A priori estimates for weak solutions of elliptic equations *

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Abstract

We give a detailed proof of the well-known De Giorgi-Nash theorem on the Hölder continuity of weak solutions of second order elliptic equations in divergence form. The proof is based on Moser iterations and on an abstract lemma of Bombieri and Giusti.

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1 Introduction

The purpose of this paper is to give a detailed proof of a version of the well-known De Giorgi-Nash theorem on the Hölder continuity of weak solutions of second order elliptic equations in divergence form. This remarkable and deep result is originally due to De Giorgi [2] in 1957 and, independently, Nash [7] in 1958. A new proof of the result was found by Moser [5] in 1960. The proof in this paper relies on Moser's iteration technique and on an elementary but subtle lemma due to Bombieri and Giusti [1], by which Moser's original proof of the Harnack inequality simplifies considerably. By means of this lemma one can avoid the well-known John-Nirenberg lemma, i.e. the exponential integrability of BMO-functions, cf. e.g. [4, Theorem 3.5]. This simplification is even of greater significance when studying parabolic equations, see Moser [6] and Saloff-Coste [8].

The plan for this paper is as follows. Section 2 collects the basic tools needed for the proof of the De Giorgi-Nash theorem. We prove first two elementary lemmas on certain functional inequalities (Section 2.1). Next we recall some facts concerning weak derivatives and L_p spaces (Section 2.2). Section 2.3 is devoted to the Moser iteration method. The results are stated in an abstract setting so that they can be also used in the parabolic case. In Section 2.4 we state and prove the lemma of Bombieri and Giusti mentioned above.

Section 3 is concerned with elliptic equations in divergence form. We derive first a local boundedness estimate for weak subsolutions (Section 3.1). In Section 3.2 we prove the weak Harnack inequality for nonnegative supersolutions. An immediate consequence of these local estimates is the Harnack inequality for nonnegative weak solutions. We state this result in Section 3.3. Finally, in Section 3.4, we apply the local estimates from the preceding sections to prove Hölder continuity of weak solutions of the equations under study.

2 Preliminaries

We begin by fixing some notation. By $B_r(y)$ we mean the open ball in $\mathbb{R}^n$ of radius r centered at y. For a ball B and $\delta > 0$, we denote by δB the ball with the same center as B and radius equal to δ times the radius of B. The sup's and inf's below must be understood as essential sup's and inf's. If Ω is a measurable subset of some measure space endowed with the measure μ and if f is a scalar-valued measurable function on Ω, we will write, for $0 < p < \infty$, $|f|_{L_p(\Omega)} = (\int_\Omega |f|^p \, d\mu)^{1/p}$ for short, provided it is clear which measure is meant. The positive part of a real-valued function u is denoted by u^+. The symbol Du stands for the gradient of u.

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2.1 Functional inequalities

The following simple lemma can be found e.g. in [4, Lemma 4.3].

Lemma 2.1 *Let* $0 \le \tau_0 < \tau_1$ *and* $f : [\tau_0, \tau_1] \to [0, \infty)$ *be bounded. Suppose that*

$$f(s) \le \theta f(t) + \frac{A}{(t-s)^\alpha} + B, \quad \tau_0 \le s < t \le \tau_1,$$

where $\theta \in [0,1)$*,* $\alpha > 0$*, and* $A, B \ge 0$*. Then*

$$f(s) \le C\Big(\frac{A}{(t-s)^\alpha} + B\Big), \quad \tau_0 \le s < t \le \tau_1,$$

where $C = C(\alpha, \theta)$ *is a positive constant.*

Proof: Fix $\tau_0 \le s < t \le \tau_1$. For some $\tau \in (0,1)$ we consider the sequence $\{t_i\}$ defined by

$$t_0 = s, \qquad t_{i+1} = t_i + (1-\tau)\tau^i(t-s), \quad i = 0, 1, 2, \ldots$$

Note that

$$t_{i+1} = s + (1-\tau)(t-s)\sum_{j=0}^{i} \tau^j, \quad i = 0, 1, 2, \ldots,$$

and thus $t_i \to t$ as $i \to \infty$. By iteration, we have

$$\begin{aligned} f(s) = \ & f(t_0) \le \theta f(t_1) + \frac{A}{(t_1 - t_0)^\alpha} + B \\ \le \ & \theta\Big(\theta f(t_2) + \frac{A}{(t_2 - t_1)^\alpha} + B\Big) + \frac{A}{(t_1 - t_0)^\alpha} + B \ \le \ \ldots \\ \le \ & \theta^k f(t_k) + \Big(\frac{A}{(1-\tau)^\alpha (t-s)^\alpha} + B\Big)\sum_{j=0}^{k-1} \theta^j \tau^{-j\alpha}, \end{aligned}$$

for all $k = 1, 2, \ldots$ Choose $\tau < 1$ such that $\theta\tau^{-\alpha} < 1$. As $k \to \infty$ we then obtain

$$f(s) \le \Big(\frac{A}{(1-\tau)^\alpha (t-s)^\alpha} + B\Big) c(\alpha, \theta),$$

which proves the assertion of the lemma. $\Box$

The next result has been taken from [3, Lemma 8.23].

Lemma 2.2 *Let* f *be a nondecreasing function on an interval* $(0, T]$ *satisfying*

$$f(\tau t) \le \theta f(t) + g(t), \quad 0 < t \le T,$$

where g *is also nondecreasing and* $\tau, \theta \in (0,1)$*. Then, for any* $\mu \in (0,1)$*,*

$$f(t) \le \frac{1}{\theta}\Big(\frac{t}{T}\Big)^{(1-\mu)\frac{\log\theta}{\log\tau}} f(T) + \frac{g(T^{1-\mu}t^\mu)}{1-\theta}, \quad 0 < t \le T.$$

Proof: Fix initially some number $0 < t_1 \le T$. Then

$$f(\tau t) \le \theta f(t) + g(t_1), \quad 0 < t \le t_1,$$

since g is nondecreasing. We iterate this inequality to obtain

$$\begin{aligned} f(\tau^m t_1) \le \ & \theta f(\tau^{m-1} t_1) + g(t_1) \ \le \ \theta[\theta f(\tau^{m-2} t_1) + g(t_1)] + g(t_1) \\ \le \ & \ldots \ \le \ \theta^m f(t_1) + g(t_1)\sum_{i=0}^{m-1} \theta^i \ \le \ \theta^m f(T) + \frac{g(t_1)}{1-\theta} \end{aligned}$$

for all $m = 1, 2, \dots$ For any $0 < t \le t_1$, we can select m such that

$$\tau^m t_1 < t \le \tau^{m-1} t_1.$$

Then

$$\theta^m \le (\frac{t}{t_1})^{\log\theta/\log\tau}.$$

In fact,

$$\tau^m t_1 < t \;\Leftrightarrow\; e^{m\log\tau} < \frac{t}{t_1} \;\Leftrightarrow\; m\log\tau < \log(t/t_1) \;\Leftrightarrow\; m\log\tau\log\theta > \log\theta\log(t/t_1)$$
$$\Leftrightarrow\; \log(\theta^{m\log\tau}) > \log[(t/t_1)^{\log\theta}] \;\Leftrightarrow\; \theta^{m\log\tau} > (t/t_1)^{\log\theta} \;\Leftrightarrow\; \theta^m < (t/t_1)^{\log\theta/\log\tau}.$$

Therefore,

$$f(t) \;\le\; f(\tau^{m-1}t_1) \;\le\; \theta^{m-1}f(T) + \frac{g(t_1)}{1-\theta} \;\le\; \frac{1}{\theta}(\frac{t}{t_1})^{\log\theta/\log\tau} f(T) + \frac{g(t_1)}{1-\theta}.$$

The assertion follows now by taking $t_1 = T^{1-\mu}t^{\mu}$. $\square$

2.2 Some basic facts about weak derivatives and L_p spaces

We will repeatedly apply the following inequality, which is a consequence of Young's inequality and the interpolation inequality for L_p norms, see e.g. [3, Section 7.1].

$$|u|_{L_q} \le \epsilon|u|_{L_r} + \epsilon^{-\mu}|u|_{L_p}, \quad 1 \le p \le q \le r,\, \epsilon > 0, \tag{1}$$

where

$$\mu = (\frac{1}{p} - \frac{1}{q})/(\frac{1}{q} - \frac{1}{r}).$$

We recall that the classical product formula for derivatives, $D(uv) = uDv + vDu$, extends to weak derivatives, cf. [3, Section 7.3]. We recall further the following chain rule for weak derivatives, see [3, Theorem 7.8].

Theorem 2.1 *Let f be a continuous function on $\mathbb{R}$ with piecewise continuous first derivatives, and assume that $f' \in L_\infty(\mathbb{R})$. Then if u is weakly differentiable, the composition $f \circ u$ is weakly differentiable as well. Moreover, letting K denote the set of corner points, one has*

$$D(f\circ u) = \begin{cases} f'(u)Du : u \notin K \\ 0 \qquad\qquad : u \in K. \end{cases}$$

2.3 Moser iterations

Throughout this subsection U_σ, $0 < \sigma \le 1$, will denote a collection of measurable subsets of a fixed finite measure space endowed with a measure μ, such that $U_{\sigma'} \subset U_\sigma$ if $\sigma' \le \sigma$.

The first Moser iteration result reads as follows.

Lemma 2.3 *Let $\kappa > 1$, $\bar p \ge 1$, $C \ge 1$, and $\gamma > 0$. Suppose f is a measurable function on U_1 such that*

$$|f|_{L_{\alpha\kappa}(U_{\sigma'})} \le \Big(\frac{C(1+\alpha)^\gamma}{(\sigma-\sigma')^\gamma}\Big)^{1/\alpha}|f|_{L_\alpha(U_\sigma)}, \quad 0 < \sigma' < \sigma \le 1,\ \alpha > 0. \tag{2}$$

Then there exist constants $M = M(C,\gamma,\kappa,\bar p)$ and $\gamma_0 = \gamma_0(\gamma,\kappa)$ such that

$$\sup_{U_\delta}|f| \le \Big(\frac{M}{(1-\delta)^{\gamma_0}}\Big)^{1/p}|f|_{L_p(U_1)} \quad \textit{for all } \delta \in (0,1),\ p \in (0,\bar p].$$

Proof: For $q > 0$ and $0 < \sigma \leq 1$, let

$$\Phi(q, \sigma) = (\int_{U_\sigma} |f|^q \, d\mu)^{1/q}.$$

Let $0 < p \leq \bar{p}$ and $\delta \in (0,1)$. Set $p_i = p\kappa^i$, $i = 0, 1, \ldots$ and define the sequence $\{\sigma_i\}$, $i = 0, 1, \ldots$, by $\sigma_0 = 1$ and $\sigma_i = 1 - \sum_{j=1}^{i} 2^{-j}(1-\delta)$, $i = 1, 2, \ldots$; observe that $1 = \sigma_0 > \sigma_1 > \ldots > \sigma_i > \sigma_{i+1} > \delta$ as well as $\sigma_{i-1} - \sigma_i = 2^{-i}(1-\delta)$, $i \geq 1$. Suppose now $N \in \mathbb{N}$. By using (2) with $\alpha = p_i$, $i = 0, 1, \ldots, N-1$, we obtain

$$\begin{aligned}
\Phi(p_N, \delta) \; &\leq \Phi(p_N, \sigma_N) \; = \; \Phi(p_{N-1}\kappa, \sigma_N) \; \leq \; \Big(\frac{C(1 + p\kappa^{N-1})^\gamma}{[2^{-N}(1-\delta)]^\gamma} \Big)^{\frac{1}{p}\kappa^{-(N-1)}} \Phi(p_{N-1}, \sigma_{N-1}) \\
&\leq \Big(\frac{C(2\bar{p}\kappa^{N-1})^\gamma}{[2^{-N}(1-\delta)]^\gamma} \Big)^{\frac{1}{p}\kappa^{-(N-1)}} \Phi(p_{N-1}, \sigma_{N-1}) \\
&\leq \Big(\frac{\tilde{C}(C, \bar{p}, \gamma)^N \kappa^{\gamma(N-1)}}{(1-\delta)^\gamma} \Big)^{\frac{1}{p}\kappa^{-(N-1)}} \Phi(p_{N-1}, \sigma_{N-1}) \; \leq \; \ldots \\
&\leq \Big(\tilde{C}^{\sum_{j=0}^{N-1}(j+1)\kappa^{-j}} \kappa^{\gamma \sum_{j=0}^{N-1} j\kappa^{-j}} (1-\delta)^{-\gamma \sum_{j=0}^{N-1} \kappa^{-j}} \Big)^{1/p} \Phi(p_0, \sigma_0) \\
&\leq \Big(\frac{M(C, \bar{p}, \gamma, \kappa)}{(1-\delta)^{\frac{\gamma\kappa}{\kappa-1}}} \Big)^{1/p} \Phi(p, 1).
\end{aligned}$$

We let now N tend to ∞ and use the fact that

$$\lim_{N \to \infty} \Phi(p_N, \delta) = \sup_{U_\delta} |f|$$

to get

$$\sup_{U_\delta} |f| \leq \Big(\frac{M(C, \bar{p}, \gamma, \kappa)}{(1-\delta)^{\frac{\gamma\kappa}{\kappa-1}}} \Big)^{1/p} |f|_{L_p(U_1)}.$$

Hence the proof is complete. $\square$

By inspection of the above proof, one immediately obtains the following variant of Lemma 2.3.

Lemma 2.4 *Let $\kappa > 1$, $C \geq 1$, $\gamma > 0$, and $p > 0$. Suppose $f \in L_p(U_1)$ satisfies*

$$|f|_{L_{\alpha\kappa}(U_{\sigma'})} \leq \Big(\frac{C(1+\alpha)^\gamma}{(\sigma - \sigma')^\gamma} \Big)^{1/\alpha} |f|_{L_\alpha(U_\sigma)}, \quad 0 < \sigma' < \sigma \leq 1, \; \alpha \geq p. \tag{3}$$

Then there exist constants $M = M(C, \gamma, \kappa, p)$ and $\gamma_0 = \gamma_0(\gamma, \kappa)$ such that

$$\sup_{U_\delta} |f| \leq \Big(\frac{M}{(1-\delta)^{\gamma_0}} \Big)^{1/p} |f|_{L_p(U_1)} \quad \textit{for all } \; \delta \in (0,1).$$

We come now to the second Moser iteration scheme.

Lemma 2.5 *Assume that $\mu(U_1) \leq 1$. Let $\kappa > 1$, $0 < p_0 < \kappa$, and $C \geq 1$, $\gamma > 0$. Suppose f is a measurable function on U_1 such that*

$$|f|_{L_{\alpha\kappa}(U_{\sigma'})} \leq \Big(\frac{C}{(\sigma - \sigma')^\gamma} \Big)^{1/\alpha} |f|_{L_\alpha(U_\sigma)}, \quad 0 < \sigma' < \sigma \leq 1, \; 0 < \alpha \leq \frac{p_0}{\kappa} < 1. \tag{4}$$

Then there exist constants $M = M(C, \gamma, \kappa)$ and $\gamma_0 = \gamma_0(\gamma, \kappa)$ such that

$$|f|_{L_{p_0}(U_\delta)} \leq \Big(\frac{M}{(1-\delta)^{\gamma_0}} \Big)^{1/p - 1/p_0} |f|_{L_p(U_1)} \quad \textit{for all } \; \delta \in (0,1), \; p \in (0, \frac{p_0}{\kappa}].$$

Proof: Set $p_i = p_0\kappa^{-i}$, $i = 1, 2, \ldots$. Given $\delta \in (0,1)$ we take again the sequence $\{\sigma_i\}$, $i = 0, 1, 2, \ldots$, defined by $\sigma_0 = 1$ and $\sigma_i = 1 - \sum_{j=1}^{i} 2^{-j}(1-\delta)$, $i \geq 1$. Suppose now $N \in \mathbb{N}$. By using (4) with $\alpha = p_i$, $i = 1, \ldots, N$, we obtain

$$\begin{aligned}
\Phi(p_0,\delta) \leq \Phi(p_0,\sigma_N) &= \Phi(p_1\kappa,\sigma_N) \leq \frac{C^{\kappa/p_0}}{[2^{-N}(1-\delta)]^{\gamma\kappa/p_0}}\,\Phi(p_1,\sigma_{N-1}) \\
&\leq \frac{C^{\kappa/p_0}}{[2^{-N}(1-\delta)]^{\gamma\kappa/p_0}}\,\frac{C^{\kappa^2/p_0}}{[2^{-(N-1)}(1-\delta)]^{\gamma\kappa^2/p_0}}\,\Phi(p_2,\sigma_{N-2}) \leq \ldots \\
&\leq \frac{C^{\frac{1}{p_0}(\kappa+\kappa^2+\ldots+\kappa^N)}}{2^{-\frac{\gamma}{p_0}(N\kappa+(N-1)\kappa^2+\ldots+2\kappa^{N-1}+\kappa^N)}(1-\delta)^{\frac{\gamma}{p_0}(\kappa+\kappa^2+\ldots+\kappa^N)}}\,\Phi(p_N,\sigma_0).
\end{aligned}$$

Since $p_i = p_0\kappa^{-i}$, we have

$$\frac{1}{p_0}\sum_{j=1}^{N}\kappa^j = \frac{\kappa(\kappa^N-1)}{p_0(\kappa-1)} = \frac{\kappa}{p_0(\kappa-1)}\,(\frac{p_0}{p_N}-1) = \frac{\kappa}{\kappa-1}\,(\frac{1}{p_N}-\frac{1}{p_0}).$$

Employing the formula

$$\sum_{j=1}^{N} j\kappa^{j-1} = \frac{1-(N+1)\kappa^N+N\kappa^{N+1}}{(\kappa-1)^2}$$

we have further

$$\begin{aligned}
\sum_{j=1}^{N}(N+1-j)\kappa^j &= (N+1)\sum_{j=1}^{N}\kappa^j - \sum_{j=1}^{N} j\kappa^j \\
&= (N+1)\kappa\,\frac{\kappa^N-1}{\kappa-1} - \kappa\,\frac{1-(N+1)\kappa^N+N\kappa^{N+1}}{(\kappa-1)^2} \\
&= \kappa\,\frac{\kappa^{N+1}-(N+1)\kappa+N}{(\kappa-1)^2} \leq \frac{\kappa}{(\kappa-1)^2}\kappa^{N+1} \\
&\leq \frac{\kappa^3}{(\kappa-1)^3}\,(\kappa^N-1) \leq \frac{\kappa^3}{(\kappa-1)^3}\,(\frac{p_0}{p_N}-1),
\end{aligned}$$

which yields

$$\frac{1}{p_0}\sum_{j=1}^{N}(N+1-j)\kappa^j \leq \frac{\kappa^3}{(\kappa-1)^3}\,(\frac{1}{p_N}-\frac{1}{p_0}).$$

Therefore

$$\Phi(p_0,\delta) \leq \Big[\frac{2^{\frac{\gamma\kappa^3}{(\kappa-1)^3}}C^{\frac{\kappa}{\kappa-1}}}{(1-\delta)^{\frac{\gamma\kappa}{\kappa-1}}}\Big]^{\frac{1}{p_N}-\frac{1}{p_0}}\Phi(p_N,\sigma_0).$$

Given $p \in (0, p_0/\kappa]$ there exists $N \geq 2$ such that $p_N < p \leq p_{N-1}$. We then have

$$\begin{aligned}
\frac{1}{p_N}-\frac{1}{p_0} &= \frac{\kappa^N-1}{p_0} \leq \frac{\kappa^N+\kappa^{N-1}-\kappa-1}{p_0} = \frac{(1+\kappa)(\kappa^{N-1}-1)}{p_0} \\
&= (1+\kappa)(\frac{1}{p_{N-1}}-\frac{1}{p_0}) \leq (1+\kappa)(\frac{1}{p}-\frac{1}{p_0}),
\end{aligned}$$

as well as

$$\Phi(p_N,\sigma_0) = \Phi(p_N,1) \leq \Phi(p,1),$$

by Hölder's inequality and the assumption $\mu(U_1) \leq 1$. All in all, we obtain

$$\Phi(p_0,\delta) \leq \Big[\frac{2^{\frac{\gamma\kappa^3}{(\kappa-1)^3}}C^{\frac{\kappa}{\kappa-1}}}{(1-\delta)^{\frac{\gamma\kappa}{\kappa-1}}}\Big]^{(1+\kappa)(\frac{1}{p}-\frac{1}{p_0})}\Phi(p,1),$$

which proves the lemma. $\square$

2.4 An abstract lemma

In this section we present an abstract lemma due to Bombieri and Giusti [1], which will play an essential role in the proof of the weak Harnack inequality. We refer also to Saloff-Coste [8, Lemma 2.2.6].

Lemma 2.6 *Let U_σ, $0 < \sigma \le 1$, be a collection of measurable subsets of a fixed finite measure space endowed with a measure μ, such that $U_{\sigma'} \subset U_\sigma$ if $\sigma' \le \sigma$. Let δ, $\eta \in (0,1)$, and let γ, C be positive constants and $0 < \alpha_0 \le \infty$. Suppose f is a positive measurable function on $U_1 =: U$ which satisfies the following two conditions:*

(i)

$$|f|_{L_{\alpha_0}(U_{\sigma'})} \le [C(\sigma-\sigma')^{-\gamma}\mu(U)^{-1}]^{1/\alpha-1/\alpha_0}|f|_{L_\alpha(U_\sigma)},$$

for all σ, σ', α such that $0<\delta\le\sigma'<\sigma\le 1$ and $0<\alpha\le\min\{1,\eta\alpha_0\}$.

(ii)

$$\mu(\log f > \lambda) \le C\mu(U)\lambda^{-1}$$

for all $\lambda > 0$.

Then

$$|f|_{L_{\alpha_0}(U_\delta)} \le M\mu(U)^{1/\alpha_0},$$

where M depends only on δ, η, γ, C, and α_0.

Proof: W.l.o.g. we may assume $\mu(U) = 1$. Set

$$\psi(\sigma) = \log(|f|_{L_{\alpha_0}(U_\sigma)}), \quad 0<\delta\le\sigma\le 1.$$

The assumptions (i) and (ii) imply that

$$\psi(\sigma') \le \frac{3}{4}\psi(\sigma) + M_0(\sigma-\sigma')^{-2\gamma}, \quad \delta \le \sigma' < \sigma \le 1, \tag{5}$$

with some constant M_0 depending only on C, α_0 and η, as we want to show.

To this purpose we decompose U_σ into the sets

$$U_\sigma^1 = U_\sigma \cap \{\log f > \psi(\sigma)/2\}, \quad U_\sigma^2 = U_\sigma \cap \{\log f \le \psi(\sigma)/2\}$$

and accordingly estimate, using (ii) and Hölder's inequality,

$$\begin{aligned}\int_{U_\sigma^1} f^\alpha\, d\mu &\le \Big(\int_{U_\sigma^1} f^{\alpha_0}\, d\mu\Big)^{\alpha/\alpha_0}\mu(U_\sigma^1)^{1-\alpha/\alpha_0}\\ &\le \exp(\alpha\log(|f|_{L_{\alpha_0}(U_\sigma^1)}))\,\mu(\log f > \psi(\sigma)/2)^{\alpha(1/\alpha-1/\alpha_0)}\\ &\le e^{\alpha\psi(\sigma)}\Big(\frac{2C}{\psi(\sigma)}\Big)^{\alpha(1/\alpha-1/\alpha_0)}\end{aligned}$$

as well as

$$\int_{U_\sigma^2} f^\alpha\, d\mu = \int_{U_\sigma^2} e^{\alpha\log f}\, d\mu \le e^{\alpha\psi(\sigma)/2}.$$

So for $0<\alpha\le 1$ we deduce that

$$|f|^\alpha_{L_\alpha(U_\sigma)} \le e^{\alpha\psi(\sigma)}\Big(\frac{2C}{\psi(\sigma)}\Big)^{\alpha(1/\alpha-1/\alpha_0)} + e^{\alpha\psi(\sigma)/2}. \tag{6}$$

We want to choose α so that the two terms on the right-hand side of (6) coincide, and $0<\alpha\le\min\{1,\eta\alpha_0\}$. This is possible if $\psi(\sigma) > 2C$ and

$$(1/\alpha - 1/\alpha_0)^{-1} = \frac{2}{\psi(\sigma)}\log\Big(\frac{\psi(\sigma)}{2C}\Big) \le \min\{1,\eta\alpha_0\},$$

and these two conditions in turn are certainly satisfied when

$$\psi(\sigma) > M_1, \tag{7}$$

where M_1 is a sufficiently large number depending only on α_0, η, and C. Supposing that (7) holds and that α has been selected as above, we obtain

$$|f|^{\alpha}_{L_\alpha(U_\sigma)} \leq 2e^{\alpha\psi(\sigma)/2},$$

which together with (i), yields

$$\begin{aligned}\psi(\sigma') \leq\ & \log\Big[(C(\sigma-\sigma')^{-\gamma})^{1/\alpha-1/\alpha_0}|f|_{L_\alpha(U_\sigma)}\Big]\\ \leq\ & (\frac{1}{\alpha}-\frac{1}{\alpha_0})\log(C(\sigma-\sigma')^{-\gamma})+\frac{\psi(\sigma)}{2}+\frac{1}{\alpha}\log 2,\end{aligned}$$

for $\delta \leq \sigma' \leq \sigma \leq 1$. By our choice of α specified above, we get

$$\psi(\sigma') \leq \frac{\psi(\sigma)}{2}\Big[\frac{\log(2C(\sigma-\sigma')^{-\gamma})}{\log(\psi(\sigma)/2C)}+1\Big]+\frac{1}{\alpha_0}\log 2.$$

On the one hand, if

$$\psi(\sigma) \geq 8C^3(\sigma-\sigma')^{-2\gamma}, \tag{8}$$

we have

$$\psi(\sigma') \leq \frac{3}{4}\psi(\sigma)+\frac{1}{\alpha_0}\log 2.$$

On the other hand, if (8) does not hold, we evidently have

$$\psi(\sigma') \leq \psi(\sigma) < 8C^3(\sigma-\sigma')^{-2\gamma}.$$

Finally, if (7) is not satisfied, we may estimate

$$\psi(\sigma') \leq \psi(\sigma) \leq M_1 \leq M_1(\sigma-\sigma')^{-2\gamma}.$$

All in all, we see that ψ enjoys the property (5). We apply now Lemma 2.1 to the function ψ to get

$$\psi(\sigma') \leq M_2(\alpha_0,\gamma,C,\eta)(\sigma-\sigma')^{-2\gamma}, \quad \delta \leq \sigma' < \sigma \leq 1.$$

Choosing $\sigma' = \delta$ and $\sigma = 1$ then yields

$$\psi(\delta) \leq M_3(\alpha_0,\gamma,C,\delta,\eta),$$

whence the assertion follows. □

3 Elliptic Problems

In this section we study the equation

$$-D_i(a^{ij}D_ju) = f \tag{9}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. We assume throughout that

(H1) $a^{ij} \in L_\infty(\Omega)$, $i,j = 1,\dots,n$, and $\sum_{i,j=1}^n |a^{ij}(x)|^2 \leq \Lambda^2$, $x \in \Omega$;

(H2) $\exists\lambda > 0$ such that $a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, $x \in \Omega$, $\xi \in \mathbb{R}^n$;

(H3) $f \in L_{q/2}(\Omega)$ for some $q > n$.

A function $u \in W^{1,2}(\Omega)$ is called *weak solution* (*subsolution, supersolution*) of (9) if

$$\int_\Omega a^{ij}D_juD_iv\,dx = \ (\ \leq\ ,\ \geq\ )\int_\Omega fv\,dx \tag{10}$$

for all nonnegative $v \in W_0^{1,2}(\Omega)$.

3.1 Subsolutions

In this section we prove the subsequent local boundedness result for subsolutions. The proof follows basically the lines in [4].

Theorem 3.1 *Let the assumptions (H1)-(H3) be satisfied. Let $\delta \in (0,1)$ and $p > 0$. Then there exists a constant $C = C(n, \Lambda/\lambda, q, p, \delta)$ such that for any ball $B \subset \Omega$ of radius r and any subsolution u of (9),*

$$\sup_{\delta B} u^+ \le C\Big((1-\delta)^{-n/p}\big(\frac{1}{|B|}\int_B (u^+)^p\,dx\big)^{1/p} + \frac{1}{\lambda} r^{2(1-n/q)} |f|_{L_{q/2}(B)}\Big).$$

Proof: **1. Scaling:** Let $B = B_r(x_0) \subset \Omega$ be a ball and suppose u is a subsolution of (9). Then

$$\int_B a^{ij}(x) D_j u(x) D_i v(x)\,dx \le \int_B f(x)v(x)\,dx \tag{11}$$

for all nonnegative $v \in W_0^{1,2}(B)$. By means of the coordinate transformation $y = (x - x_0)/r$, we see that (11) is equivalent to

$$\int_{B_1(0)} \tilde{a}^{ij}(y) D_j \tilde{u}(y) D_i \tilde{v}(y)\,dy \le \int_{B_1(0)} \tilde{f}(y)\tilde{v}(y)\,dy \tag{12}$$

for all nonnegative $\tilde{v} \in W_0^{1,2}(B_1(0))$, where

$$\tilde{u}(y) = u(ry + x_0), \quad y \in B_1(0),$$

and

$$\tilde{a}(y) = a(ry + x_0), \qquad \tilde{f}(y) = r^2 f(ry + x_0), \quad y \in B_1(0).$$

Observe that $\tilde{a}^{ij} \in L_\infty(B_1(0))$, $i, j = 1, \ldots, n$, and $\sum_{i,j=1}^n |\tilde{a}^{ij}(y)|^2 \le \Lambda^2$, $y \in B_1(0)$. Moreover $\tilde{a}^{ij}(y)\xi_i\xi_j \ge \lambda|\xi|^2$, $y \in B_1(0)$, $\xi \in \mathbb{R}^n$, and $\tilde{f} \in L_{q/2}(B_1(0))$. Also

$$\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} (u^+)^p\,dx = \frac{r^n}{|B_r(x_0)|}\int_{B_1(0)} (\tilde{u}^+)^p\,dy = \frac{1}{|B_1(0)|}\int_{B_1(0)} (\tilde{u}^+)^p\,dy$$

as well as

$$r^{2(1-n/q)} |f|_{L_{q/2}(B_r(x_0))} = |\tilde{f}|_{L_{q/2}(B_1(0))}.$$

All this shows that we may assume w.l.o.g. that $r = 1$.

2. The case $\delta = 1/2$ and $p \ge 2$: Let B be a ball in Ω with radius 1. Set $\bar{u} = u^+ + k$, where we choose $k = |f|_{L_{q/2}(B)}/\lambda$ if f is not identically zero in B; otherwise choose an arbitrary $k > 0$ and eventually let $k \to 0+$. For $m > 0$ we further set

$$\bar{u}_m = \begin{cases} \bar{u} & : u < m \\ k + m & : u \ge m. \end{cases}$$

Then $\bar{u}_m \le \bar{u}$ and by the remarks in Section 2.2, the weak gradient $D\bar{u}_m$ is given by

$$D\bar{u}_m = \begin{cases} Du\ (= D\bar{u}) & : 0 < u < m \\ 0 & : u \le 0 \text{ or } u \ge m. \end{cases}$$

We choose now the nonnegative test function $v = \psi^2(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \in W_0^{1,2}(\Omega)$ with $\beta \ge 0$ and nonnegative $\psi \in C_0^1(\Omega)$ such that $\operatorname{supp}\psi \subset B$. Direct calculation gives

$$\begin{aligned} Dv &= 2\psi D\psi(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) + \beta\psi^2 \bar{u}_m^{\beta-1} D\bar{u}_m \bar{u} + \psi^2 \bar{u}_m^\beta D\bar{u} \\ &= 2\psi D\psi(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) + \psi^2 \bar{u}_m^\beta(\beta D\bar{u}_m + D\bar{u}); \end{aligned}$$

here we use the fact that $\bar u_m = \bar u$ if $u < m$ and $D\bar u_m = 0$ if $u \ge m$. Observe that $v = 0$ and $Dv = 0$ in the set $\{u \le 0\}$. Thus if we substitute such v in (10) we integrate over $B^+ := B \cap \{u > 0\}$ and obtain, by assumption (H1),

$$\int_{B^+} a^{ij} D_j \bar u (\beta D_i \bar u_m + D_i \bar u)\psi^2 \bar u_m^\beta \, dx \ \le \int_{B^+} 2\Lambda\psi |D\psi|\, |D\bar u| (\bar u_m^\beta \bar u - k^{\beta+1})\, dx + \int_{B^+} |f| \psi^2 (\bar u_m^\beta \bar u - k^{\beta+1})\, dx.$$

Clearly, $\bar u_m^\beta \bar u - k^{\beta+1} \le \bar u_m^\beta \bar u$. Using this and (H2) we deduce further

$$\lambda\beta \int_{B^+} \psi^2 \bar u_m^\beta |D\bar u_m|^2\, dx + \lambda \int_{B^+} \psi^2 \bar u_m^\beta |D\bar u|^2\, dx \le \int_{B^+} 2\Lambda\psi|D\psi|\, |D\bar u| \bar u_m^\beta \bar u\, dx + \int_{B^+} |f|\psi^2 \bar u_m^\beta \bar u\, dx.$$

Young's inequality yields

$$2\Lambda\psi|D\psi|\, |D\bar u|\bar u_m^\beta \bar u \le \epsilon\psi^2 |D\bar u|^2 \bar u_m^\beta + \epsilon^{-1}\Lambda^2 |D\psi|^2 \bar u_m^\beta \bar u^2, \quad \epsilon > 0.$$

So if we take $\epsilon = \lambda/2$ we obtain

$$\lambda\beta \int_{B^+} \psi^2 \bar u_m^\beta |D\bar u_m|^2\, dx + \frac{\lambda}{2} \int_{B^+} \psi^2 \bar u_m^\beta |D\bar u|^2\, dx \le \frac{2\Lambda^2}{\lambda} \int_{B^+} |D\psi|^2 \bar u_m^\beta \bar u^2\, dx + \int_{B^+} |f|\psi^2 \bar u_m^\beta \bar u\, dx.$$

Dividing by λ and using $k \le \bar u$, as well as $D\bar u_m = D\bar u = 0$ in $\{u \le 0\}$ we conclude

$$\beta \int_B \psi^2 \bar u_m^\beta |D\bar u_m|^2\, dx + \frac{1}{2} \int_B \psi^2 \bar u_m^\beta |D\bar u|^2\, dx \le \frac{2\Lambda^2}{\lambda^2} \int_B |D\psi|^2 \bar u_m^\beta \bar u^2\, dx + \int_B \frac{|f|}{\lambda k} \psi^2 \bar u_m^\beta \bar u^2\, dx.$$

Set now $w = \bar u_m^{\beta/2} \bar u$. Then

$$Dw = \frac{\beta}{2}\, \bar u_m^{\frac{\beta}{2}-1} D\bar u_m\, \bar u + \bar u_m^{\beta/2} D\bar u = \frac{\beta}{2}\, \bar u_m^{\frac{\beta}{2}} D\bar u_m + \bar u_m^{\beta/2} D\bar u,$$

thanks to $\bar u D\bar u_m = \bar u_m D\bar u_m$, and hence

$$|Dw|^2 \le \frac{\beta^2}{2}\, \bar u_m^\beta |D\bar u_m|^2 + 2\bar u_m^\beta |D\bar u|^2 \le 4(1+\beta)(\beta \bar u_m^\beta |D\bar u_m|^2 + \frac{1}{2}\bar u_m^\beta |D\bar u|^2).$$

Therefore we have

$$\int_B \psi^2 |Dw|^2\, dx \le (1+\beta)\, \frac{8\Lambda^2}{\lambda^2} \int_B |D\psi|^2 w^2\, dx + 4(1+\beta) \int_B \frac{|f|}{\lambda k}\, \psi^2 w^2\, dx. \tag{13}$$

By the Sobolev inequality, applied to the function $\psi w \in W_0^{1,2}(B)$, we obtain

$$|\psi w|^2_{L_{2\hat n/(\hat n-2)}(B)} \le C(\hat n) |D(\psi w)|^2_{L_2(B)} \le 2C(\hat n) \int_B (\psi^2 |Dw|^2 + |D\psi|^2 w^2)\, dx, \tag{14}$$

where $\hat n = n$ for $n > 2$ and $\hat 2$ is a fixed number in the interval $(2, q)$. Applying Hölder's inequality and the interpolation inequality (1), we get, for any $\epsilon > 0$,

$$\begin{aligned} \int_B \frac{|f|}{\lambda k} \psi^2 w^2\, dx \ &\le \ \frac{1}{\lambda k}\, |f|_{L_{q/2}(B)} |\psi^2 w^2|_{L_{q/(q-2)}(B)} \ \le \ |\psi w|^2_{L_{2q/(q-2)}(B)} \\ &\le \ (\epsilon |\psi w|_{L_{2\hat n/(\hat n-2)}(B)} + \epsilon^{-\mu} |\psi w|_{L_2(B)})^2 \\ &\le \ 2(\epsilon^2 |\psi w|^2_{L_{2\hat n/(\hat n-2)}(B)} + \epsilon^{-2\mu} |\psi w|^2_{L_2(B)}), \end{aligned} \tag{15}$$

where $\mu = \frac{\hat{n}}{q-\hat{n}}$. By substitution of (14) and (15) into (13), we obtain

$$\begin{aligned}|\psi w|^2_{L_{2\hat{n}/(\hat{n}-2)}(B)} &\leq 2C(\hat{n})\Big(1 + \frac{8\Lambda^2(1+\beta)}{\lambda^2}\Big)\int_B |D\psi|^2 w^2\,dx \\ &\quad + 16C(\hat{n})(1+\beta)(\epsilon^2|\psi w|^2_{L_{2\hat{n}/(\hat{n}-2)}(B)} + \epsilon^{-2\mu}|\psi w|^2_{L_2(B)}).\end{aligned}$$

We choose then $\epsilon^2 = [32C(\hat{n})(1+\beta)]^{-1}$, thereby getting

$$|\psi w|^2_{L_{2\hat{n}/(\hat{n}-2)}(B)} \leq 4C(\hat{n})\Big(1 + \frac{8\Lambda^2(1+\beta)}{\lambda^2}\Big)\int_B |D\psi|^2 w^2\,dx + [32C(\hat{n})(1+\beta)]^{\mu+1}|\psi w|^2_{L_2(B)},$$

that is,

$$|\psi w|^2_{L_{2\hat{n}/(\hat{n}-2)}(B)} \leq C(1+\beta)^{\mu+1}|(\psi + |D\psi|)w|^2_{L_2(B)}, \tag{16}$$

where $C = C(\hat{n}, \Lambda/\lambda, q)$.

Given $0 < \rho' < \rho \leq 1$, we pick $\psi \in C_0^1(B)$ so that $0 \leq \psi \leq 1$, $\psi = 1$ in $\rho' B$, $\operatorname{supp}\psi \subset \rho B$, and $|D\psi| \leq 2/(\rho - \rho')$. From (16) we then infer that

$$|w|^2_{L_{2\hat{n}/(\hat{n}-2)}(\rho' B)} \leq \frac{5C(1+\beta)^{\mu+1}}{(\rho-\rho')^2}\,|w|^2_{L_2(\rho B)};$$

here we use also the trivial estimate $1 \leq (\rho-\rho')^{-2}$. Recalling the definition of w and setting $\hat{\kappa} = \hat{n}/(\hat{n}-2)$ we have

$$\Big(\int_{\rho' B} \bar{u}^{2\hat{\kappa}}\bar{u}_m^{\beta\hat{\kappa}}\,dx\Big)^{\frac{1}{\hat{\kappa}}} \leq \frac{5C(1+\beta)^{\mu+1}}{(\rho-\rho')^2}\int_{\rho B}\bar{u}^2\bar{u}_m^{\beta}\,dx.$$

Since $\bar{u}_m \leq \bar{u}$ we obtain with $\gamma = \beta + 2$,

$$\Big(\int_{\rho' B} \bar{u}_m^{\gamma\hat{\kappa}}\,dx\Big)^{\frac{1}{\hat{\kappa}}} \leq \frac{5C(\gamma-1)^{\mu+1}}{(\rho-\rho')^2}\int_{\rho B}\bar{u}^{\gamma}\,dx$$

provided the integral on the right-hand side is bounded. We let now $m \to \infty$ and conclude, by the monotone convergence theorem, that

$$\Big(\int_{\rho' B} \bar{u}^{\gamma\hat{\kappa}}\,dx\Big)^{\frac{1}{\hat{\kappa}}} \leq \frac{5C(\gamma-1)^{\mu+1}}{(\rho-\rho')^2}\int_{\rho B}\bar{u}^{\gamma}\,dx$$

whenever $\bar{u} \in L_\gamma(\rho B)$. Hence, with $\tau = \max\{2, 1+\mu\}$, we have

$$|\bar{u}|_{L_{\gamma\hat{\kappa}}(\rho' B)} \leq \Big(\frac{5C(1+\gamma)^{\tau}}{(\rho-\rho')^{\tau}}\Big)^{1/\gamma}|\bar{u}|_{L_\gamma(\rho B)}, \quad 0 < \rho' < \rho \leq 1,\ \gamma \geq 2,$$

provided that $\bar{u} \in L_\gamma(\rho B)$. Thus, since $\bar{u} \in L_2(B)$, we can apply Lemma 2.4 with $p = 2$ and $U_\sigma = \sigma B$, thereby obtaining

$$\sup_{\delta B}\bar{u} \leq \Big(\frac{M_0}{(1-\delta)^{\tau_0}}\Big)^{1/2}|\bar{u}|_{L_2(B)} \quad \text{for all } \delta \in (0,1),$$

where $M_0 = M_0(\hat{n}, \Lambda/\lambda, q)$ and $\tau_0 = \tau_0(\hat{n}, q)$. In particular, we get

$$\sup_{\frac{1}{2}B}\bar{u} \leq M_1(\hat{n}, \Lambda/\lambda, q)\,|\bar{u}|_{L_2(B)},$$

and thus

$$\begin{aligned}\sup_{\frac{1}{2}B}u^+ \leq \sup_{\frac{1}{2}B}\bar{u} &\leq M_1|u^+ + k|_{L_2(B)} \leq M_1(|u^+|_{L_2(B)} + k|B|^{1/2}) \\ &\leq M_2(n, \hat{2}, \Lambda/\lambda, q)\Big(|u^+|_{L_2(B)} + \frac{1}{\lambda}|f|_{L_{q/2}(B)}\Big).\end{aligned}$$

Hence, if $B = B_r \subset \Omega$ has radius r, we obtain for $p \geq 2$,

$$\sup_{\frac{1}{2}B_r} u^+ \leq M\Big(\frac{1}{r^{n/p}}\,|u^+|_{L_p(B_r)} + \frac{1}{\lambda}\,r^{2(1-n/q)}|f|_{L_{q/2}(B_r)}\Big), \tag{17}$$

where $M = M(n, \hat{2}, \Lambda/\lambda, q)$.

3. The case $\delta \in (0,1)$ and $p \geq 2$: Let $\delta \in (0,1)$ and $B = B_r \subset \Omega$. The desired estimate in δB can be derived by applying (17) to $B_{(1-\delta)r}(y)$ for any center $y \in \delta B_r$. In fact,

$$\begin{aligned}
\sup_{\delta B_r} u^+ &\leq \sup_{y\in\delta B_r}\Big(\sup_{B_{(1-\delta)r/2}(y)} u^+\Big) \\
&\leq \sup_{y\in\delta B_r} M\Big(\frac{1}{[(1-\delta)r]^{n/p}}\,|u^+|_{L_p(B_{(1-\delta)r}(y))} + \frac{1}{\lambda}\,[(1-\delta)r]^{2(1-n/q)}|f|_{L_{q/2}(B_{(1-\delta)r}(y))}\Big) \\
&\leq M\Big(\frac{1}{[(1-\delta)r]^{n/p}}\,|u^+|_{L_p(B_r)} + \frac{1}{\lambda}\,r^{2(1-n/q)}|f|_{L_{q/2}(B_r)}\Big).
\end{aligned}$$

4. The case $\delta \in (0,1)$ and $p \in (0,2)$: Let $B \subset \Omega$ be a ball with radius r. We assume again w.l.o.g. that $r = 1$. We know already, cf. (17), that for any $\delta \in (0,1)$ and $0 < \tau \leq 1$, one has

$$\begin{aligned}
|u^+|_{L_\infty(\delta\tau B)} &\leq M\Big(\frac{1}{[(1-\delta)\tau]^{n/2}}\,|u^+|_{L_2(\tau B)} + \frac{1}{\lambda}\,\tau^{2(1-n/q)}|f|_{L_{q/2}(\tau B)}\Big) \\
&\leq M\Big(\frac{1}{[(1-\delta)\tau]^{n/2}}\,|u^+|_{L_2(\tau B)} + \frac{1}{\lambda}\,|f|_{L_{q/2}(B)}\Big).
\end{aligned}$$

Suppose now that $p \in (0,2)$. Then

$$\int_{\tau B}(u^+)^2\,dx \leq |u^+|^{2-p}_{L_\infty(\tau B)}\int_{\tau B}(u^+)^p\,dx \leq |u^+|^{2-p}_{L_\infty(\tau B)}\int_B (u^+)^p\,dx,$$

and hence

$$|u^+|_{L_\infty(\delta\tau B)} \leq M\Big(\frac{1}{[(1-\delta)\tau]^{n/2}}\,|u^+|^{1-\frac{p}{2}}_{L_\infty(\tau B)}(\int_B (u^+)^p\,dx)^{\frac{1}{2}} + \frac{1}{\lambda}\,|f|_{L_{q/2}(B)}\Big).$$

By applying Young's inequality, $ab \leq a^s/s + b^{s'}/s'$, with $s = 2/(2-p)$ and $s' = 2/p$ to

$$a = \frac{1}{2}\,|u^+|^{1-\frac{p}{2}}_{L_\infty(\tau B)}, \qquad b = \frac{2M}{[(1-\delta)\tau]^{n/2}}\,(\int_B (u^+)^p\,dx)^{\frac{1}{2}},$$

and noting that $(1/2)^s/s \leq 1/2$, we get

$$|u^+|_{L_\infty(\delta\tau B)} \leq \frac{1}{2}\,|u^+|_{L_\infty(\tau B)} + \frac{p(2M)^{2/p}}{2[(1-\delta)\tau]^{n/p}}\,(\int_B (u^+)^p\,dx)^{\frac{1}{p}} + \frac{M}{\lambda}\,|f|_{L_{q/2}(B)}.$$

Set next $g(\tau) = |u^+|_{L_\infty(\tau B)}$ for $\tau \in (0,1]$. Then g is a bounded nonnegative function and we have for any $0 < \tau' < \tau \leq 1$,

$$g(\tau') \leq \frac{1}{2}\,g(\tau) + \frac{\tilde{M}}{(\tau-\tau')^{n/p}}\,|u^+|_{L_p(B)} + \frac{M}{\lambda}\,|f|_{L_{q/2}(B)},$$

where $\tilde{M} = \tilde{M}(n, \hat{2}, \Lambda/\lambda, q, p)$. We fix now $\delta \in (0,1)$ and apply Lemma 2.1 with $\tau_0 = \delta$ and $\tau_1 = 1$, thereby getting for any $\delta \leq \tau' < \tau \leq 1$,

$$g(\tau') \leq C\Big(\frac{\tilde{M}}{(\tau-\tau')^{n/p}}\,|u^+|_{L_p(B)} + \frac{M}{\lambda}\,|f|_{L_{q/2}(B)}\Big),$$

where $C = C(n,p)$. In particular, if we choose $\tau' = \delta$ and $\tau = 1$, we see that there exists a constant $\tilde{C} = \tilde{C}(n, \hat{2}, \Lambda/\lambda, q, p)$ such that

$$|u^+|_{L_\infty(\delta B)} \leq \tilde{C}\Big((1-\delta)^{-n/p}\,|u^+|_{L_p(B)} + \frac{1}{\lambda}\,|f|_{L_{q/2}(B)}\Big).$$

Hence the proof is complete. □

3.2 Supersolutions

The aim of this section is to prove the following *weak Harnack inequality* for nonnegative supersolutions of (9). A variant of the proof below in the homogeneous case can be found in [8].

Theorem 3.2 *Let the assumptions (H1)-(H3) be satisfied. Let $\delta \in (0,1)$ and assume that $0 < p < \kappa = \frac{n}{n-2}$ if $n > 2$, $0 < p < \infty$ in case $n = 2$. Then there exists a constant $C = C(n, \Lambda/\lambda, q, p, \delta)$ such that for any ball $B \subset \Omega$ of radius r and any nonnegative supersolution u of (9),*

$$(\frac{1}{|\delta B|}\int_{\delta B} u^p\,dx)^{1/p} \le C\Big(\inf_{\delta B} u + \frac{1}{\lambda}\, r^{2(1-n/q)}|f|_{L_{q/2}(B)}\Big).$$

Proof: **1. Notation and plan:** Let δ and p be as in the statement of the theorem, and let B be a ball in Ω. Put $\eta = (1+\delta)/2$ and $B_1 = \eta B$. W.l.o.g. we may assume that $|B_1| = 1$. Set $\bar u = u + k$, where we choose $k = |f|_{L_{q/2}(B)}/\lambda$ if f is not identically zero in B; otherwise choose an arbitrary $k > 0$ and eventually let $k \to 0+$. The idea of the proof is to apply Lemma 2.6 to $e^{-c(\bar u)}\bar u$ and $e^{c(\bar u)}\bar u^{-1}$ where $c(\bar u) = |\log \bar u|_{L_1(B_1)}$. Set $U = U_1 = B_1$ and $U_\sigma = \sigma B_1$ for $\sigma \in [\delta', 1]$, where $\delta' = \delta/\eta$. Note that

$$U_{\delta'} = \frac{\delta}{\eta}\,B_1 = \frac{\delta}{\eta}\,\eta B = \delta B.$$

Fix now σ' and σ such that $\delta' \le \sigma' < \sigma \le 1$, and set $B' = U_\sigma$.

2. Basic estimates: We choose the test function $v = \psi^2 \bar u^\beta$ with $\beta < 0$, $\beta \neq -1$, and nonnegative $\psi \in C_0^1(\Omega)$ such that $\operatorname{supp}\psi \subset B'$. Clearly v is nonnegative and belongs to the space $W_0^{1,2}(\Omega)$. We have

$$Dv = 2\psi D\psi \bar u^\beta + \beta\psi^2\bar u^{\beta-1}D\bar u,$$

so that by substitution into

$$\int_{B'} a^{ij}D_j u D_i v\,dx \ge \int_{B'} fv\,dx$$

and assumption (H1) we obtain

$$|\beta|\int_{B'} a^{ij}D_j\bar u D_i\bar u\,\psi^2\bar u^{\beta-1}\,dx \le \int_{B'} 2\Lambda\psi|D\psi|\,|D\bar u|\bar u^\beta\,dx + \int_{B'}|f|\psi^2\bar u^\beta\,dx. \tag{18}$$

By Young's inequality,

$$2\Lambda\psi|D\psi|\,|D\bar u|\bar u^\beta = 2\Lambda\psi|D\psi|\,|D\bar u|\bar u^{\frac{\beta-1}{2}}\bar u^{\frac{\beta+1}{2}} \le \epsilon\psi^2|D\bar u|^2\bar u^{\beta-1} + \frac{1}{\epsilon}\Lambda^2|D\psi|^2\bar u^{\beta+1} \tag{19}$$

for all $\epsilon > 0$. Since $1 \le \bar u/k$, we may further estimate

$$\int_{B'}|f|\psi^2\bar u^\beta\,dx \le \int_{B'}\frac{|f|}{k}\psi^2\bar u^{\beta+1}\,dx. \tag{20}$$

Using (H2), (19) with $\epsilon = \lambda|\beta|/2$, and (20), we deduce from (18) that

$$\frac{\lambda|\beta|}{2}\int_{B'}\psi^2\bar u^{\beta-1}|D\bar u|^2\,dx \le \frac{2\Lambda^2}{\lambda|\beta|}\int_{B'}|D\psi|^2\bar u^{\beta+1}\,dx + \int_{B'}\frac{|f|}{k}\psi^2\bar u^{\beta+1}\,dx. \tag{21}$$

Setting $w = \bar u^{\frac{\beta+1}{2}}$, we have $Dw = \frac{\beta+1}{2}\bar u^{\frac{\beta-1}{2}}D\bar u$ and so we get from (21),

$$\int_{B'}\psi^2|Dw|^2\,dx \le \frac{\Lambda^2(\beta+1)^2}{\lambda^2|\beta|^2}\int_{B'}|D\psi|^2w^2\,dx + \frac{(\beta+1)^2}{2|\beta|}\int_{B'}\frac{|f|}{\lambda k}\psi^2w^2\,dx. \tag{22}$$

By the Sobolev inequality, applied to the function $\psi w \in W_0^{1,2}(B')$, we obtain

$$|\psi w|^2_{L_{2\hat n/(\hat n-2)}(B')} \le C(\hat n)|D(\psi w)|^2_{L_2(B')} \le 2C(\hat n)\int_{B'}(\psi^2|Dw|^2 + |D\psi|^2w^2)\,dx, \tag{23}$$

where $\hat{n} = n$ for $n > 2$ and $2 < \hat{2} < q$. The number $\hat{2}$ is chosen in such a way that $p < \hat{n}/(\hat{n}-2)$. Using Hölder's inequality and the interpolation inequality (1), we obtain (cp. (15))

$$\int_{B'} \frac{|f|}{\lambda k}\psi^2 w^2\,dx \leq 2(\epsilon^2 |\psi w|^2_{L_{2\hat{n}/(\hat{n}-2)}(B')} + \epsilon^{-2\mu}|\psi w|^2_{L_2(B')}), \quad \epsilon > 0,$$

where $\mu = \frac{\hat{n}}{q-\hat{n}}$. Hence, by substitution of this estimate and (23) into (22), we get

$$\begin{aligned}|\psi w|^2_{L_{2\hat{n}/(\hat{n}-2)}(B')} &\leq 2C(\hat{n})\Big(1+\frac{\Lambda^2(\beta+1)^2}{\lambda^2|\beta|^2}\Big)\int_{B'}|D\psi|^2 w^2\,dx \\ &\quad + 2C(\hat{n})\frac{(\beta+1)^2}{|\beta|}(\epsilon^2|\psi w|^2_{L_{2\hat{n}/(\hat{n}-2)}(B')} + \epsilon^{-2\mu}|\psi w|^2_{L_2(B')}).\end{aligned}$$

We take then $\epsilon^2 = \frac{|\beta|}{4C(\hat{n})(\beta+1)^2}$ to the result

$$|\psi w|^2_{L_{2\hat{n}/(\hat{n}-2)}(B')} \leq 4C(\hat{n})\Big(1+\frac{\Lambda^2(\beta+1)^2}{\lambda^2|\beta|^2}\Big)\int_{B'}|D\psi|^2w^2\,dx + \Big(\frac{4C(\hat{n})(\beta+1)^2}{|\beta|}\Big)^{\mu+1}|\psi w|^2_{L_2(B')}.$$

Letting $\gamma = 1+\beta$, we therefore have

$$|\psi w|_{L_{2\hat{n}/(\hat{n}-2)}(B')} \leq C(1+|\gamma|)^{\mu+1}|(\psi+|D\psi|)w|_{L_2(B')}, \tag{24}$$

where $C = C(\hat{n}, \Lambda/\lambda, q, |\beta|)$ is bounded provided that $|\beta|$ is bounded away from zero.

Given $0 < \rho' < \rho \leq 1$, we pick $\psi \in C^1_0(B')$ so that $0 \leq \psi \leq 1$, $\psi = 1$ in $\rho' B'$, $\operatorname{supp}\psi \subset \rho B'$, and $|D\psi| \leq 2(\sigma\zeta_n)^{-1}/(\rho-\rho')$, ζ_n denoting the radius of B_1. From (24) we then obtain

$$|w|_{L_{2\hat{n}/(\hat{n}-2)}(\rho' B')} \leq \frac{C(1+4(\sigma\zeta_n)^{-2})^{1/2}(1+|\gamma|)^{\mu+1}}{\rho-\rho'}\,|w|_{L_2(\rho B')};$$

here we use also the trivial estimate $1 \leq (\rho-\rho')^{-2}$. Note that

$$(\sigma\zeta_n)^{-1} \leq (\delta'\zeta_n)^{-1} \leq C(\delta, n).$$

Thus

$$|w|_{L_{2\hat{n}/(\hat{n}-2)}(\rho' B')} \leq \frac{\bar{C}(1+|\gamma|)^{\mu+1}}{\rho-\rho'}\,|w|_{L_2(\rho B')},$$

where $\bar{C} = \bar{C}(n, \hat{2}, \Lambda/\lambda, q, \delta, |\beta|)$ is bounded provided that $|\beta|$ is bounded away from zero. Returning to the function $\bar{u}$ and setting $\hat{\kappa} = \hat{n}/(\hat{n}-2)$, we have

$$\Big(\int_{\rho' B'} \bar{u}^{\gamma\hat{\kappa}}\,dx\Big)^{\frac{1}{2\hat{\kappa}}} \leq \frac{\bar{C}(1+|\gamma|)^{\mu+1}}{\rho-\rho'}\Big(\int_{\rho B'}\bar{u}^{\gamma}\,dx\Big)^{\frac{1}{2}}.$$

Hence we obtain

$$\Big(\int_{\rho' B'} \bar{u}^{-|\gamma|\hat{\kappa}}\,dx\Big)^{\frac{1}{|\gamma|\hat{\kappa}}} \leq \Big(\frac{C_1(1+|\gamma|)^{\mu+1}}{\rho-\rho'}\Big)^{2/|\gamma|}\Big(\int_{\rho B'}\bar{u}^{-|\gamma|}\,dx\Big)^{\frac{1}{|\gamma|}}, \quad \gamma < 0, \tag{25}$$

where the constant $C_1 = C_1(n, \hat{2}, \Lambda/\lambda, q, \delta)$ does not depend on γ, since $|\beta|$ is bounded away from zero ($\beta < -1$). Furthermore,

$$\Big(\int_{\rho' B'} \bar{u}^{\gamma\hat{\kappa}}\,dx\Big)^{\frac{1}{\gamma\hat{\kappa}}} \leq \Big(\frac{C_2(1+\gamma)^{\mu+1}}{\rho-\rho'}\Big)^{2/\gamma}\Big(\int_{\rho B'}\bar{u}^{\gamma}\,dx\Big)^{\frac{1}{\gamma}}, \quad 0 < \gamma \leq p/\hat{\kappa}, \tag{26}$$

where $C_2 = C_2(n, \hat{2}, \Lambda/\lambda, q, \delta)$ is independent of γ, since γ is bounded away from 1, by the restriction $\gamma \leq p/\hat{\kappa}$.

3. Moser iterations: For $\rho \in (0,1]$ we set $V_\rho = \rho B' = \rho\sigma B_1$. Then (25) leads to

$$|\bar{u}^{-1}|_{L_{\alpha\hat{\kappa}}(V_{\rho'})} \leq \Big(\frac{C(1+\alpha)^\tau}{(\rho-\rho')^\tau}\Big)^{1/\alpha} |\bar{u}^{-1}|_{L_\alpha(V_\rho)}, \quad 0<\rho'<\rho\leq 1,\ \alpha>0,$$

where $C = C(n, \hat{2}, \Lambda/\lambda, q, \delta)$ and $\tau = 2(1+\mu)$. Employing the first Moser iteration, Lemma 2.3 (with $\bar{p}=1$), it follows that there exist constants $M_0 = M_0(n, \hat{2}, \Lambda/\lambda, q, \delta)$ and $\tau_0 = \tau_0(\hat{n}, q)$ such that

$$\sup_{V_\theta} \bar{u}^{-1} \leq \Big(\frac{M_0}{(1-\theta)^{\tau_0}}\Big)^{1/\alpha} |\bar{u}^{-1}|_{L_\alpha(V_1)} \quad \text{for all } \theta\in(0,1),\ \alpha\in(0,1].$$

Thus if we take $\theta = \sigma'/\sigma$ and notice that

$$\frac{1}{1-\theta} = \frac{\sigma}{\sigma-\sigma'} \leq \frac{1}{\sigma-\sigma'},$$

we obtain

$$\sup_{U_{\sigma'}} \bar{u}^{-1} \leq \Big(\frac{M_0}{(\sigma-\sigma')^{\tau_0}}\Big)^{1/\alpha} |\bar{u}^{-1}|_{L_\alpha(U_\sigma)}, \quad \alpha\in(0,1]. \tag{27}$$

This shows that the assumption (i) of Lemma 2.6 is satisfied by any positive constant multiple of $\bar{u}$ with $\alpha_0 = \infty$.

From (26) we infer that

$$|\bar{u}|_{L_{\alpha\hat{\kappa}}(V_{\rho'})} \leq \Big(\frac{C}{(\rho-\rho')^2}\Big)^{1/\alpha} |\bar{u}|_{L_\alpha(V_\rho)}, \quad 0<\rho'<\rho\leq 1,\ 0<\alpha\leq p/\hat{\kappa},$$

where $C = C(n, \hat{2}, \Lambda/\lambda, q, p, \delta)$. Using the second Moser iteration, Lemma 2.5, it follows that there exist constants $M_1 = M_1(n, \hat{2}, \Lambda/\lambda, q, p, \delta)$ and $\tau_1 = \tau_1(\hat{n}, q)$ such that

$$|\bar{u}|_{L_p(V_\theta)} \leq \Big(\frac{M_1}{(1-\theta)^{\tau_1}}\Big)^{1/\alpha - 1/p} |\bar{u}|_{L_\alpha(V_1)} \quad \text{for all } \theta\in(0,1),\ 0<\alpha\leq p/\hat{\kappa}.$$

Hence with $\theta = \sigma'/\sigma$ we get

$$|\bar{u}|_{L_p(U_{\sigma'})} \leq \Big(\frac{M_1}{(\sigma-\sigma')^{\tau_1}}\Big)^{1/\alpha-1/p} |\bar{u}|_{L_\alpha(U_\sigma)}, \quad \alpha\in(0, p/\hat{\kappa}], \tag{28}$$

which shows that the assumption (i) of Lemma 2.6 is fulfilled by any positive constant multiple of $\bar{u}^{-1}$ with $\alpha_0 = p$ and $\eta = \kappa^{-1}$.

4. The $\log \bar{u}$ estimate: We choose the test function $v = \psi^2 \bar{u}^{-1}$ with $\psi \in C_0^1(\Omega)$ such that $\operatorname{supp}\psi \subset B$, $\psi = 1$ in B_1, and $|D\psi| \leq 2/(\zeta_n/\eta - \zeta_n) = C(n,\delta)$. We proceed then as in step 2 with $\beta = -1$ and with B' replaced by B and infer from the analogue of (21) that

$$\int_B \psi^2 |D(\log\bar{u})|^2\,dx \leq \frac{4\Lambda^2}{\lambda^2}\int_B |D\psi|^2\,dx + \frac{2}{\lambda}\int_B \frac{|f|}{k}\psi^2\,dx.$$

By Hölder's inequality and the definition of k,

$$\begin{aligned} \frac{2}{\lambda}\int_B \frac{|f|}{k}\psi^2\,dx \ &\leq \frac{2}{\lambda k}|f|_{L_{q/2}(B)}|B|^{(q-2)/q} \leq 2(\eta^{-n}|\eta B|)^{(q-2)/q} \\ &\leq 2\eta^{-n(q-2)/q} = C(n,q,\delta). \end{aligned}$$

We thus have

$$\int_{B_1} |D(\log\bar{u})|^2\,dx \leq \frac{4\Lambda^2}{\lambda^2}C(n,\delta)|B| + C(n,q,\delta) \leq C(n,\Lambda/\lambda,q,\delta). \tag{29}$$

The L_2-Poincaré inequality applied in the ball $U = B_1$ to the function $\log \bar{u}$ yields

$$\int_{B_1} |\log \bar{u} - c(\bar{u})|^2 \, dx \leq C_n \int_{B_1} |D(\log \bar{u})|^2 \, dx,$$

where

$$c(\bar{u}) = \int_{B_1} \log \bar{u} \, dx;$$

recall that $|B_1| = 1$. This estimate along with (29) implies for any $t > 0$,

$$\begin{aligned} t|B_1 \cap \{|\log \bar{u} - c(\bar{u})| \geq t\}| &\leq \int_{B_1} |\log \bar{u} - c(\bar{u})| \, dx \\ &\leq (\int_{B_1} |\log \bar{u} - c(\bar{u})|^2 \, dx)^{1/2} \\ &\leq \tilde{C}(n, \Lambda/\lambda, q, \delta). \end{aligned}$$

This shows that the assumption (ii) of Lemma 2.6 is satisfied by the functions $g_1 = e^{-c(\bar{u})}\bar{u}$ and $g_2 = e^{c(\bar{u})}\bar{u}^{-1}$. In fact,

$$|B_1 \cap \{|\log g_1 > t\}| = |B_1 \cap \{\log \bar{u} - c(\bar{u}) > t\}| \leq |B_1 \cap \{|\log \bar{u} - c(\bar{u})| \geq t\}| \leq \frac{\tilde{C}}{t};$$

the desired inequality for g_2 follows by the same argument.

5. Application of the abstract lemma: By step 3 and 4, we may apply Lemma 2.6 to g_1 with $\alpha_0 = p$ and conclude that

$$|g_1|_{L_p(U_{\delta'})} \leq M_2$$

with $M_2 = M_2(n, \hat{2}, \Lambda/\lambda, q, p, \delta)$. Since $U_{\delta'} = \delta B$, we thus get

$$|\bar{u}|_{L_p(\delta B)} \leq M_2 e^{c(\bar{u})}. \tag{30}$$

On the other hand, step 3 and 4 allow us also to apply Lemma 2.6 to g_2 with $\alpha_0 = \infty$; thereby we obtain

$$\sup_{U_{\delta'}} g_2 \leq M_3$$

with $M_3 = M_3(n, \hat{2}, \Lambda/\lambda, q, p, \delta)$. Hence

$$e^{c(\bar{u})} \leq M_3 \inf_{\delta B} \bar{u}. \tag{31}$$

Combining (30) and (31) then yields

$$|\bar{u}|_{L_p(\delta B)} \leq M_2 M_3 \inf_{\delta B} \bar{u},$$

which in turn implies

$$|u|_{L_p(\delta B)} \leq |\bar{u}|_{L_p(\delta B)} \leq M_2 M_3 \Big(\inf_{\delta B} u + k \Big) = M_2 M_3 \Big(\inf_{\delta B} u + \frac{1}{\lambda} |f|_{L_{q/2}(B)} \Big).$$

The proof is complete. □

3.3 The Harnack inequality

The following form of the Harnack inequality follows immediately from Theorem 3.1 and Theorem 3.2.

Theorem 3.3 *Let the assumptions (H1)-(H3) be satisfied and let $\delta \in (0,1)$. Then there exists a constant $C = C(n, \Lambda/\lambda, q, \delta)$ such that for any ball $B \subset \Omega$ of radius r and any nonnegative weak solution u of (9),*

$$\sup_{\delta B} u \leq C \Big(\inf_{\delta B} u + \frac{1}{\lambda} r^{2(1-n/q)} |f|_{L_{q/2}(B)} \Big).$$

3.4 Hölder continuity

We derive first a local estimate for the oscillation of weak solutions of (9), cp. [4], [3], [8].

Theorem 3.4 *Let the assumptions (H1)-(H3) be satisfied and let $\delta \in (0,1)$. Then there exist $\alpha = \alpha(n, \Lambda/\lambda, q) \in (0,1)$ and $C = C(n, \Lambda/\lambda, q, \delta)$ such that for any ball $B \subset \Omega$ of radius r and any weak solution u of (9),*

$$\sup_{\rho B} u - \inf_{\rho B} u \le C\rho^{\alpha}\Big((\frac{1}{|B|}\int_B u^2\,dx)^{1/2} + \frac{1}{\lambda}\, r^{2(1-n/q)} |f|_{L_{q/2}(B)}\Big), \quad 0 < \rho \le \delta.$$

Proof: We prove the estimate for $r=1$. Let $M(\rho) = \sup_{\rho B} u$ and $m(\rho) = \inf_{\rho B} u$ for $\rho \in (0,1)$. Then $M(\rho) < \infty$ and $m(\rho) > -\infty$ for all $\rho \in (0,1)$, by Theorem 3.1. Fix now $\delta \in (0,1)$ and let $\rho \in (0, \delta]$. Clearly, $M(\rho) - u \ge 0$ in ρB as well as

$$\int_{\rho B} a^{ij} D_j[M(\rho) - u]\, D_i v\, dx = -\int_{\rho B} f v\, dx$$

for all nonnegative $v \in W_0^{1,2}(\rho B)$. So we may apply Theorem 3.2 with $p=1$ to $M(\rho) - u \ge 0$ in ρB to get

$$\frac{1}{|\frac{\rho}{2}B|}\int_{\frac{\rho}{2}B} (M(\rho) - u)\, dx \le C\Big(\inf_{\frac{\rho}{2}B} (M(\rho) - u) + \frac{1}{\lambda}\,\rho^{2(1-n/q)} |f|_{L_{q/2}(\rho B)}\Big),$$

where $C = C(n, \Lambda/\lambda, q)$. From this we deduce that

$$\frac{1}{|\frac{\rho}{2}B|}\int_{\frac{\rho}{2}B} (M(\rho) - u)\, dx \le C\Big(M(\rho) - M(\rho/2) + \frac{1}{\lambda}\,\rho^{2(1-n/q)} |f|_{L_{q/2}(\rho B)}\Big). \tag{32}$$

We further have $u - m(\rho) \ge 0$ in ρB and

$$\int_{\rho B} a^{ij} D_j[u - m(\rho)]\, D_i v\, dx = \int_{\rho B} f v\, dx$$

for all nonnegative $v \in W_0^{1,2}(\rho B)$. By applying Theorem 3.2 with $p=1$ to $u - m(\rho) \ge 0$ in ρB we obtain

$$\frac{1}{|\frac{\rho}{2}B|}\int_{\frac{\rho}{2}B} (u - m(\rho))\, dx \le C\Big(\inf_{\frac{\rho}{2}B} (u - m(\rho)) + \frac{1}{\lambda}\,\rho^{2(1-n/q)} |f|_{L_{q/2}(\rho B)}\Big),$$

which gives

$$\frac{1}{|\frac{\rho}{2}B|}\int_{\frac{\rho}{2}B} (u - m(\rho))\, dx \le C\Big(m(\rho/2) - m(\rho) + \frac{1}{\lambda}\,\rho^{2(1-n/q)} |f|_{L_{q/2}(\rho B)}\Big); \tag{33}$$

here the constant C is the same as in (32). We add now (32) and (33) and find that

$$M(\rho) - m(\rho) \le C\Big(M(\rho) - m(\rho) + m(\rho/2) - M(\rho/2) + \frac{2}{\lambda}\,\rho^{2(1-n/q)} |f|_{L_{q/2}(\rho B)}\Big). \tag{34}$$

Introducing the oscillation $\omega(\rho) = M(\rho) - m(\rho)$, we infer from (34) that

$$\omega(\rho/2) \le \frac{C-1}{C}\,\omega(\rho) + \frac{2}{\lambda}\,\rho^{2(1-n/q)} |f|_{L_{q/2}(\delta B)}.$$

Note that $\theta := (C-1)/C < 1$. Put

$$g(\rho) = \frac{2}{\lambda}\,\rho^{2(1-n/q)} |f|_{L_{q/2}(\delta B)}.$$

Evidently, ω and g are nondecreasing on $(0,\delta]$. Hence it follows from Lemma 2.2 that

$$\omega(\rho) \le \frac{1}{\theta}\,(\frac{\rho}{\delta})^{(1-\mu)\frac{\log\theta}{\log(1/2)}}\omega(\delta) + \frac{g(\delta^{1-\mu}\rho^{\mu})}{1-\theta}, \quad 0<\rho\le\delta,\, 0<\mu<1.$$

Choose $\mu \in (0,1)$ such that

$$\alpha := (1-\mu)\frac{\log\theta}{\log(1/2)} < 2\mu(1-\frac{n}{q}).$$

Then $\alpha = \alpha(n,\Lambda/\lambda,q)$ and we obtain

$$\begin{aligned}
\omega(\rho) &\le \rho^{\alpha}\Big(\theta^{-1}\delta^{-\alpha}\omega(\delta) + \frac{2}{\lambda(1-\theta)}\,\delta^{2(1-\mu)(1-n/q)}\rho^{2\mu(1-n/q)-\alpha}|f|_{L_{q/2}(\delta B)}\Big)\\
&\le \rho^{\alpha}\Big(\theta^{-1}\delta^{-\alpha}\omega(\delta) + \frac{2}{\lambda(1-\theta)}\,|f|_{L_{q/2}(\delta B)}\Big)\\
&\le \tilde{C}\rho^{\alpha}\Big(\omega(\delta) + \frac{1}{\lambda}\,|f|_{L_{q/2}(B)}\Big), \qquad 0<\rho\le\delta,
\end{aligned}$$

where $\tilde{C} = \tilde{C}(n,\Lambda/\lambda,q,\delta)$. By Theorem 3.1 with $p=2$,

$$\omega(\delta) = \sup_{\delta B} u + \sup_{\delta B}(-u) \le C_1\Big(|u|_{L_2(B)} + \frac{1}{\lambda}\,|f|_{L_{q/2}(B)}\Big)$$

with $C_1 = C_1(n,\Lambda/\lambda,q,\delta)$. Hence

$$\omega(\rho) \le C_2(n,\Lambda/\lambda,q,\delta)\rho^{\alpha}\Big(|u|_{L_2(B)} + \frac{1}{\lambda}\,|f|_{L_{q/2}(B)}\Big), \quad 0<\rho\le\delta.$$

This proves the statement if $r=1$. The general case can be recovered through the coordinate transformation discussed in step 1 of the proof of Theorem 3.1. □

By means of the above oscillation estimate, we can now prove the interior Hölder continuity of weak solutions of (9).

Theorem 3.5 *Let the assumptions (H1)-(H3) be satisfied and let $\Omega' \subset\subset \Omega$. Then there exist $\alpha = \alpha(n,\Lambda/\lambda,q) \in (0,1)$ and $C = C(n,\Lambda/\lambda,q,|\Omega'|,d')$, $d' = \operatorname{dist}(\Omega',\partial\Omega)$, such that for any weak solution u of (9),*

$$|u|_{C^{\alpha}(\overline{\Omega'})} \le C\Big(|u|_{L_2(\Omega)} + \frac{1}{\lambda}\,|f|_{L_{q/2}(\Omega)}\Big).$$

Proof: Put $r_0 = d'/4$. By boundedness of Ω', we can cover $\overline{\Omega'}$ by finitely many balls $B_i = B_{r_0}(x_i)$, $x_i \in \overline{\Omega'}$, $i=1,\dots,N = N(r_0,|\Omega'|)$. Clearly, $B_{2r_0}(x_i) \subset \Omega$ and therefore, by Theorem 3.1, we have

$$\begin{aligned}
\sup_{\overline{\Omega'}}|u| \le \sup_{i=1,\dots,N}\,\sup_{\bar{B}_i}|u| \;&\le\; C_1 \sup_{i=1,\dots,N}\Big(|u|_{L_2(2B_i)} + \frac{1}{\lambda}\,|f|_{L_{q/2}(2B_i)}\Big)\\
&\le C_1\Big(|u|_{L_2(\Omega)} + \frac{1}{\lambda}\,|f|_{L_{q/2}(\Omega)}\Big),
\end{aligned}$$

where $C_1 = C_1(n,\Lambda/\lambda,q,r_0)$.

Next let x, $y \in \Omega'$. We distinguish two cases.

Case 1: Suppose $r := |x-y| \le 2r_0$. By applying Theorem 3.4 with $\delta = 1/2$ to the function u in $B_{4r_0}(x) \subset \Omega$ we obtain

$$\sup_{\rho B_{4r_0}(x)} u - \inf_{\rho B_{4r_0}(x)} u \le C_2\rho^{\alpha}\Big(|u|_{L_2(B_{4r_0}(x))} + \frac{1}{\lambda}\,|f|_{L_{q/2}(B_{4r_0}(x))}\Big), \quad 0<\rho\le 1/2;$$

here $\alpha = \alpha(n, \Lambda/\lambda, q) \in (0,1)$ and $C_2 = C_2(n, \Lambda/\lambda, q, r_0)$. Choosing $\rho = r/(4r_0) \leq 1/2$ we deduce that

$$|u(x) - u(y)| \leq \sup_{B_r(x)} u - \inf_{B_r(x)} u \leq C_2 r^\alpha (4r_0)^{-\alpha} \Big(|u|_{L_2(B_{4r_0}(x))} + \frac{1}{\lambda} |f|_{L_{q/2}(B_{4r_0}(x))} \Big),$$

which yields

$$|u(x) - u(y)| \leq C_3(n, \Lambda/\lambda, q, r_0)|x - y|^\alpha \Big(|u|_{L_2(\Omega)} + \frac{1}{\lambda} |f|_{L_{q/2}(\Omega)} \Big). \tag{35}$$

Case 2: Suppose now $r := |x - y| > 2r_0$. Then we find a chain of balls $B_{p(1)}, \ldots, B_{p(k)}$ with $k \leq N$ such that $x \in B_{p(1)}$, $y \in B_{p(k)}$, and $B_{p(i)} \cap B_{p(i+1)} \neq \emptyset$ for all $i = 1, \ldots, k-1$. By relabeling, we may assume that $B_{p(i)} = B_i$, $i = 1, \ldots, k$. We use now the triangle inequality and (35) to estimate

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(x_1)| + \sum_{j=1}^{k-1} |u(x_j) - u(x_{j+1})| + |u(x_k) - u(y)| \\ &\leq C_3 (|x - x_1|^\alpha + \sum_{j=1}^{k-1} |x_j - x_{j+1}|^\alpha + |x_k - y|^\alpha) \Big(|u|_{L_2(\Omega)} + \frac{1}{\lambda} |f|_{L_{q/2}(\Omega)} \Big) \\ &\leq C_3 (k+1)|x - y|^\alpha \Big(|u|_{L_2(\Omega)} + \frac{1}{\lambda} |f|_{L_{q/2}(\Omega)} \Big), \end{aligned}$$

which, together with (35), implies the desired inequality for the Hölder seminorm, because $k \leq N(d', |\Omega'|)$. $\square$

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