Delta v|_{rp}|\Phi''(v)|_{r'p} \\
&\quad + |\nabla u|_{2\sigma p}|\Phi'''(u)-\Phi'''(v)|_{\sigma' p} + ||\nabla u|^2 - |\nabla v|^2|_{\sigma p}|\Phi'''(v)|_{\sigma' p} \\
&\le T^{1/r'p}\left(|\Delta u|_{rp}|\Phi''(u)-\Phi''(v)|_\infty + |\Delta u-\Delta v|_{rp}|\Phi''(v)|_\infty\right) \\
&\quad + T^{1/\sigma' p}\left(|\nabla u|_{2\sigma p}|\Phi'''(u)-\Phi'''(v)|_\infty + ||\nabla u|^2 - |\nabla v|^2|_{\sigma p}|\Phi'''(v)|_\infty\right),
\end{aligned}
$$

where we also used the fact that all functions belonging to $\mathbb{B}_R(u^*)$ are uniformly bounded. We have

$$\nabla w \in H_p^{3\theta_1/4}(J;H_p^{3(1-\theta_1)}(\Omega)) \hookrightarrow L_{2\sigma p}(J\times\Omega), \quad \theta_1\in[0,1],$$

and

$$\Delta w \in H_p^{\theta_2/2}(J;H_p^{2(1-\theta_2)}(\Omega)) \hookrightarrow L_{rp}(J\times\Omega), \quad \theta_2\in[0,1],$$

for every function $w\in\mathbb{B}_R(u^*)$, since $r,\sigma>1$ may be chosen close to 1. Therefore we have

$$|\Delta\Phi'(u)-\Delta\Phi'(v)|_p \le \mu_1(T)\,(R+|u^*|_1)\,|u-v|_1,$$

due to the assumption $\Phi\in C^{4-}(\mathbb{R})$. The function μ_1 is given by $\mu_1(T)=\max\{T^{1/r'p},T^{1/\sigma' p}\}$. This yields (i).

(ii) Firstly we observe that

$$\Delta(\lambda'(w)F(w)) = (\Delta w\lambda''(w) + |\nabla w|^2\lambda'''(w))F(w) + 2\lambda''(w)\nabla w\cdot\nabla F(w) + \lambda'(w)\Delta F(w),$$

for all $w\in\mathbb{B}_R(u^*)$. Secondly by (3.6), the embeddings

$$F(w)\in H_p^{s+\theta-1}(H_p^{3-2\theta}) \hookrightarrow L_{2p}(J\times\Omega) \quad\text{and}\quad \nabla F(w)\in H_p^{s+\theta-1}(H_p^{2-2\theta}) \hookrightarrow L_{4p/3}(J\times\Omega),$$

with $s\in\left[\frac{1}{2},\frac{3}{4}\right)$, $\theta\in[0,1]$, are valid, whenever

$$p > \frac{2}{2s+1}(\frac{n}{4}+\frac{1}{2}) \quad\text{and}\quad p > \frac{1}{s}(\frac{n}{8}+\frac{1}{4}),$$

respectively. It is obvious, that these conditions are fulfilled for every $s\in\left[\frac{1}{2},\frac{3}{4}\right)$, whenever $p>n/4+1$. An easy computation shows that $\nabla w\in L_{4p}(J\times\Omega)$ and $\Delta w\in L_{2p}(J\times\Omega)$, if $p>n/4+1$ (here we use strict embeddings). If $1/\sigma + 1/\sigma' = 1$ and $\sigma > 1$ is sufficiently small, then Hölder's inequality and Proposition 3.1 lead to the estimate

$$\begin{aligned}|\lambda''(u)\Delta uF(u) - \lambda''(v)\Delta vF(v)|_p &\le C|\Delta uF(u) - \Delta vF(v)|_p + |\lambda''(u) - \lambda''(v)|_\infty|\Delta vF(v)|_p\\ &\le T^{1/2\sigma' p}(|\Delta u|_{2\sigma p}|F(u) - F(v)|_{2p} + |\Delta u - \Delta v|_{2\sigma p}|F(v)|_{2p}\\ &\qquad\qquad + |\lambda''(u) - \lambda''(v)|_\infty|\Delta v|_{2\sigma p}|F(v)|_{2p})\\ &\le \mu_2(T)(1 + |u^*|_1)|u - v|_1.\end{aligned}$$

In a similar way we obtain

$$|\lambda'''(u)|\nabla u|^2F(u) - \lambda'''(v)|\nabla v|^2F(v)|_p \le \mu_2(T)(1 + |u^*|_1)|u - v|_1,$$

$$|\lambda''(u)\nabla u\nabla F(u) - \lambda''(v)\nabla v\nabla F(v)|_p \le \mu_2(T)(1 + |u^*|_1)|u - v|_1,$$

as well as

$$|\lambda'(u)\Delta F(u) - \lambda'(v)\Delta F(v)|_p \le \mu_2(T)(1 + |u^*|_1)|u - v|_1,$$

for all $u, v \in \mathbb{B}_R(u^*)$. This proves (ii).
(iii) This is an easy consequence of (i) and (3.5), since by trace-theory we obtain

$$|\partial_\nu\Phi'(u) - \partial_\nu\Phi'(v)|_{Y_1(T)} \le C\left(|\Phi'(u) - \Phi'(v)|_{H_p^{1/2}(L_p)} + |\Phi'(u) - \Phi'(v)|_{L_p(H_p^2)}\right).$$

(iv) In a similar way as in (iii) we obtain

$$\begin{aligned}&|\partial_\nu(\lambda'(u)F(u)) - \partial_\nu(\lambda'(v)F(v))|_{Y_1(T)}\\ &\qquad\le C\left(|\lambda'(u)F(u) - \lambda'(v)F(v)|_{H_p^{1/2}(L_p)} + |\lambda'(u)F(u) - \lambda'(v)F(v)|_{L_p(H_p^2)}\right).\end{aligned}$$

The desired Lipschitz-estimate for the second term follows from (ii). The first term will be rewritten in the usual way, i.e.

$$\begin{aligned}&|\lambda'(u)F(u) - \lambda'(v)F(v)|_{H_p^{1/2}(L_p)}\\ &\qquad\le \left(|(\lambda'(u) - \lambda'(v))F(u)|_{H_p^{1/2}(L_p)} + |\lambda'(v)(F(u) - F(v))|_{H_p^{1/2}(L_p)}\right).\end{aligned}$$

Applying (3.4) we obtain

$$\begin{aligned}&|(\lambda'(u) - \lambda'(v))F(u)|_{H_p^{1/2}(L_p)}\\ &\qquad\le C\left(|\lambda'(u) - \lambda'(v)|_{H_{rp}^{1/2}(L_{rp})}|F(u)|_{L_{r'p}(L_{r'p})} + T^{1/\sigma'}|\lambda'(u) - \lambda'(v)|_\infty|F(u)|_{H_{\sigma p}^{1/2}(L_{\sigma p})}\right),\end{aligned}$$

as well as

$$\begin{aligned}&|\lambda'(v)(F(u) - F(v))|_{H_p^{1/2}(L_p)}\\ &\qquad\le C\left(|\lambda'(v)|_{H_{rp}^{1/2}(L_{rp})}|F(u) - F(v)|_{L_{r'p}(L_{r'p})} + T^{1/\sigma'}|\lambda'(v)|_\infty|F(u) - F(v)|_{H_{\sigma p}^{1/2}(L_{\sigma p})}\right).\end{aligned}$$

Let $w \in \mathbb{B}_R(u^*)$. It is obvious that $F(w) \in H_{\sigma p}^{1/2}(J; L_{\sigma p}(\Omega))$, since $\sigma > 1$ may be chosen arbitrarily close to 1. So it remains to check if $\lambda'(w) \in H_{rp}^{1/2}(J; L_{rp}(\Omega))$ and $F(w) \in L_{r'p}(J; L_{r'p}(\Omega))$. It holds

$$H_p^\theta(J; H_p^{4(1-\theta)}(\Omega)) \hookrightarrow H_{rp}^{1/2}(J; L_{rp}(\Omega)) \quad\text{and}\quad H_p^{s+\theta-1}(J; H_p^{3-2\theta}(\Omega)) \hookrightarrow L_{r'p}(J; L_{r'p}(\Omega))$$

if $p > \frac{2}{r'}(\frac{n}{4} + 1)$ and $p > \frac{2}{r(2s+1)}(\frac{n}{2} + 1)$, respectively. Thus we set $r' = r = 2$. Now the claim follows from (3.5) and Proposition 3.1.
(v) With the help of Hölder's inequality we compute

$$|(\Delta\lambda'(u) - \Delta\lambda'(v))\eta|_p \le |\eta|_{Z^2}(|\Delta u\lambda''(u) - \Delta v\lambda''(v)|_{2p} + ||\nabla u|^2\lambda'''(u) - |\nabla v|^2\lambda'''(v)|_{2p}), \tag{6.1}$$

for each $\eta \in Z^2$. Since $\lambda \in C^{4-}(\mathbb{R})$, it follows from the uniform boundedness of $u, v \in \mathbb{B}_R(u^*)$ that

$$\begin{aligned}|\Delta u\lambda''(u) - \Delta v\lambda''(v)|_{2p} &\le |\lambda''(u) - \lambda''(v)|_\infty|\Delta u|_{2p} + |\lambda''(v)|_\infty|\Delta u - \Delta v|_{2p}\\ &\le C(1 + |u^*|_1)|u - v|_1.\end{aligned}$$

In a similar way the second term in (6.1) can be treated, obtaining

$$||\nabla u|^2\lambda'''(u) - |\nabla v|^2\lambda'''(v)|_{2p} \le C(1 + |u^*|_1)|u - v|_1.$$

Furthermore we have

$$\begin{aligned}|(\nabla u\lambda''(u)-\nabla v\lambda''(v))\nabla\eta|_p &\le |\nabla\eta|_{4p/3}|\nabla u\lambda''(u)-\nabla v\lambda''(v)|_{4p}\\ &\le |\eta|_{Z^2}(|\lambda''(u)-\lambda''(v)|_\infty|\nabla u|_{4p}+|\lambda''(v)|_\infty|\nabla u-\nabla v|_{4p})\\ &\le C|\eta|_{Z^2}(1+|u^*|_1)|u-v|_1\end{aligned}$$

and

$$|(\lambda'(u)-\lambda'(v))\Delta\eta|_p\le|\lambda'(u)-\lambda'(v)|_\infty|\Delta\eta|_p\le C|\eta|_{Z^2}|u-v|_1,$$

by Hölder's inequality and the Lipschitz-property of λ', λ''.
(vi) Finally we apply trace-theory to obtain

$$|\partial_\nu(\lambda'(u)\eta)-\partial_\nu(\lambda'(v)\eta)|_{Y_1(T)}\le C\left(|(\lambda'(u)-\lambda'(v))\eta|_{H^{1/2}_p(L_p)}+|(\lambda'(u)-\lambda'(v))\eta|_{L_p(H^2_p)}\right).$$

The estimate for the second term is clear by (v). Again we will use (3.4) to estimate the first term. This yields

$$\begin{aligned}&|(\lambda'(u)-\lambda'(v))\eta|_{H^{1/2}_p(L_p)}\\ &\quad\le C(T_0)\left(|\lambda'(u)-\lambda'(v)|_{H^{1/2}_{rp}(L_{rp})}|\eta|_{L_{r'p}(L_{r'p})}+|\lambda'(u)-\lambda'(v)|_{L_{\sigma'p}(L_{\sigma'p})}|\eta|_{H^{1/2}_{\sigma p}(L_{\sigma p})}\right).\end{aligned}$$

As in (iv), it follows that $r=2$. Furthermore we have

$$H^\theta_p(J;H^{2(1-\theta)}_p(\Omega))\hookrightarrow L_{2p}(J;L_{2p}(\Omega)),$$

if $p>\frac{n}{4}+1$. Last but not least we apply (3.5). The proof is complete.

(b) Proof of Lemma 4.1.
Step 1. We start with $\Delta\Phi'(\psi)=\Delta\psi\Phi''(\psi)+|\nabla\psi|^2\Phi'''(\psi)$. Using the Gagliardo-Nirenberg inequality and (4.3) we obtain

$$|\Phi''(\psi)\Delta\psi|_p\le|\psi|^{\beta+1}_{2(\beta+1)p}|\Delta\psi|_{2p}\le C|\psi|^{a+b(\beta+1)}_{H^4_p}|\psi|_q^{1-a+(1-b)(\beta+1)}, \tag{6.2}$$

where q will be chosen in such a way that $H^1_2(\Omega)\hookrightarrow L_q$, i.e. $\frac{n}{q}\ge\frac{n}{2}-1$ and

$$(a+(\beta+1)b)\left(4-\frac{n}{p}+\frac{n}{q}\right)=2-\frac{n}{p}+\frac{n}{q}(\beta+2).$$

The second term $\Phi'''(\psi)|\nabla\psi|^2$ will be treated in a similar way. The Gagliardo-Nirenberg inequality and (4.3) again yield

$$|\Phi'''(\psi)|\nabla\psi|^2|_p\le|\psi|^\beta_{2\beta p}|\nabla\psi|^2_{4p}\le C|\psi|^{2a+b\beta}_{H^4_p}|\psi|_q^{2-2a+(1-b)\beta}, \tag{6.3}$$

with $\frac{n}{q}\ge\frac{n}{2}-1$ and

$$(2a+\beta b)\left(4-\frac{n}{p}+\frac{n}{q}\right)=2-\frac{n}{p}+(\beta+2)\frac{n}{q}.$$

It turns out that the condition $\beta<3$ in case $n=3$ ensures that either $a+(\beta+1)b<1$ and $2a+\beta b<1$ in (6.2) and (6.3), respectively. Integrating (6.2) and (6.3) with respect to t and using Hölders inequality we obtain the desired estimate. Now we estimate $\partial_\nu\Phi'(\psi)$ in Y_1. By trace-theory we obtain

$$|\partial_\nu\Phi'(\psi)|_{Y_1}\le C(|\Phi'(\psi)|_{H^{1/2}_p(L_p)}+|\Phi'(\psi)|_{L_p(H^2_p)}).$$

The estimate in $L_p(H^2_p)$ follows from the considerations above. By the mean-value theorem and (4.3) we obtain

$$|\Phi'(\psi)|_{H^{1/2}_p(L_p)}\le C(|\psi|^{\beta+2}_{L_{(\beta+2)p}(L_{(\beta+2)p})}+|\psi|_{H^{1/2}_p(L_p)}+|\psi|^{\beta+1}_{L_{\sigma'p}(L_{r'p})}|\psi|_{H^{1/2}_{\sigma p}(L_{rp})}), \tag{6.4}$$

where $1/\sigma+(\beta+1)/\sigma'=1/r+(\beta+1)/r'=1$. This follows similarly to (3.4) from the characterization of H^s_p via differences and Hölders inequality. The Gagliardo-Nirenberg inequality implies

$$|\psi|_{L_{\sigma'p}(L_{r'p})}\le c|\psi|^a_{Z^1}|\psi|^{1-a}_{L_\infty(H^1_2)},$$

if $a\in[0,1/\sigma']$ and $a(3-n/p+n/2)\ge n/2-1-n/r'p$. Therefore we set

$$a=1/\sigma'=[n/2-1-n/r'p]_+/(3-n/p+n/2)$$

and choose $r'=(\beta+1)n/p$ if $p\le n$, $r'=2(\beta+1)n/p$ if $n<p\le 2n$ and $r'=\infty$ if $p>2n$. Observe that

$$Z^1\hookrightarrow H^{1-\theta}_p(H^{4\theta}_p)\hookrightarrow H^s_p(L_{rp}),$$

if $s = 1 - n/4r'p$. Hence complex interpolation yields

$$|\psi|_{H^{1/2}_{\sigma p}(L_{rp})} \le c|\psi|^b_{Z^1}|\psi|^{1-b}_{L_{\tau p}(L_{rp})},$$

provided $b = 1/2s$ and

$$1/\sigma \ge b + (1-b)/\tau. \tag{6.5}$$

Finally we apply the Gagliardo-Nirenberg inequality one more time to obtain

$$|\psi|_{L_{\tau p}(L_{rp})} \le c|\psi|^d_{Z^1}|\psi|^{1-d}_{L_\infty(H^1_2)},$$

with $d(3 - n/p + n/2) \ge n/2 - 1 - n/rp$ and $d \in [0, 1/\tau]$. We set

$$d = 1/\tau = [n/2 - 1 - n/rp]_+/(3 - n/p + n/2).$$

Suppose now that (6.5) holds. Then we have

$$(\beta+1)a + b + (1-b)d = \frac{\beta+1}{\sigma'} + b + \frac{1-b}{\tau} \le \frac{\beta+1}{\sigma'} + \frac{1}{\sigma} = 1.$$

Hence the desired estimate follows if the inequality (6.5) is strict. We have to distinguish three cases, namely $p \le n$, $n < p \le 2n$ and $p > 2n$. In the first case we have $r' = (\beta+1)n/p$, $s = (4\beta+3)/(4\beta+4)$ and $b = (2\beta+2)/(4\beta+3)$. Since $p \ge 2$, (6.5) is equivalent to

$$\frac{6\beta+3}{4\beta+3} - (\beta+1)[n/2 - 1/(\beta+1) - 1]_+ \ge 0.$$

We see that $[n/2 - 1/(\beta+1) - 1]_+ = 0$, if either $n = 1, 2$ or $n = 3$ and $\beta \le 1$; then we are done. So let $n = 3$ and $\beta > 1$. An easy calculation shows that

$$\frac{6\beta+3}{4\beta+3} > \frac{\beta-1}{2},$$

for all $1 < \beta < 3$. In the second case we have $r' = 2(\beta+1)n/p$, $s = (8\beta+7)/(8\beta+8)$ and $b = (4\beta+4)/(8\beta+7)$, thus (6.5) is equivalent to

$$\frac{4\beta+3}{8\beta+7}(3 - n/p + n/2) - (\beta+1)[n/2 - 1/2(\beta+1) - 1]_+ \ge \frac{4\beta+3}{8\beta+7}[n/2 - 1/2 - n/p]_+.$$

Since $p \le 2n$ we see that this inequality is strict for $n = 1, 2$. If $n = 3$ and due to $p > n$, the inequality reduces to

$$\frac{4\beta+3}{8\beta+7} \ge \beta/7$$

and we have again strict inequality, if $\beta < 3$. In the last case, we have $r' = \infty$, $s = 1$ and $b = 1/2$. Therefore (6.5) is equivalent to

$$(3 - n/p + n/2) - 2(\beta+1)[n/2 - 1]_+ \ge [n/2 - n/p - 1]_+.$$

Again for $n = 1, 2$ we have $[n/2 - 1]_+ = [n/2 - n/p - 1]_+ = 0$, thus we set $n = 3$. Since $p > 2n$ this yields

$$4 - (\beta+1) \ge 0,$$

hence strict inequality if $\beta < 3$. Note that the second term on the right hand side of (6.4) is dominated by the third term. Furthermore the desired estimate for the first term is a simple consequence of the Gagliardo-Nirenberg inequality as long as $q > (\beta+1)n/4$.

Step 2. Next we estimate the term $\Delta(\lambda'(\psi)F(\psi))$ in X. Observe that

$$\Delta(\lambda'(\psi)F(\psi)) = F(\psi)(\Delta\psi\lambda''(\psi) + |\nabla\psi|^2\lambda'''(\psi)) + 2\lambda''(\psi)\nabla\psi\nabla F(\psi) + \lambda'(\psi)\Delta F(\psi).$$

As in section 2 we will solve the homogeneous heat equation (2.2). Obviously $|F(\psi)|_{L_p(L_p)} \le C(1 + |\psi|_{L_p(L_p)})$, for every $1 < p < \infty$, since $|\lambda(s)| \le C(1 + |s|)$. Applying the Gagliardo-Nirenberg one more time we obtain

$$\begin{aligned}|\lambda''(\psi)F(\psi)\Delta\psi|_{L_p(L_p)} &\le C(|\Delta\psi|_{L_{3p/2}(L_{3p/2})} + |\Delta\psi|_{L_{3p/2}(L_{3p/2})}|\psi|_{L_{3p}(L_{3p})}) \\ &\le C(|\psi|^a_{Z^1}|\psi|^{1-a}_{L_\infty(L_q)} + |\psi|^{a+b}_{Z^1}|\psi|^{2-(a+b)}_{L_\infty(L_q)}),\end{aligned}$$

if $a(4 - n/p + n/q) = 2 - 2n/3p + n/q$ and $b(4 - n/p + n/q) = n/q - n/3p$ and $a \in [1/2, 1]$, $b \in [0, 1]$. The two latter conditions are fulfilled if $q \le 3p$. We require furthermore $a < 2/3$ and $b < 1/3$. This leads to the condition $q > n/2$, which is true. Then we also have $a + b < 1$. In a similar way we estimate $\lambda'''(\psi)F(\psi)|\nabla\psi|^2$, to obtain

$$\begin{aligned}|\lambda'''(\psi)F(\psi)|\nabla\psi|^2|_{L_p(L_p)} &\le C(1 + |\psi|_{L_{3p}(L_{3p})})|\nabla\psi|^2_{L_{3p}(L_{3p})} \\ &\le C(|\psi|^{2a}_{Z^1}|\psi|^{2(1-a)}_{L_\infty(L_q)} + |\psi|^{2a+b}_{Z^1}|\psi|^{3-(2a+b)}_{L_\infty(L_q)}),\end{aligned} \tag{6.6}$$

whenever $a(4-n/p+n/q) = 1+n/q-n/3p$ and $b(4-n/p+n/q) = n/q-n/3p$ and $a \in [1/4, 1]$, $b \in [0, 1]$. The two latter conditions are satisfied if $q \le 3p$. It is easy to verify that $a \le 1/3$ and $b < 1/3$, whenever $q \ge 2n$, i.e. $q = 6$. Finally it holds $2a + b < 1$.
Note that the representation of $F(\psi)$ implies

$$|\nabla F(\psi)|_{L_{2p}(L_{2p})} \le C(1 + |\nabla\lambda(\psi)|_{L_{2p}(L_{2p})}) \le C(1 + |\nabla\psi|_{L_{2p}(L_{2p})}),$$

hence by the inequalities of Hölder and Gagliardo-Nirenberg we obtain

$$\begin{aligned}|\lambda''(\psi)\nabla F(\psi)\nabla\psi|_{L_p(L_p)} &\le |\nabla F(\psi)|_{L_{2p}(L_{2p})}|\nabla\psi|_{L_{2p}(L_{2p})}\\ &\le C(1 + |\nabla\psi|^2_{L_{2p}(L_{2p})}) \le C(1 + |\psi|^{2a}_{Z^1}|\psi|^{2(1-a)}_{L_\infty(L_q)}),\end{aligned}$$

with

$$2a\left(4 - \frac{n}{p} + \frac{n}{q}\right) = 2 + \frac{2n}{q} - \frac{n}{p}, \quad a \in [1/4, 1].$$

Since $q \le 3p$ we see that $a \ge 1/4$. Furthermore we have $2a < 1$, if $n < 6$. The estimate of $\lambda'(\psi)\Delta F(\psi)$ in L_p is more involved. With the help of (3.6) with $s = \theta = 1/2$, we obtain

$$|\lambda'(\psi)\Delta F(\psi)|_{L_p(L_p)} \le C(1 + |\lambda(\psi)|_{H^{1/2}_p(H^1_p)}) \le C(1 + |\psi|_{H^{1/2}_p(L_p)} + |\nabla\psi\lambda'(\psi)|_{H^{1/2}_p(L_p)}).$$

A similar estimate for ψ in $H^{1/2}_p(L_p)$ has already been done in (6.4). For the term $\nabla\psi\lambda'(\psi)$ we will use (3.4). This leads to

$$|\nabla\psi\lambda'(\psi)|_{H^{1/2}_p(L_p)} \le c(|\nabla\psi|_{H^{1/2}_p(L_p)} + |\nabla\psi|_{L_{\sigma'p}(L_{r'p})}|\psi|_{H^{1/2}_{\sigma p}(L_{rp})}),$$

since $\lambda' \in L_\infty(\mathbb{R})$. We will use the same strategy as in (6.4). First we observe that complex-interpolation and the Gagliardo-Nirenberg inequality lead to the desired estimate for $|\nabla\psi|_{H^{1/2}_p(L_p)}$. Secondly we make again use of the Gagliardo-Nirenberg inequality to obtain

$$|\nabla\psi|_{L_{\sigma'p}(L_{r'p})} \le c|\psi|^a_{Z^1}|\psi|^{1-a}_{L_\infty(H^1_2)},$$

where $a = 1/\sigma' = (n/2 - n/r'p)/(3 + n/2 - n/p)$. Complex interpolation yields

$$|\psi|_{H^{1/2}_{\sigma p}(L_{rp})} \le c|\psi|^b_{H^s_p(L_{rp})}|\psi|^{1-b}_{L_{\tau p}(L_{rp})}, \tag{6.7}$$

where $b = 1/2s$, $s = 1 - n/4r'p$ and

$$1/\sigma \ge b + (1-b)/\tau. \tag{6.8}$$

One more time the Gagliardo-Nirenberg inequality leads to the estimate

$$|\psi|_{L_{\tau p}(L_{rp})} \le c|\psi|^d_{Z^1}|\psi|^{1-d}_{L_\infty(H^1_2)}, \tag{6.9}$$

with $d = 1/\tau = [n/2 - 1 - n/rp]_+/(3 - n/p + n/2)$. Finally we have to check if (6.8) is valid and if in this case the inequality is strict. We distinguish two cases. If $p \le n$ we set $r' = 2n/p$ and if $p > n$ we set $r' = 2$ (then $r' \in [2, 3]$). In the first case we have $s = 7/8$, $b = 4/7$ and $[n/2 - 1/2 - n/p]_+ = 0,\ n = 1, 2, 3$, since $p \ge 2$. Thus (6.8) is equivalent to the condition

$$p \ge \frac{6n}{25 - 4n}\ ,$$

which is always fulfilled and strict inequality holds if $n \le 3$. In the second case we have $r' = 2$, thus $s = 1 - n/8p > 7/8$. We set $s = 7/8$ and therefore $b = 4/7$. Then (6.8) is equivalent to

$$18 - 4n + n/p \ge 6[n/2 - 1 - n/2p]_+.$$

This inequality is obviously fulfilled and additionally strict, if $n = 1, 2$. So let $n = 3$. Then $n/2-1-n/2p > 0$ and we obtain $1 + 4/p > 0$, which is certainly true.
The next estimate will be done for the term $\partial_\nu(\lambda'(\psi)F(\psi))$ in Y_1. Again we use trace-theory to obtain

$$|\partial_\nu(\lambda'(\psi)F(\psi))|_{Y_1} \le C(|\lambda'(\psi)F(\psi)|_{H^{1/2}_p(L_p)} + |\lambda'(\psi)F(\psi)|_{L_p(H^2_p)}).$$

The estimate of $\lambda'(\psi)F(\psi)$ in $L_p(H^2_p)$ has already been done. Making use of (3.4) and (3.6), with $s = 1/2$ and L_p instead of H^1_p, we obtain

$$|\lambda'(\psi)F(\psi)|_{H^{1/2}_p(L_p)} \le C\left(1 + |\psi|_{H^{1/2}_p(L_p)} + (1 + |\psi|_{L_{\sigma'p}(L_{r'p})})(1 + |\psi|_{H^{1/2}_{\sigma p}(L_{rp})})\right),$$

since $\lambda', \lambda'' \in L_\infty(\mathbb{R})$. Therefore the estimate follows immediately from Step 1 .

Step 3. Last but not least we have to consider $\Delta(\lambda'(\psi)\eta)$ in X and $\partial_\nu(\lambda'(\psi)\eta)$ in Y_1, where $\eta \in Z^2$ is a fixed function. We compute

$$\Delta(\lambda'(\psi)\eta) = \eta(\Delta\psi\lambda''(\psi) + |\nabla\psi|^2\lambda'''(\psi)) + 2\lambda''(\psi)\nabla\psi\nabla\eta + \lambda'(\psi)\Delta\eta.$$

Since $p \geq 2$ we have $Z^2 \hookrightarrow L_{3p}(J \times \Omega)$, hence the estimate for the first term follows from Step 2. Moreover by (6.6) we obtain

$$|\nabla\psi\nabla\eta|_{L_p(L_p)} \leq |\nabla\psi|_{L_{3p}(L_{3p})}|\nabla\eta|_{L_{3p/2}(L_{3p/2})} \leq c|\psi|^{\delta}_{Z^1(T)}|\psi|^{1-\delta}_{L_\infty(J;H^1_2(\Omega))}, \ \delta < 1,$$

since $\nabla\eta$ is a fixed function and

$$H^{1/2}_p(J; L_p(\Omega)) \cap L_p(J; H^1_p(\Omega)) \hookrightarrow L_{3p/2}(J \times \Omega).$$

Finally the last term $\lambda'(\psi)\Delta\eta$ is dominated by the fixed function $\Delta\eta \in L_p(J \times \Omega)$, since $\lambda' \in L_\infty(\mathbb{R})$. A last time we apply trace-theory to obtain

$$|\partial_\nu(\lambda'(\psi)\eta)|_{Y_1} \leq C(|\lambda'(\psi)\eta|_{H^{1/2}_p(L_p)} + |\lambda'(\psi)\eta|_{L_p(H^2_p)})$$

and then (3.4) leads to

$$|\lambda'(\psi)\eta|_{H^{1/2}_p(L_p)} \leq C(|\eta|_{H^{1/2}_p(L_p)} + |\eta|_{L_{\sigma' p}(L_{r'p})}|\psi|_{H^{1/2}_{\sigma p}(L_{rp})}).$$

Since $\eta \in L_\infty(J; H^1_p(\Omega))$ for all $p \geq 2$ we may choose $r' \in [2,3]$, i.e. $r \in [3/2, 2]$ and σ' may be arbitrarily large. Then the claim follows from (6.7) and (6.9). The proof is complete.

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