Coherent Hedging in Incomplete Markets

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Abstract

In incomplete financial markets not every given contingent claim can be replicated by a self-financing strategy. The risk of the resulting shortfall can be measured by coherent risk measures, introduced by Artzner et al. [1]. The dynamic optimization problem of finding a self-financing strategy that minimizes the coherent risk of the shortfall can be split into a static optimization problem and a representation problem.
In this paper, we will deduce necessary and sufficient optimality conditions for the static problem using convex duality methods. The solution of the static optimization problem tourns out to be a randomized test with a typical 0-1-structure. Our results improve the ones obtained by Nakano [7].

Keywords and phrases: hedging, shortfall risk, coherent risk measures, convex duality, generalized Neyman-Pearson lemma

JEL Classification: G10, G13, D81

Mathematics Subject Classification (2000): 60H30, 62F03, 91B28

1 Introduction

Measures of risk play a crucial role in optimization under uncertainty, especially in coping with the losses that might be incurred in finance and insurance industry. The current industry standard for risk quantification is Value-at-Risk (VaR). One serious shortcoming of VaR is that it takes into account only the probability of a loss and not its actual size. This leaves the position unprotected against losses beyond the VaR. In order to develop more appropriate measures of risk, recent research has taken an axiomatic approach in which the structure of so called coherent risk measures is derived from a set of economically desirable properties, cf. Artzner et al. [1].
We want to apply these advanced risk measures to the problem of pricing and hedging contingent claims. This problem is well understood in the context of no-arbitrage models (e.g. Black-Scholes-Merton model) in complete markets. In such models every

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contingent claim is attainable, i.e., it can be replicated by a self-financing trading strategy. The cost of replication defines the price of the claim, and it can be computed as the expectation of the claim under the unique equivalent martingale measure. Research in mathematical finance shows that the assumption of market completeness is a restrictive idealization based on the Black-Scholes-Merton model of a financial market: transaction costs and taxes are neglected, the future volatility of stock price fluctuations is assumed to be known in advance, prices evolve independently of traders' actions, asset prices follow a log-normal distribution. Relaxing any of these assumptions typically leads to models where financial products bear an intrinsic risk which cannot be hedged away completely. This is called the incomplete market.

In an incomplete market the equivalent martingale measure is no longer unique and not every contingent claim is attainable. Thus, a perfect hedge as in the Black-Scholes-Merton model is not possible any longer. Therefore we are faced with the problem of searching strategies which reduce the risk as much as possible.

One can still stay on the safe side using a "superhedging" strategy, cf. Föllmer, Leukert [5]. The cost of superhedging is given by the supremum of the expected values over all equivalent martingale measures. The corresponding value process is a supermartingale under all equivalent martingale measures, thus the superhedging strategy is determined by the "optional decomposition", cf. Föllmer [5], [6]. But from a practical point of view the cost of superhedging is often too high.

For this reason, Föllmer, Leukert [5], [6] and Nakano [8], [7] investigate the possibility of investing less capital than the superhedging price of the liability. This leads to a shortfall, the risk of which, measured by a suitable risk measure, should be minimized. The mentioned studies differ in the choice of the risk measure used to quantify the shortfall risk. Föllmer and Leukert [5] used the so called quantile hedging to determine a portfolio strategy which minimizes the probability of loss. This idea leads to partial hedges. However, in this approach, losses could be very substantial, even if they occur with a very small probability. Therefore, Föllmer and Leukert [6] proposed to use the expected loss function as risk measure instead. Nakano [8] followed the method of Föllmer, Leukert but used coherent risk measures.

The resulting dynamic optimization problem of finding a self-financing strategy that minimizes the coherent risk of the shortfall can be split into a static optimization problem and a representation problem. The optimal strategy consists in (super-)hedging a modified claim $\widetilde{\varphi}H$, where H is the payoff of the claim and $\widetilde{\varphi}$ is the solution of the static optimization problem, the optimal randomized test. This splitting is also possible in a discrete time setting (see [7]).

Nakano [8] showed the existence to a solution of the static optimization problem and applied in [7] the method of Cvitanić and Karatzas [2]. In this paper we use another method and deduce a result about the structure of the optimal randomized test via the direct application of convex duality. We state the relationship between our results and the results of [7] and show that our method gives more detailed information about the solution.

The paper is organized as follows: In Section 2 we state the formulation of the shortfall problem and the axiomatic definition of coherent risk measures and review the decomposition of the dynamic optimization problem into a static and a representation problem. In Section 3 we analyze the static optimization problem. We deduce the dual problem and show that strong duality holds. Thus, the optimal solution is a saddle point of a functional, specified in Section 3.1. First we consider the inner problem in 3.2 and deduce an extended Neyman-Pearson lemma. Then we solve the saddle point problem. The optimal solution of the static optimization problem is a randomized test with the typical 0-1-structure. In Section 3.3 we deduce the relationship to the results of [7]. In Section 3.4 we give an example. Proofs of the stated results are relegated to the Appendix.

2 Formulation of the Problem

The discounted price process of the underlying asset is described as a semimartingale $S = (S_t)_{t\in[0,T]}$ on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{t\in[0,T]}$. We write L^1 and L^∞ for $L^1(\Omega, \mathcal{F}, P)$ and $L^\infty(\Omega, \mathcal{F}, P)$, respectively. Unless stated otherwise, we regard L^1 and L^∞ as a duality pair associated with the nondegenerated bilinear form $\langle X, Y\rangle = E[XY]$ for all $X \in L^1$, $Y \in L^\infty$ (see Robertson, Robertson [9]), where E denotes the mathematical expectation with respect to P. This means we endow L^∞ with the weak*-topology.
Let $\widehat{\mathcal{Q}}$ be the set of all probability measures on $(\Omega, \mathcal{F})$ absolutely continuous with respect to P. For $Q \in \widehat{\mathcal{Q}}$ we denote the expectation with respect to Q by E^Q and the Radon-Nikodym derivative dQ/dP by Z_Q. Let $\mathcal{P}$ denote the set of equivalent martingale measures with respect to P. Since we assume the absence of arbitrage opportunities, $\mathcal{P} \neq \emptyset$. Equations and inequalities between random variables are always understood as $P - a.s.$
A self-financing strategy is given by an initial capital $V_0 \geq 0$ and a predictable process ξ such that the resulting value process

$$V_t = V_0 + \int_0^t \xi_s dS_s, \quad t \in [0, T]$$

is well defined. A strategy (V_0, ξ) is called admissible if the corresponding value process V satisfies $V_t \geq 0$ for all $t \in [0, T]$.
Consider a contingent claim. Its payoff is given by a $\mathcal{F}_T$ -measurable, nonnegative random variable $H \in L^1$. We assume

$$U_0 = \sup_{P^* \in \mathcal{P}} E^{P^*}[H] < +\infty.$$

The above equation is the dual characterization of the superhedging price U_0, the smallest amount V_0 such that there exists an admissible strategy (V_0, ξ) with value process V_t satisfying $V_T \geq H$.
In the complete case, where the equivalent martingale measure P^* is unique, $U_0 =$

$E^{P^*}[H]$ is the unique arbitrage-free price of the contingent claim.
Since superhedging can be quite expensive in the incomplete case we search for the best hedge an investor can achieve with a smaller amount $\widetilde{V}_0 < U_0$. In other words, we look for an admissible strategy (V_0, ξ) with $0 < V_0 \leq \widetilde{V}_0$ that minimizes the risk of the shortfall $(H - V_T)^+$. The risk will be measured by a coherent risk measure ρ, introduced by Artzner et al. [1] and Delbaen [3]. In contrast to [1], [3], [8] and [7] we consider coherent risk measures that can also attain the value $+\infty$ for investments that are in any way not acceptable.

2.1 Coherent Risk Meaures

In this subsection, we shall introduce the concept of coherent risk measures on $L^1(\Omega, \mathcal{F}, P)$ and give a theorem about the structure of the subdifferential of a lower semicontinuous coherent risk measure. For the definition of the subdifferential and lower semicontinuity see for example [4], Section 2.2 and 5.

Definition 1 (coherent risk measure). *A function $\rho : \ L^1 \to \mathbb{R} \cup \{+\infty\}$ is a coherent risk measure if it satisfies*

(i) monotonicity $\quad (X_1, X_2 \in L^1,\ X_1 \geq X_2) \implies \rho(X_1) \leq \rho(X_2)$

(ii) translation invariance $\quad \forall X \in L^1,\ c \in \mathbb{R} :\ \rho(X + c\mathbf{1}) = \rho(X) - c$

(iii) positive homogeneity $\quad \forall X \in L^1,\ t > 0 :\ \rho(tX) = t\rho(X)$ *and* $\rho(0) = 0$

(iv) subadditivity $\quad \forall X_1, X_2 \in L^1 :\ \rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.

The random variable equal to 1 almost surely is denoted by **1** in (ii).

Remark 1. *For a proper, positive homogeneous and lower semicontinuous function ρ we always have $\rho(0) = 0$.*

For lower semicontinuous coherent risk measures there exists a dual representation (see Theorem 1.2 in Nakano [8], corresponding to Proposition 4.1 in [1] and Theorem 2.3 in [3]).

Theorem 1. *A function $\rho : \ L^1 \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous, coherent risk measure if and only if there exists a non-emty subset of probability measures $\mathcal{Q}$ of $\widehat{\mathcal{Q}}$, such that*

$$\rho(X) = \sup_{Q \in \mathcal{Q}} E^Q(-X). \tag{1}$$

Remark 2. *The corresponding Radon-Nikodym derivatives Z_Q (Rockafellar et al. [10] called them risk envelopes) satisfy*

$$\{Z_Q | Q \in \mathcal{Q}\} = \{Z_Q \in L^\infty \mid \forall X \in L^1 : -\langle Z_Q, X\rangle \leq \rho(X)\}. \tag{2}$$

The set $\{Z_Q | Q \in \mathcal{Q}\}$ is non-empty, weakly-closed and convex.*

In the following theorem the subdifferential of a lower semicontinuous coherent risk measure is deduced. The proof can be found in the Appendix.

Theorem 2. *Let $\rho: L^1 \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous coherent risk measure. Then the subdiffenrential of ρ at $X \in L^1$ is the set*

$$\partial\rho(X) = \{-\widetilde{Z}_Q \in L^\infty : \widetilde{Z}_Q \in \arg\min_{Q\in\mathcal{Q}} E[Z_Q X]\}. \tag{3}$$

Remark 3. *This means that the subgradients of a lower semicontinuous coherent risk measure are the negatives of the Radon-Nikodym derivatives $Z_Q = dQ/dP$ that yield the extremal value of the dual representation (1) of the risk measure.*

2.2 The Dynamic Optimization Problem

Let ρ be a coherent risk measure. We consider the dynamic optimization problem of finding an admissible strategy that minimizes the coherent shortfall risk

$$\min_{(V_0,\xi)} \rho\left(-(H - V_T)^+\right) \tag{4}$$

under the capital constraint of investing less capital than the superhedging price

$$0 < V_0 \leq \widetilde{V}_0 < U_0. \tag{5}$$

In the next subsection we review the decomposition of this dynamic optimization problem and then study the resulting static optimization problem.

2.3 Decomposition of the Dynamic Problem

The dynamic optimization problem (4), (5) can be split into the following two problems:

1. Static optimization problem: Find an optimal modified claim $\widetilde{\varphi}H$, where $\widetilde{\varphi}$ is a randomized test solving
$$\min_{\varphi\in R_0} \rho\left((\varphi - 1)H\right), \tag{6}$$
$$R_0 = \{\varphi : \Omega \to [0,1],\ \mathcal{F}_T - \text{measurable},\ \sup_{P^*\in\mathcal{P}} E^{P^*}[\varphi H] \leq \widetilde{V}_0\} \tag{7}$$

2. Representation problem: Find a superhedging strategy for the modified claim $\widetilde{\varphi}H$.

This idea was introduced by Föllmer et al. [5], [6] using expected loss function as risk measure and was used for coherent risk measures in Nakano [8], [7] analogously. The following theorem results (Theorem 1.5 in Nakano [8]):

Theorem 3. *Let $\widetilde{\varphi}$ be a solution of the minimization problem (6) and let $(\widetilde{V}_0, \widetilde{\xi})$ be the admissible strategy, where $\widetilde{\xi}$ is determined by the optional decomposition of the claim $\widetilde{\varphi}H$. Then the strategy $(\widetilde{V}_0, \widetilde{\xi})$ solves the optimization problem (4), (5).*

Remark 4. *All strategies $(V_0, \widetilde{\xi})$ with $\widetilde{U}_0 = \sup_{P^*\in\mathcal{P}} E^{P^*}[\widetilde{\varphi}H] \leq V_0 \leq \widetilde{V}_0$ solve (4), (5). In Theorem 6 we will show that $\widetilde{U}_0 = \widetilde{V}_0$, thus the optimal strategy is $(\widetilde{V}_0, \widetilde{\xi})$.*

2.4 The Static Optimization Problem

The existence of a solution $\widetilde{\varphi}$ to the static optimization problem (6) was shown in Proposition 1.3 in Nakano [8].
For strictly convex risk measures one can additionally show that any two solutions coincide $P-a.s.$ on $\{\omega : H > 0\}$ (see Proposition 3.1 in Föllmer [6]). A coherent risk measure cannot be strictly convex since translation invariance of ρ (Definition 1 (ii)) implies the linearity of ρ on the one dimensional subspace of L^1 generated by the random variable equal to 1 a.s. This means that for coherent risk measures one can only show the existence, not the essential uniqueness of the solution.
Furthermore, a direct construction of $\widetilde{\varphi}$ via the Neyman-Pearson lemma as it is done for some special risk measures in [5], [6] is not possible anymore for coherent risk measures. Therefore we will derive a generalized Neyman-Pearson lemma for the inner problem of the saddle point problem spezified in Section 3.1 and then solve the whole problem. We obtain necessary and sufficient optimality conditions for the solution $\widetilde{\varphi}$ of the static optimization problem (6). In Section 3.3 we discuss this results with respect to the results of [7].

2.5 The Representation Problem

Let $\widetilde{\varphi}$ be the solution of the problem defined by (6). We introduce the modified claim

$$\widetilde{H} = \widetilde{\varphi} H$$

and define $\widetilde{U}$ as a right-continuous version of the process

$$\widetilde{U}_t = \text{ess.} \sup_{P^* \in \mathcal{P}} E^{P^*}[\widetilde{\varphi} H | \mathcal{F}_t].$$

The process $\widetilde{U}$ is a $\mathcal{P}$-supermartingale, i.e., a supermartingale with respect to any equivalent martingale measure $P^* \in \mathcal{P}$. By the optional decomposition theorem (see Föllmer [5], [6]) there exists an admissible strategy $(\widetilde{U}_0, \widetilde{\xi})$ and an increasing optional process $\widetilde{C}$ with $\widetilde{C}_0 = 0$ such that

$$\widetilde{U}_t = \widetilde{U}_0 + \int_0^t \widetilde{\xi}_s dS_s - \widetilde{C}_t.$$

Remark 5. *In the complete case where the equivalent martingale measure is unique, $(\widetilde{U}_0, \widetilde{\xi})$ is simply the replicating strategy for the modified claim $\widetilde{H} = \widetilde{\varphi} H$ ($\widetilde{U}_0 = E^{P^*}[\widetilde{\varphi} H]$ is the unique arbitrage-free price of the contingent claim $\widetilde{H}$), i.e.,*

$$E^{P^*}[\widetilde{\varphi} H | \mathcal{F}_t] = E^{P^*}[\widetilde{\varphi} H] + \int_0^t \widetilde{\xi}_s dS_s, \quad t \in [0, T].$$

Remark 6. *In the case of risk measures that allow the construction of $\widetilde{\varphi}$ via the Neyman-Pearson lemma directly (Föllmer [5], [6]), one can see that $\widetilde{U}_0 = \widetilde{V}_0$ since the optimal test $\widetilde{\varphi}$ attains the bound $\widetilde{V}_0$ in (7).*
In Section 3.2 of this paper we will show that in the case of coherent hedging the bound $\widetilde{V}_0$ is as well attained by the optimal test.

3 The Static Optimization Problem

We consider the static optimization problem (6) as the primal problem with value p :

$$
\begin{aligned}
p &= \min_{\varphi \in R_0} \rho\left((\varphi - 1)H\right) \\
&= \min_{\varphi \in L^\infty} \{\rho\left((\varphi - 1)H\right) + \mathcal{I}_{R_0}(\varphi)\},
\end{aligned}
$$

$$
R_0 = \{\varphi : \ \Omega \to [0,1], \ \mathcal{F}_T - \text{measurable}, \ \sup_{P^* \in \mathcal{P}} E^{P^*}[\varphi H] \leq \widetilde{V}_0\},
$$

where $\mathcal{I}_{R_0}(\varphi)$ is the indicator function equal to zero for $\varphi \in R_0$ and $+\infty$ otherwise. To solve this problem it is necessary to impose the following assumption that has to be satisfied throughout this paper:

Assumption 1. *ρ is a lower semicontinuous coherent risk measures that is continuous in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$.*

Remark 7. *A lower semicontinuous coherent risk measure ρ is continuous in some $(\varphi_0 - 1)H$ if $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$ is an inner point of the domain of ρ (see [4], Corollary 2.5). Especially, if $\rho(X) < +\infty$ for all $X \in L^1$, a lower semicontinuous coherent risk measure is continuous (cf. [7], Remark 3.3).*

The dual representation (1) of ρ enables us to rewrite p as follows

$$
p = \min_{\varphi \in R_0} \{\sup_{Q \in \mathcal{Q}} E^Q[(1-\varphi)H]\}. \tag{8}
$$

In this section, we will construct the dual problem of p and establish necessary and sufficient optimality conditions for the solution of (6). The proofs of Theorem 4 and 5 can be found in the Appendix.

3.1 The Dual Problem

Theorem 4. *The dual problem of (6) is the following problem with value d*

$$
d = \sup_{Q \in \mathcal{Q}} \{\inf_{\varphi \in R_0} E^Q[(1-\varphi)H]\}. \tag{9}
$$

Strong duality holds: $p = d$ and $(\widetilde{Z}_Q, \widetilde{\varphi})$ is a saddle point of the function $E^Q[(1-\varphi)H]$, where $\widetilde{\varphi}$ is the solution of (8) and $\widetilde{Z}_Q = \frac{d\widetilde{Q}}{dP}$ is the solution of (9). Thus

$$
\min_{\varphi \in R_0} \{\max_{Q \in \mathcal{Q}} E^Q[(1-\varphi)H]\} = \max_{Q \in \mathcal{Q}} \{\min_{\varphi \in R_0} E^Q[(1-\varphi)H]\}. \tag{10}
$$

In the next subsection we consider the inner problem of the dual problem (9) for an arbitrary $Q \in \mathcal{Q}$. We give a result about the structure of the solution and then deduce in our main theorem a result about the saddle point of Theorem 4, that means a result about the solution of the static optimization problem (6).

3.2 The Saddle Point

First let us consider the inner problem of the dual problem (9) for a $Q \in \mathcal{Q}$:

$$p^i(Q) := \max_{\varphi \in R_0} E^Q[\varphi H].$$

Let $R = \{\varphi : \ \Omega \to [0,1], \ \mathcal{F}_T - \text{measurable}\}$ be the set of randomized tests. Then, $p^i(Q)$ is

$$\max_{\varphi \in R} E^Q[\varphi H], \tag{11}$$

$$\forall P^* \in \mathcal{P} : \quad E^{P^*}[\varphi H] \le \widetilde{V}_0. \tag{12}$$

This optimization problem can be ranged in test theory: We define the measures O and $O^* = O^*(P^*)$ by

$$\frac{dO}{dQ} = H \quad \text{and} \quad \frac{dO^*}{dP^*} = H, \quad P^* \in \mathcal{P}.$$

The objective function (11) turns into

$$\max_{\varphi \in R} E^O[\varphi], \tag{13}$$

and the constraints (12) take the form

$$\forall P^* \in \mathcal{P} : \quad E^{O^*}[\varphi] \le \widetilde{V}_0 =: \alpha. \tag{14}$$

This is equivalent of looking for an optimal test $\widetilde{\varphi}$ when testing the compound hypothesis $H_0 = \{O^*(P^*) : P^* \in \mathcal{P}\}$, parameterized by the class of equivalent martingale measures, against the simple alternative $H_1 = \{O\}$ in a generalized sense. In the generalized Neyman-Pearson lemma (Theorem 2.79, Witting [11]) O and O^* are not necessarily probability measures, but measures and the significance level α is generalized to be a bounded measurable function $\alpha(P^*)$.
The existence of an optimal test $\widetilde{\varphi}$ of (13), (14) follows from standard theory, since $\alpha(P^*) \equiv \widetilde{V}_0 > 0$ is a bounded and measurable function and $\sup_{P^* \in \mathcal{P}} E^{P^*}[H] = U_0 < \infty$ (see Witting [11], Theorem 2.80 b). We now want to show that strong duality holds. In this case, the typical 0-1-structure of $\widetilde{\varphi}$ is sufficient and necessary for optimality. The dual problem of $p^i(Q)$ is (see Section 4.3, equation (36))

$$d^i(Q) = \inf_{\lambda \in \Lambda_+} \Big\{ \int_\Omega [HZ_Q - H \int_{\mathcal{P}} Z_{P^*} d\lambda]^+ dP + \widetilde{V}_0 \|\lambda\| \Big\}, \tag{15}$$

where Λ_+ is the set of finite measure on $\mathcal{P}$ and $\|\lambda\| = \int_{\mathcal{P}} d\lambda$.

Theorem 5. *The optimal randomized test $\widetilde{\varphi}_Q$ for (11), (12) has the following structure:*

$$\widetilde{\varphi}_Q(\omega) = \begin{cases} 1 & : \quad HZ_Q > H\int_{\mathcal{P}} Z_{P^*}d\widetilde{\lambda}_Q \\ 0 & : \quad HZ_Q < H\int_{\mathcal{P}} Z_{P^*}d\widetilde{\lambda}_Q \end{cases} \qquad P-a.s. \tag{16}$$

and

$$E^{P^*}[\widetilde{\varphi}_Q H] = \widetilde{V}_0 \qquad \widetilde{\lambda}_Q - a.s., \tag{17}$$

where $\widetilde{\lambda}_Q \in \Lambda_+$, a finite measure on $\mathcal{P}$, is the solution of (15). There is no duality gap: $d^i(Q) = p^i(Q)$ holds true for each $Q \in \mathcal{Q}$.

The proof of the theorem can be found in the Appendix, Section 4.3. For each $Q \in \mathcal{Q}$ there exist a primal and a dual solution $\widetilde{\varphi}_Q$, $\widetilde{\lambda}_Q$, respectively. If $Q = \widetilde{Q}$ is the solution of the outer problem of (9), $\widetilde{\varphi}_{\widetilde{Q}}$ is the solution of the static optimization problem (6). The outer problem of (9) and Theorem 2 show that $-\widetilde{Z}_Q H$ is an element of the subdifferential of ρ at $(\widetilde{\varphi}_{\widetilde{Q}} - 1)H$, where $\widetilde{Z}_Q = d\widetilde{Q}/dP$. Now, let us consider the saddle point problem (10). With Theorem 5 it follows that

$$\begin{aligned} \max_{Q\in\mathcal{Q}} \min_{\varphi\in R_0} E^Q[(1-\varphi)H] &= \max_{Q\in\mathcal{Q}}\{E^Q[H] - p^i(Q)\} = \max_{Q\in\mathcal{Q}}\{E^Q[H] - d^i(Q)\} \\ &= \max_{Q\in\mathcal{Q}}\Big\{\max_{\lambda\in\Lambda_+}\Big\{-E^P[(HZ_Q - H\int_{\mathcal{P}} Z_{P^*}d\lambda)^+] - \widetilde{V}_0\|\lambda\|\Big\} + E^Q[H]\Big\} \\ &= \max_{Q\in\mathcal{Q},\lambda\in\Lambda_+}\Big\{E^P[HZ_Q \wedge H\int_{\mathcal{P}} Z_{P^*}d\lambda] - \widetilde{V}_0\|\lambda\|\Big\}, \end{aligned} \tag{18}$$

where $x \wedge y = min(x, y)$. With Theorem 4 it follows that $\widetilde{Q}$ attains the maximum w.r.t. $Q \in \mathcal{Q}$. Theorem 5 shows the existence of a $\widetilde{\lambda} = \widetilde{\lambda}_{\widetilde{Q}}$ that attains the maximum w.r.t. $\lambda \in \Lambda_+$. Thus, there exists a pair $(\widetilde{Q}, \widetilde{\lambda})$ such that

$$E^P[H\widetilde{Z}_Q \wedge H\int_{\mathcal{P}} Z_{P^*}d\widetilde{\lambda}] - \widetilde{V}_0\|\widetilde{\lambda}\| = \max_{Q\in\mathcal{Q},\lambda\in\Lambda_+}\Big\{E^P[HZ_Q \wedge H\int_{\mathcal{P}} Z_{P^*}d\lambda] - \widetilde{V}_0\|\lambda\|\Big\}. \tag{19}$$

Now, our main theorem follows.

Theorem 6. *Let $(\widetilde{Q}, \widetilde{\lambda})$ be the optimal pair in (19).*

- *The solution of the static optimization problem (6) is*

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad H\widetilde{Z}_Q > H\int_{\mathcal{P}} Z_{P^*}d\widetilde{\lambda} \\ 0 & : \quad H\widetilde{Z}_Q < H\int_{\mathcal{P}} Z_{P^*}d\widetilde{\lambda} \end{cases} \qquad P-a.s. \tag{20}$$

  with

$$E^{P^*}[\widetilde{\varphi}H] = \widetilde{V}_0 \qquad \widetilde{\lambda} - a.s. \tag{21}$$

- $(\widetilde{\varphi}, \widetilde{Z}_Q)$ *is the saddle point of Theorem 4.*
- $(\widetilde{V}_0, \widetilde{\xi})$ *solves the dynamic coherent hedging problem, where* $\widetilde{\xi}$ *is the superhedging strategy of the modified claim* $\widetilde{\varphi}H$*.*

Proof. The results follow from Theorem 4 and 5 and the considerations in Section 3.2. □

Remark 8. *It follows that there exists a* $[0,1]$*-valued random variable* B *such that* $\widetilde{\varphi}$ *as in Theorem 6 admits*

$$\widetilde{\varphi} = I_{\{H\widetilde{Z}_Q > H\int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda}\}} + BI_{\{H\widetilde{Z}_Q = H\int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda}\}},$$

where $I_A(\omega)$ *is the stochastic indicator function equal to one for* $\omega \in A$ *and zero otherwise.*

3.3 Links to known Results

Now, we want to range our results in the theory of Nakano [7] and Cvitanić, Karatzas [2]. In [7] the static optimization problem (6) has been considered as well, but another method is used to solve it. In this section we will deduce the relationship between our results and the results of [7] and we will show that our considerations lead to a more precise result about the structure of the solution. In [2] saddle point problems in test theory has been considered and a generalized Neyman-Pearson lemma has been deduced. Nakano [7] followed the method of [2] to show that the solution of the static optimization problem is a Neyman-Pearson test. The inequality (4.8) in [7] and (3.4) in [2] coincides with weak duality

$$\forall Q \in \mathcal{Q}, \forall \lambda \in \Lambda_+, \forall \varphi \in R_0: \quad -p^i(\varphi, Q) + E^Q[H] \geq -d^i(\lambda, Q) + E^Q[H].$$

That is (see (18))

$$\forall Q \in \mathcal{Q}, \forall \lambda \in \Lambda_+, \forall \varphi \in R_0: \quad E^Q[(1-\varphi)H] \geq E^P[HZ_Q \wedge H \int_{\mathcal{P}} Z_{P^*} d\lambda] - \widetilde{V}_0 \|\lambda\|.$$

In contrast to the method used in Theorem 5 and 6 to prove the validity of strong duality directly, [2] and [7] prove the existence of a solution to the dual problem. To do this, it is necessary to consider larger sets than the ones of the saddle point problem (10). These enlarged sets are

$$\mathcal{Z} = \{Z \in L^\infty_+ \mid E[Z] \leq 1, \forall X \in L^1_+ : E[XZ] \leq \rho(-X)\}$$

and

$$\mathcal{G} = \{G \in L^1_+ \mid E[G] \leq 1, E[GH] \leq U_0, \forall \varphi \in R_0 : E[G\varphi H] \leq \widetilde{V}_0\},$$

see Section 4 of [7]. $\mathcal{Z}$ contains the risk envelopes of the coherent risk measure ρ: $\mathcal{Q} \subset \mathcal{Z}$. The set $\mathcal{G}$ contains the equivalent martingal measures $P^* \in \mathcal{P}$: $\mathcal{P} \subset \mathcal{G}$. In

Theorem 4.11 in [7] it is shown that the optimal randomized test $\widetilde{\varphi}$ has as well a 0-1-structure, but the elements in the sets that define $\widetilde{\varphi}$ are chosen from the larger sets $\mathcal{Z}$ and $\mathcal{G}$:

$$\widetilde{\varphi} = I_{\{\widehat{y}\widehat{G}<\widehat{Z}\}} + BI_{\{\widehat{y}\widehat{G}=\widehat{Z}\}},$$

where $(\widehat{Z}, \widehat{G})$ attain the supremum of

$$f(y) = \sup_{Z\in\mathcal{Z}, G\in\mathcal{G}} E[H(Z \wedge yG)] \quad (y \geq 0)$$

and $\widehat{y}$ attains the supremum of

$$g(\widetilde{V}_0) = \sup_{y\geq 0}(f(y) - \widetilde{V}_0 y).$$

With the method used in our paper, it is not necessary to consider the larger sets $\mathcal{Z}$ and $\mathcal{G}$ to prove the strong duality $p^i = d^i$ in Theorem 5 and $p = d$ in Theorem 6. Furthermore the representation of an optimal randomized test $\widetilde{\varphi}$ is possible with elements of the smaller sets $\mathcal{Q}$ and $\mathcal{P}$. The application of Theorem 6 and Theorem 5 shows that there is a one-to-one relationship between the optimal elements $\widehat{Z}$, $\widehat{G}$ and $\widehat{y}$ of [7] and elements of $\mathcal{Q}$ and $\mathcal{P}$:

$$\widehat{Z} = \begin{cases} \widetilde{Z}_Q & : \quad \{H > 0\} \\ 0 & : \quad \{H = 0\} \end{cases}, \tag{22}$$

$$\widehat{G} = \|\widetilde{\lambda}\|^{-1} \int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda}, \tag{23}$$

$$\widehat{y} = \|\widetilde{\lambda}\|, \tag{24}$$

where $(\widetilde{Q}, \widetilde{\lambda})$ is the optimal pair in (19) and $\widetilde{Z}_Q = \frac{d\widetilde{Q}}{dP}$. Thus, the direct application of convex duality gives more detailed information about the structure of the optimal randomized test $\widetilde{\varphi}$. Another difference to [7] is that we consider coherent risk measures that can also attain the value $+\infty$. Furthermore, we could now show in Theorem 6 that the upper bound of the constraint in (7) is attained, that means $\widetilde{U}_0 = \sup_{P^*\in\mathcal{P}} E^{P^*}[\widetilde{\varphi}H] = \widetilde{V}_0$. Thus, $(\widetilde{V}_0, \widetilde{\xi})$ solves the optimization problem (4), (5) (see Remark 4).

3.4 Examples

Let us consider the problem of minimizing the risk of the shortfall $(H - V_T)^+$, where the risk is measured by the coherent risk measure

$$\rho(X) = -E^Q[X].$$

That means the risk envelope of the coherent risk measure is a singleton, $\mathcal{Q} = \{Q\}$ with $Z_Q \in L^\infty$. Thus we look for an admissible strategy $(V_0, \widetilde{\xi})$ that minimizes

$$\rho(-(H - V_T)^+) = E^Q[(H - V_T)^+] \tag{25}$$

under the constraint

$$V_0 \leq \widetilde{V}_0. \tag{26}$$

Theorem 3 shows that the corresponding static optimization problem is

$$\max_{\varphi \in R} E^Q[\varphi H] \tag{27}$$

under the constraint

$$\forall P^* \in \mathcal{P} : \quad E^{P^*}[\varphi H] \leq \widetilde{V}_0. \tag{28}$$

The same optimization problem with $HZ_Q = Z_P$ arises in [5], Section 4, where the problem of Quantile Hedging in the incomplete case is considered. The risk measure used there is just the probability of the shortfall.
In Föllmer, Leukert [6] the expected loss function is used as risk measure. In Section 4 they consider the minimizing of the expected shortfall. This means the linear loss function $l(x) = x$ is used. This leads to the optimization problem (27), (28) with $Q = P$.
Under the assumption $\widetilde{V}_0 > 0$ we obtain with Theorem 5 the following structure of the optimal solution $\widetilde{\varphi}$ of (27), (28):

$$\widetilde{\varphi}(\omega) = \begin{cases} 1 & : \quad HZ_Q > H\int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda} \\ 0 & : \quad HZ_Q < H\int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda} \end{cases} \qquad P - a.s.$$

with

$$E^{P^*}[\widetilde{\varphi} H] = \widetilde{V}_0 \qquad \widetilde{\lambda} - a.s.,$$

where $\widetilde{\lambda}$, a finite measure on $\mathcal{P}$, is the solution of the dual problem of (27), (28). The solution of the coherent hedging problem (25), (26) is $(\widetilde{V}_0, \widetilde{\xi})$, where $\widetilde{\xi}$, obtained by the optional decomposition theorem (see Section 2.5), is the superhedging strategy of the modified claim $\widetilde{\varphi} H$.
If additionally $\mathcal{P} = \{P^*\}$ is a singleton as in Nakano [8], Proposition 4.1, the static problem can be solved explicitly. The optimal solution is

$$\widetilde{\varphi}(\omega) = I_{\{Z_Q > \widetilde{a} Z_{P^*}\}}(\omega) + \gamma I_{\{Z_Q = \widetilde{a} Z_{P^*}\}}(\omega),$$

where

$$\widetilde{a} = \inf\{a \mid E^{P^*}[HI_{\{Z_Q > aZ_{P^*}\}}] \leq \widetilde{V}_0\}$$

and

$$\gamma = \begin{cases} \dfrac{\widetilde{V}_0 - E^{P^*}[HI_{\{Z_Q > \widetilde{a} Z_{P^*}\}}]}{E^{P^*}[HI_{\{Z_Q = \widetilde{a} Z_{P^*}\}}]} & : \quad P^*(\{Z_Q = \widetilde{a} Z_{P^*}\} \cap \{H > 0\}) > 0 \\ c \in [0,1] \text{ arbitrarily} & : \quad P^*(\{Z_Q = \widetilde{a} Z_{P^*}\} \cap \{H > 0\}) = 0. \end{cases}$$

When Q is equal to P this coincides with Proposition 4.1 in Föllmer [6].

4 Appendix

4.1 Proof of Theorem 2

This proof corresponds to the one of Rockafellar et al. [10], where the subdifferential of lower semicontinuous expectation-bounded risk measures $\mathcal{R}: L^2 \to \mathbb{R} \cup \{+\infty\}$ is deduced. Here, we consider lower semicontinuous coherent risk measures $\rho: L^1 \to \mathbb{R} \cup \{+\infty\}$.
By definition of the subdifferential ([4], Section 5),

$$-Z \in \partial\rho(\widehat{X}) \iff \forall X \in L^1: \ \rho(X) \geq \rho(\widehat{X}) + E[(X - \widehat{X})(-Z)], \tag{29}$$

in particular for $X = \lambda\widehat{X}$ when $\lambda > 0$. By positive homogeneity of ρ, we obtain

$$\forall\lambda > 0: \ (\lambda - 1)E[-\widehat{X}Z] \leq (\lambda - 1)\rho(\widehat{X}).$$

Since $(\lambda - 1)$ can be either positive or negative, it follows

$$-E[\widehat{X}Z] = \rho(\widehat{X}).$$

The general subgradient inequality (29) for $-Z$ reduces then to

$$\forall X \in L^1: \ \rho(X) \geq -E[XZ].$$

From relation (2) we obtain that Z is a Radon-Nikodym derivative of a measure $\widetilde{Q} \in \mathcal{Q}$, hence we will denote it by $\widetilde{Z}_Q$.
Conversely, if $\widetilde{Z}_Q$ with $\widetilde{Q} \in \mathcal{Q}$ and $-E[\widehat{X}\widetilde{Z}_Q] = \rho(\widehat{X})$ we see from (2) that $\rho(X) \geq \rho(\widehat{X}) + E[(X - \widehat{X})(-\widetilde{Z}_Q)]$. This means that $-\widetilde{Z}_Q \in \partial\rho(\widehat{X})$. Hence

$$\partial\rho(\widehat{X}) = \{-\widetilde{Z}_Q \in L^\infty: \ \widetilde{Q} \in \mathcal{Q} \text{ and } \rho(\widehat{X}) = -E[\widehat{X}\widetilde{Z}_Q]\}.$$

Since ρ admits the dual representation (1)

$$\rho(\widehat{X}) = \sup_{Q \in \mathcal{Q}} E^Q(-\widehat{X}) = -\inf_{Q \in \mathcal{Q}} E[\widehat{X}Z_Q],$$

one can see that

$$\widetilde{Z}_Q \in \arg\min_{Q \in \mathcal{Q}} E[Z_Q\widehat{X}].$$

Hence the subdifferential of the lower semicontinuous coherent risk measure $\rho: L^1 \to \mathbb{R} \cup \{+\infty\}$ is

$$\partial\rho(\widehat{X}) = \{-\widetilde{Z}_Q \in L^\infty: \widetilde{Z}_Q \in \arg\min_{Q \in \mathcal{Q}} E[Z_Q\widehat{X}]\}.$$

4.2 Proof of Theorem 4

We consider the primal problem

$$p = \min_{\varphi \in L^\infty} \{\rho\left((\varphi - 1)H\right) + \mathcal{I}_{R_0}(\varphi)\},$$

$$R_0 = \{\varphi: \ \Omega \to [0,1], \ \mathcal{F}_T - \text{measurable}, \ \sup_{P^* \in \mathcal{P}} E^{P^*}[\varphi H] \leq \widetilde{V}_0\}.$$

We define the linear and continuous operator $A : L^\infty \to L^1$ by $A\varphi := H\varphi$. The function $f: \ L^\infty \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(\varphi) := \mathcal{I}_{R_0}(\varphi)$$

is convex because of the convexity of R_0. The function $g: \ L^1 \to \mathbb{R} \cup \{+\infty\}$

$$g(A\varphi) := \rho\left(A\varphi - H\right) = \rho\left((\varphi - 1)H\right)$$

is convex since ρ is convex (positive homogeneous and subadditive, see Definition 1). Since ρ is assumed to be continuous in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$ (Assumption 1, Section 3) we have strong duality $p = d$ (Theorem 2.7.1, Condition (iii) in Zălinescu [13]).

The operator $A : L^\infty \to L^1$ is self-adjoined since by definition and use of the bilinear form of the duality pair (L^1, L^∞) with the weak*-topology on L^∞ the following equations have to be satisfied:

$$\begin{aligned} \forall \varphi \in L^\infty, \forall Y \in L^\infty: \quad \langle A\varphi, Y\rangle &= \langle \varphi, A^*Y\rangle, \\ \forall \varphi \in L^\infty, \forall Y \in L^\infty: \quad \int_\Omega H\varphi Y dP &= \int_\Omega \varphi A^* Y dP. \end{aligned} \tag{30}$$

Suppose $A^*Y < HY$ on $\Omega_1 \subseteq \Omega$ with $P(\Omega_1) > 0$. Define $\varphi(\omega) = 1$ on Ω_1 and 0 otherwise. This φ violates (30). The case $A^*Y > HY$ on $\Omega_2 \subseteq \Omega$ with $P(\Omega_2) > 0$ is analogous. We conclude $A^*Y = HY = AY$, i.e., the operator A is self-adjoined.

To establish the dual problem (Zălinescu [13], Theorem 2.7.1)

$$d = \sup_{Y^* \in L^\infty} \{-f^*(A^*Y^*) - g^*(-Y^*)\},$$

we derive the conjugate functions f^*, g^* of f and g, respectively.

$$\begin{aligned} f^*(A^*Y^*) &= \sup_{X \in L^\infty} \{\langle A^*Y^*, X\rangle - f(X)\} \\ &= \sup_{X \in L^\infty} \{\langle HY^*, X\rangle - \mathcal{I}_{R_0}(X)\} \\ &= \sup_{\varphi \in R_0} \{\langle HY^*, \varphi\rangle\} \\ &= \sup_{\varphi \in R_0} E[\varphi H Y^*] = \delta_{R_0}(HY^*), \end{aligned}$$

where δ_{R_0} is the support function of R_0. Since the dual representation of a coherent risk measure (see (1)) is also a support function $\delta_{-\mathcal{Q}}(X)$

$$\rho(X) \;=\; \sup_{\{Z_Q | Q \in -\mathcal{Q}\}} \langle X, Z_Q \rangle = \delta_{-\mathcal{Q}}(X),$$

the conjugate function is the indicator function $\mathcal{I}_{-\mathcal{Q}}(Y^*)$ (application of the biconjugate theorem, Theorem 2.3.3 in [13] for $\mathcal{I}_{-\mathcal{Q}}(Y^*)$ since $\mathcal{Q}$ is non-empty, convex and closed)

$$\rho^*(Y^*) = \delta^*_{-\mathcal{Q}}(Y^*) = \mathcal{I}_{-\mathcal{Q}}(Y^*).$$

The function g was defined by $g(Y) = \rho(Y - H)$. Its conjugate function is (Theorem 2.3.1 (vi) in Zălinescu [13]):

$$\begin{aligned} g^*(Y^*) &= \rho^*(Y^*) + \langle Y^*, H \rangle \\ &= \mathcal{I}_{-\mathcal{Q}}(Y^*) + \langle Y^*, H \rangle. \end{aligned}$$

Since $-g^*(-Y^*) = -\mathcal{I}_{\mathcal{Q}}(Y^*) + \langle Y^*, H \rangle$, the dual problem with value d is

$$d = \sup_{Y^* \in L^\infty} \{-\delta_{R_0}(HY^*) + \langle Y^*, H \rangle - \mathcal{I}_{\mathcal{Q}}(Y^*)\}. \tag{31}$$

Note the similar structure of the primal problem

$$p = \inf_{\varphi \in L^\infty} \{\delta_{\mathcal{Q}}((1-\varphi)H) + \mathcal{I}_{R_0}(\varphi)\}.$$

Rewriting (31) we obtain the first result of Theorem 4:

$$d = \sup_{Q \in \mathcal{Q}} \{ \inf_{\varphi \in R_0} E^Q[(1-\varphi)H]\}.$$

Since ρ is continuous in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$, we have strong duality $p = d$. Let $\widetilde{\varphi}$ be the solution of the primal problem (8) and $\widetilde{Z}_Q$ the solution of the dual problem (9). Since $p = \sup_{Q \in \mathcal{Q}} E^Q[(1-\widetilde{\varphi})H] \geq E[(1-\widetilde{\varphi})H\widetilde{Z}_Q]$ and $d = \inf_{\varphi \in R_0} E[(1-\varphi)H\widetilde{Z}_Q] \leq E[(1-\widetilde{\varphi})H\widetilde{Z}_Q]$ and because of strong duality we have $E[(1-\widetilde{\varphi})H\widetilde{Z}_Q] \leq p = d \leq E[(1-\widetilde{\varphi})H\widetilde{Z}_Q]$. Hence

$$\min_{\varphi \in R_0} \{\max_{Q \in \mathcal{Q}} E^Q[(1-\varphi)H]\} = \max_{Q \in \mathcal{Q}} \{\min_{\varphi \in R_0} E^Q[(1-\varphi)H]\} :$$

$(\widetilde{Z}_Q, \widetilde{\varphi})$ is a saddle point of the function $E^Q[(1-\varphi)H]$.

4.3 Proof of Theorem 5

The constraint (12) can be rewritten as

$$\begin{aligned} \widetilde{V}_0\mathbf{1} - E^{P^*}[\varphi H] &\geq \mathbf{0} \\ -E^{P^*}[\varphi H] &\in \mathrm{I\!R}_+^{\mathcal{P}} - \widetilde{V}_0\mathbf{1}, \end{aligned}$$

for $\mathbf{1}$ and $\mathbf{0} \in \mathrm{I\!R}^{\mathcal{P}}$.
We define the linear operator $B:\ L^\infty \to \mathrm{I\!R}^{\mathcal{P}}$ by $B\varphi := -E^{P^*}[\varphi H]$. Then the primal problem (11), (12) with value $p^i = p^i(Q)$ is

$$-p^i = \min_{\varphi \in L^\infty} \Big\{ - E^Q[\varphi H] + \mathcal{I}_R(\varphi) + \mathcal{I}_{\mathrm{I\!R}_+^{\mathcal{P}} - \widetilde{V}_0\mathbf{1}}(B\varphi) \Big\}. \tag{32}$$

$B\varphi$ is bounded

$$\forall \varphi \in L^\infty: \quad \sup_{P^* \in \mathcal{P}} |B\varphi| = \sup_{P^* \in \mathcal{P}} |E^{P^*}[\varphi H]| \leq \|\varphi\|_\infty \sup_{P^* \in \mathcal{P}} E^{P^*}[H] = U_0 \|\varphi\|_\infty$$

and measurable on the measurable space $(\mathcal{P}, \mathcal{S})$, where $\mathcal{S}$ is the σ-field generated by the integrals $\int_\Omega f dP^*$ for bounded and measurable functions f on $(\Omega, \mathcal{F}_T)$. Thus $B:\ L^\infty \to \mathcal{L}$, where $\mathcal{L} \subset \mathrm{I\!R}^{\mathcal{P}}$ is the space of bounded and measurable functions on $(\mathcal{P}, \mathcal{S})$ with the norm $\|l\| = \sup_{P^* \in \mathcal{P}} |l(P^*)|$. We endow L^∞ with the strong topology generated with respect to the norm $\|\cdot\|_\infty$. Then the operator B is linear and continuous. Equation (32) reduces to

$$-p^i = \min_{\varphi \in L^\infty} \Big\{ \underbrace{-E^Q[\varphi H] + \mathcal{I}_R(\varphi)}_{=:\, f(\varphi)} + \underbrace{\mathcal{I}_{\mathcal{L}_+ - \widetilde{V}_0\mathbf{1}}(B\varphi)}_{=:\, g(B\varphi)} \Big\}, \tag{33}$$

where $\mathcal{L}_+ = \{ l \in \mathcal{L} |\ \forall P^* \in \mathcal{P}:\ l(P^*) \geq 0 \}$.
Let Λ be the space of finite, signed measures on $(\mathcal{P}, \mathcal{S})$. We regard $\mathcal{L}$ and Λ as the duality pair associated with the bilinear form $\langle l, \lambda \rangle = \int_{\mathcal{P}} l d\lambda$ for $l \in \mathcal{L}$ and $\lambda \in \Lambda$, see [11], Example 1.63. Now we want to establish the dual problem of (33)

$$-d^i = \sup_{\lambda \in \Lambda} \Big\{ - f^*(B^*\lambda) - g^*(-\lambda) \Big\}$$

analogously to Section 4.2. The conjugate function of g is

$$\begin{aligned} g^*(\lambda) &= \sup_{\tilde{l} \in \mathcal{L}} \Big\{ \langle \tilde{l}, \lambda \rangle - \mathcal{I}_{\mathcal{L}_+ - \widetilde{V}_0\mathbf{1}}(\tilde{l}) \Big\} = \sup_{\tilde{l} \in \mathcal{L}_+ - \widetilde{V}_0\mathbf{1}} \langle \tilde{l}, \lambda \rangle = \sup_{l \in \mathcal{L}_+} \langle l - \widetilde{V}_0\mathbf{1}, \lambda \rangle \\ &= \sup_{l \in \mathcal{L}_+} \langle l, \lambda \rangle - \widetilde{V}_0 \int_{\mathcal{P}} d\lambda = \mathcal{I}_{\mathcal{L}_+^*}(\lambda) - \widetilde{V}_0 \int_{\mathcal{P}} d\lambda, \end{aligned}$$

where $\mathcal{L}_+^*$ is the negative dual cone of $\mathcal{L}_+$. To establish the conjugate function of f

$$f^*(B^*\lambda) = \sup_{\varphi \in L^\infty} \Big\{ \langle B^*\lambda, \varphi \rangle + E^Q[\varphi H] - \mathcal{I}_R(\varphi) \Big\},$$

we have to calculate $\langle B^*\lambda, \varphi\rangle$, where $B^*: \Lambda \to (L^\infty)^*$ is the adjoined operator of B. $(L^\infty)^*$, the dual space of L^∞, is the space of finitely additive measures on $(\Omega, \mathcal{F}, P)$ absolutely continuous to P (see [12], Chapter IV, 9, Example 5). For B^* being the adjoined operator of B, the following equations have to be satisfied:

$$\begin{aligned}\forall \varphi \in L^\infty, \forall\, \lambda \in \Lambda: \ \langle B^*\lambda, \varphi\rangle &= \langle \lambda, B\varphi\rangle,\\ \forall \varphi \in L^\infty, \forall\, \lambda \in \Lambda: \ \langle B^*\lambda, \varphi\rangle &= \int_{\mathcal{P}} -E^{P^*}[\varphi H] d\lambda = E\Big[-\varphi H \int\limits_{\mathcal{P}} Z_{P^*} d\lambda\Big].\end{aligned}$$

Hence the conjugate function of f is

$$f^*(B^*\lambda) = \sup_{\varphi \in L^\infty} \Big\{ \langle B^*\lambda, \varphi\rangle + E^Q[\varphi H] - \mathcal{I}_R(\varphi)\Big\} = \sup_{\varphi \in R} E\Big[\varphi\Big(HZ_Q - H\int\limits_{\mathcal{P}} Z_{P^*} d\lambda\Big)\Big].$$

The dual problem $d^i = d^i(Q)$ is as follows:

$$-d^i = \sup_{\lambda \in \Lambda} \Big\{ -\sup_{\varphi \in R} E\Big[\varphi\Big(HZ_Q - H\int\limits_{\mathcal{P}} Z_{P^*} d\lambda\Big)\Big] - \mathcal{I}_{-\mathcal{L}_+^*}(\lambda) - \widetilde{V}_0 \int\limits_{\mathcal{P}} d\lambda \Big\}, \quad (34)$$

$$d^i = \inf_{\lambda \in -\mathcal{L}_+^*} \Big\{ \sup_{\varphi \in R} \Big\{ \int\limits_{\Omega} \varphi \Big[HZ_Q - H\int\limits_{\mathcal{P}} Z_{P^*} d\lambda\Big] dP\Big\} + \widetilde{V}_0 \int\limits_{\mathcal{P}} d\lambda\Big\}, \quad (35)$$

where $-\mathcal{L}_+^* = \{\lambda \in \Lambda : \forall l \in \mathcal{L}_+ : \langle l, \lambda\rangle \geq 0\} = \Lambda_+$ is the set of finite measures on $(\mathcal{P}, \mathcal{S})$. Since $\varphi \in R$ is a randomized test, the supremum over all $\varphi \in R$ in (35) is attained by

$$\overline{\varphi}(\omega) = \begin{cases} 1 & : \quad HZ_Q > H\int_{\mathcal{P}} Z_{P^*} d\lambda \\ 0 & : \quad HZ_Q < H\int_{\mathcal{P}} Z_{P^*} d\lambda \end{cases} \qquad P-a.s.$$

If we denote $HZ_Q - H\int\limits_{\mathcal{P}} Z_{P^*} d\lambda =: \nu_\lambda(\omega)$, the value of the dual problem is

$$d^i = \inf_{\lambda \in \Lambda_+} \Big\{ \int\limits_{\Omega} \nu_\lambda^+(\omega) dP + \widetilde{V}_0 \int\limits_{\mathcal{P}} d\lambda\Big\}. \quad (36)$$

Strong duality holds for f and g convex and g continuous in some $B\varphi_0$ with $\varphi_0 \in \operatorname{dom} f$. The function f is convex since R is a convex set and g is convex since the set $\mathcal{L}_+ - \widetilde{V}_0\mathbf{1}$ is convex. The function g is continuous in some $B\varphi_0$ with $\varphi_0 \in \operatorname{dom} f$ if $B\varphi_0 \in \operatorname{int}(\mathcal{L}_+ - \widetilde{V}_0\mathbf{1})$. If we take $\varphi_0 \equiv 0,\ 0 \in \operatorname{dom} f$, we see that $B\varphi_0 \equiv 0 \in \operatorname{int}(\mathcal{L}_+ - \widetilde{V}_0\mathbf{1})$ for $\widetilde{V}_0 > 0$, since $\operatorname{int}\mathcal{L}_+ \neq \emptyset$.
To demonstrate the dependence from the selected measure $Q \in \mathcal{Q}$ we use the notation $\widetilde{\varphi}_Q$ and $\widetilde{\lambda}_Q$ for the primal and dual solution, respectively. Since $\widetilde{V}_0 > 0$ as assumed in (5), strong duality holds. The existence of a solution $\widetilde{\varphi}_Q \in R_0$ of the primal problem follows from Witting [11], Theorem 2.80. Now with strong duality the existence of a dual solution $\widetilde{\lambda}_Q$ follows and the values of the primal and dual objective function at $\widetilde{\varphi}_Q$,

respectively $\widetilde{\lambda}_Q$, coincide. This leads to a necessary and sufficient optimality condition. We consider the primal objective function

$$\begin{aligned} E[\varphi H Z_Q] &= \int_\Omega \varphi H Z_Q dP \\ &= \int_\Omega \varphi \Big[H Z_Q - H \int_{\mathcal{P}} Z_{P^*} d\lambda \Big] dP + \int_{\mathcal{P}} \int_\Omega \varphi H Z_{P^*} dP d\lambda \\ &= \int_\Omega \varphi \nu_\lambda^+(\omega) dP - \int_\Omega \varphi \nu_\lambda^-(\omega) dP + \int_{\mathcal{P}} \int_\Omega \varphi H Z_{P^*} dP d\lambda \end{aligned}$$

and subtract it from the dual objective function. Because of strong duality the difference has to be zero at $\widetilde{\varphi}_Q$, respectively $\widetilde{\lambda}_Q$:

$$\int_\Omega \Big[1 - \widetilde{\varphi}_Q\Big] \nu^+_{\widetilde{\lambda}_Q}(\omega) dP + \int_\Omega \widetilde{\varphi}_Q \nu^-_{\widetilde{\lambda}_Q}(\omega) dP + \int_{\mathcal{P}} \Big[\widetilde{V}_0 - \int_\Omega \widetilde{\varphi}_Q H Z_{P^*} dP\Big] d\widetilde{\lambda}_Q = 0.$$

The sum of this three nonnegative integrals is zero if $\widetilde{\varphi}_Q \in R_0$ satisfies condition (16) and (17) of Theorem 5.

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