Maximal regularity of type L_p for abstract parabolic Volterra equations

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Abstract

We prove maximal regularity results of type L_p for abstract parabolic Volterra equations including problems with inhomogeneous boundary data. Our approach is purely operator theoretic. It uses the inversion of the convolution, the Dore-Venni theorem, the Mikhlin theorem in the operator-valued version, and real interpolation. Known results on L_p-regularity of abstract Cauchy problems and abstract parabolic pde's with inhomogeneous boundary conditions are recovered. As an application we consider the heat equation of memory type with inhomogeneous boundary condition.

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1 Introduction

The purpose of this paper is to establish maximal regularity results of type L_p for two classes of problems: (I) for abstract Volterra equations of the form

$$u(t) + \int_0^t a(t-s)Au(s)\,ds = f(t), \quad t \geq 0, \tag{1}$$

where a is a scalar kernel and A is a closed linear operator in a Banach space X; and (II) for a class of abstract linear Volterra equations with *inhomogeneous* boundary condition of Dirichlet resp. Robin type, see (21) and (30) below. Such problems arise in a variety of applications as model problems. Well-known techniques like localization, perturbation, and coordinate transformation allow to transfer results for such model problems to parabolic integrodifferential equations on smooth domains with inhomogeneous boundary conditions.

To illustrate this point, let us consider an example from the theory of heat conduction with memory. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary

Γ which decomposes as $\Gamma = \Gamma_b \cup \Gamma_f$ with $\mathrm{dist}(\Gamma_b, \Gamma_f) > 0$. For the unknown temperature $u(t,x)$ consider the problem

$$\begin{aligned} \partial_t u(t,x) - \int_0^t \Delta u(\tau,x)\, dk(t-\tau) &= f(t,x),\ t>0,\ x \in \Omega, \\ u(t,x) &= g(t,x),\ t>0,\ x\in\Gamma_b, \\ \partial_\nu u(t,x) &= h(t,x),\ t>0,\ x\in\Gamma_f, \\ u(0,x) &= u_0(x),\ x\in\Omega. \end{aligned} \tag{2}$$

Here ∂_ν stands for the normal derivative at the boundary portion Γ_f, and the kernel k is given by $k(0)=0$, $k(t) = k_0 t^\alpha e^{-\omega t}$, $t>0$, with $\alpha \in [0,1)$, $k_0 > 0$, and $\omega \geq 0$. Note that in the case $k(t) = 1$, $t > 0$, (2) amounts to a parabolic initial-boundary value problem. Using localization, perturbation and change of coordinates the problem (2) can be reduced to three model problems, namely to the corresponding heat flow equation on $\mathbb{R}^n$ and on the half-space $\mathbb{R}^n_+$ with Dirichlet or Neumann condition.

Equation (1), and the problems under (II) are abstract versions of these model problems. Applying the corresponding results, Theorem 3.4, Theorem 5.1, and Theorem 5.2 below, we obtain the subsequent optimal regularity result for (2).

Theorem 1.1 *Let $J=[0,T]$, $1<p<\infty$. Suppose $\alpha \in [0,1) \setminus \{\frac{2}{2p-1}-1, \frac{2}{p-1}-1, \frac{1}{p}, \frac{3}{2p-1}\}$. Then (2) admits a unique solution $u \in H^{1+\alpha}_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega))$ if and only if $f \in H^\alpha_p(J; L_p(\Omega))$, $g \in B^{(1+\alpha)(1-\frac{1}{2p})}_{pp}(J; L_p(\Gamma_b)) \cap L_p(J; B^{2-\frac{1}{p}}_{pp}(\Gamma_b))$, $h \in B^{(1+\alpha)(\frac{1}{2}-\frac{1}{2p})}_{pp}(J; L_p(\Gamma_f)) \cap L_p(J; B^{1-\frac{1}{p}}_{pp}(\Gamma_f))$, $u_0 \in B^{2-\frac{2}{p(1+\alpha)}}_{pp}(\Omega)$, as well as $f|_{t=0} \in B^{\frac{2(\alpha-1/p)}{1+\alpha}}_{pp}(\Omega)$ in case $\alpha > \frac{1}{p}$, and the compatibility conditions $u_0|_{\Gamma_b} = g|_{t=0}$ for $\alpha > \frac{2}{2p-1}-1$, $\partial_\nu u_0|_{\Gamma_f} = h|_{t=0}$ for $\alpha > \frac{2}{p-1}-1$, $f|_{t=0,\Gamma_b} = \partial_t g|_{t=0}$ for $\alpha > \frac{3}{2p-1}$, are satisfied.*

We would like to point out that a more general optimal regularity result on parabolic Volterra equations of second order with variable coefficients has been proven in the author's thesis, see [24, Chapter 4]. By means of this result and the contraction mapping principle, the author was further able to obtain existence and uniqueness of local strong solutions of a class of quasilinear parabolic problems with nonlinear boundary conditions, cf. [24, Chapter 6]. The results presented in this paper have also been applied successfully to *vector*-valued problems arising in linear viscoelasticity, see [24, Chapter 5].

Our main results concerning (1), Theorem 3.4 and Theorem 3.5, rely essentially on techniques developed in Prüss [15] using the representation of the resolvent S for (1) via Laplace transform, and on the Mikhlin theorem in the operator-valued version, which has been proven by Weis [22]. They generalize the results in Prüss [14], where $A \in \mathcal{BIP}(X)$ and f is of the form $f = a * g$ with $g \in L_p([0,T]; X)$, as well as those obtained by Bajlekova [2] on fractional evolution equations. In the present paper we allow the kernel a to belong to a certain class which, roughly speaking, contains all 1-regular, sectorial kernels which behave like $t^{\alpha-1}$ ($\alpha \in (0,2)$)

near zero. Note further that we also treat the case of higher time regularity; the motivation for this comes from problems involving several independent kernels, see e.g. [24, Chapter 5].

Our approach to the problems under (II) is much inspired by Prüss [16], who studied the (abstract) pde case. It is purely operator theoretic. Important tools are the inversion of the convolution, the operator-sum method in form of the Dore-Venni theorem, and real interpolation. We emphasize that in finding the natural regularity classes of the boundary data, the results from [16] on abstract second order equations with inhomogeneous boundary condition play an essential part. We state these results in Section 4. Theorem 5.1 and Theorem 5.2, which are our main results on the problems under (II), seem to be the first towards an L_p-theory for such problems. Theorem 5.1 may be generalized employing a more recent theorem of Kalton and Weis [10]; it is sufficient to assume that A is $\mathcal{R}$-sectorial, and its power angle can be replaced with its $\mathcal{R}$-angle.

The plan for this paper is as follows. Section 2 collects basic definitions and tools concerning sectorial operators and parabolic Volterra equations. In Section 3 we study problems of the form (1). Section 4 repeats some known maximal regularity results for abstract equations on the halfline. These will be used in Section 5, which is devoted to the problems (II) with inhomogeneous boundary data.

2 Preliminaries

We begin by fixing some notations. If not indicated otherwise, by $f * g$ we mean the convolution defined by $(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$, $t \geq 0$, of two functions f, g supported on the halfline. The Laplace transform of f will be denoted by $\hat{f}$.

Recall that a Banach space X belongs to the class $\mathcal{HT}$, if the Hilbert transform is bounded on $L_2(\mathbb{R}; X)$. For an interval $J \subset \mathbb{R}$, $s > 0$ and $1 < p < \infty$, by $H_p^s(J; X)$ and $B_{pp}^s(J; X)$ we mean the vector-valued Bessel potential space resp. Sobolev-Slobodeckij space of X-valued functions on J, see [1], [19], [21], [25]. In Section 2.5 we give a definition of the spaces $H_p^s(J; X)$ in the situation where $X \in \mathcal{HT}$; this will always be the case when we are to consider vector-valued Bessel potential spaces. We will frequently use the property that the Sobolev-Slobodeckij spaces appear as real interpolation spaces between the spaces L_p and H_p^s; more precisely $(L_p(J; X), H_p^s(J; X))_{\theta, p} = B_{pp}^{\theta s}(J; X)$ for all $1 < p < \infty$, $s > 0$, and $\theta \in (0, 1)$.

2.1 Sectorial operators

Let X be a complex Banach space and A be a closed linear operator in X. Then A is called *pseudo-sectorial* if $(-\infty, 0)$ is contained in the resolvent set of A and the resolvent estimate $|t(t+A)^{-1}|_{\mathcal{B}(X)} \leq M$, $t > 0$, holds, for some constant $M > 0$. If in addition the null space $\mathcal{N}(A) = \{0\}$, and the domain $\mathcal{D}(A)$ as well as the range $\mathcal{R}(A)$ of A are dense in X then A is called *sectorial*. The class of sectorial operators in X is denoted by $\mathcal{S}(X)$. We recall that in case X is reflexive and A is pseudo-sectorial, the space X decomposes as $X = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$. Thus in such a situation A is sectorial on $\overline{\mathcal{R}(A)}$. Putting $\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}$ the *spectral angle* $\phi_A \in [0, \pi)$ of $A \in \mathcal{S}(X)$ is defined by $\phi_A = \inf\{\phi : \rho(-A) \supset \Sigma_{\pi-\phi}, \sup\{\lambda \in \Sigma_{\pi-\phi} : |\lambda(\lambda + A)^{-1}| < \infty\}\}$. We point out that for sectorial operators the Dunford functional calculus is available; in particular the complex powers A^z, $z \in \mathbb{C}$ are well-defined closed linear operators in X; cf. e.g. Komatsu [11].

At this point let us state a very useful property of the real interpolation spaces $(X, X_\alpha)_{\beta, p}$, $0 < \alpha,\ \beta < 1$, $1 \leq p < \infty$, between X and the fractional power spaces $X_\alpha = X_{A^\alpha} = (\mathcal{D}(A^\alpha), |\cdot|_\alpha)$, $|x|_\alpha = |x| + |A^\alpha x|$, $0 < \alpha < 1$, associated with an operator $A \in \mathcal{S}(X)$, defined by, e.g. the K-method. Let D_A denote the domain of A equipped with the graph norm and recall that for $A \in \mathcal{S}(X)$, $1 \leq p < \infty$, and $\gamma \in (0, 1)$, the real interpolation space $(X, D_A)_{\gamma, p}$ coincides with the space $D_A(\gamma, p)$ defined by $D_A(\gamma, p) := \{x \in X : [x]_{D_A(\gamma, p)} < \infty\}$, where the seminorm $[x]_{D_A(\gamma,p)} = (\int_0^\infty (t^\gamma |A(t + A)^{-1} x|_X)^p \frac{dt}{t})^{1/p}$, see e.g. [3, Prop. 3]. The subsequent result is due to Komatsu [12, Thm. 3.2].

Theorem 2.1 *Let $A \in \mathcal{S}(X)$. Then $D_{A^\alpha}(\beta, p) = D_A(\alpha\beta, p)$ for all $\alpha, \beta \in (0, 1)$, and $1 \leq p < \infty$.*

We say that an operator $A \in \mathcal{S}(X)$ admits bounded imaginary powers and write $A \in \mathcal{BIP}(X)$ if the imaginary powers A^{is} form a bounded C_0-group on X. The type θ_A of this group is called the *power angle* of A. The class $\mathcal{BIP}(X)$ is of overriding importance in the context of sums of commuting linear operators. This finds expression in the Dore-Venni theorem, which is one of the fundamental results in this connection, see Section 2.2.

We come now to $\mathcal{R}$-sectorial operators. First we have to recall the definition of $\mathcal{R}$-*bounded* families of bounded linear operators. Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called $\mathcal{R}$-*bounded*, if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables ε_j on a probability space $(\Omega, \mathcal{M}, \mu)$ the inequality $|\sum_{j=1}^N \varepsilon_j T_j x_j|_{L_p(\Omega; Y)} \leq C |\sum_{j=1}^N \varepsilon_j x_j|_{L_p(\Omega; X)}$ is valid. The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, we denote it by $\mathcal{R}(\mathcal{T})$.

Let X be a complex Banach space. An operator $A \in \mathcal{S}(X)$ is called $\mathcal{R}$-*sectorial* if $\mathcal{R}\{t(t + A)^{-1} : t > 0\} < \infty$. In this case the $\mathcal{R}$-*angle* ϕ_A^R of A is defined by means of $\phi_A^R := \inf\{\theta \in (0, \pi) : \mathcal{R}_A(\pi - \theta) < \infty\}$, $\mathcal{R}_A(\theta) := \mathcal{R}\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_\theta\}$. The class of $\mathcal{R}$-sectorial operators in X will be denoted by $\mathcal{RS}(X)$. The $\mathcal{R}$-angle of an $\mathcal{R}$-sectorial operator A is well-defined and is not smaller than the spectral angle of A, cp. [5, Definition 4.1].

A fundamental result, which has been obtained by Clément and Prüss [4], says that $\mathcal{BIP}(X) \subset \mathcal{RS}(X)$ and $\phi_A^R \leq \theta_A$ provided that the underlying Banach space X belongs to the class $\mathcal{HT}$.

For sectorial operators A it is known that the powers A^α with $\alpha \in \mathbb{R}$ and $|\alpha| < \pi/\phi_A$ are sectorial as well and $\phi_{A^\alpha} \leq |\alpha|\phi_A$, see e.g. [5, Thm. 2.3]. It turns out that there is a corresponding result for the class $\mathcal{RS}(X)$.

Proposition 2.1 *Let X be a complex Banach space. Suppose $A \in \mathcal{RS}(X)$ and $\alpha \in \mathbb{R}$ is such that $|\alpha| < \pi/\phi_A^R$. Then A^α is also $\mathcal{R}$-sectorial and $\phi_{A^\alpha}^R \le |\alpha|\phi_A^R$.*

This result seems to be missing in the literature. For a proof we refer to the author's thesis [24, Prop. 2.2.1].

2.2 Sums of closed linear operators

Let X be a Banach space. For closed linear operators A and B in X we define the sum operator $A+B$ by $(A+B)x = Ax + Bx$, $x \in \mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B)$. We state now a variant of the Dore-Venni theorem, cf. [6], [16], [18].

Theorem 2.2 *Suppose X is a Banach space of class $\mathcal{HT}$, and $A, B \in \mathcal{BIP}(X)$ commute in the resolvent sense and satisfy the* strong parabolicity condition *$\theta_A + \theta_B < \pi$. Let further $\mu > 0$. Then $A + \mu B$ is closed and sectorial, $A+\mu B \in \mathcal{BIP}(X)$ with $\theta_{A+\mu B} \le \max\{\theta_A, \theta_B\}$, and there exists a constant $C > 0$, independent of $\mu > 0$, such that $|Ax| + \mu|Bx| \le C|Ax + \mu Bx|$, for all $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$. In particular, if A or B is invertible, then $A + \mu B$ is invertible as well.*

The subsequent two results are very useful in connection with the method of sums. The first of these has been established by Grisvard [7], even in a more general situation. X stands for a Banach space.

Proposition 2.2 *Suppose A, $B \in \mathcal{S}(X)$ commute in the resolvent sense. Then*

$$(X, \mathcal{D}(A) \cap \mathcal{D}(B))_{\alpha, p} = (X, \mathcal{D}(A))_{\alpha, p} \cap (X, \mathcal{D}(B))_{\alpha, p}, \quad \alpha \in (0,1),\ p \in [1, \infty].$$

The following result is known as *the mixed derivative theorem* and is due to Sobolevskii [20].

Proposition 2.3 *Suppose A, $B \in \mathcal{S}(X)$ commute in the resolvent sense with $\phi_A + \phi_B < \pi$, and assume that the pair (A, B) is* coercively positive*, i.e. $A + \mu B$ is closed for each $\mu > 0$ and there is a constant $M > 0$ such that $|Ax|_X + \mu|Bx|_X \le M|Ax + \mu Bx|_X$ for all $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$, $\mu > 0$. Then there exists a constant $C > 0$ such that*

$$|A^\alpha B^{1-\alpha} x|_X \le C|Ax + Bx|_X, \quad \textit{for all}\ x \in \mathcal{D}(A) \cap \mathcal{D}(B),\ \alpha \in [0,1].$$

In particular, if A or B is invertible, then $A^\alpha B^{1-\alpha}(A+B)^{-1} \in \mathcal{B}(X)$, $\alpha \in [0,1]$.

Observe that Proposition 2.3 applies in particular to the situation of Theorem 2.2.

2.3 Kernels

In this section we collect some of the basic definitions and properties concerning scalar kernels which we need for our treatment of parabolic Volterra equations.

A function $a \in L_{1,loc}(\mathbb{R}_+)$ is called to be of *subexponential growth* if for all $\varepsilon > 0$, $\int_0^\infty e^{-\varepsilon t}|a(t)|\,dt < \infty$. Following [14, Def. 3.3] we say that a kernel $a \in L_{1,loc}(\mathbb{R}_+)$

of subexponential growth is *k-regular* ($k \in \mathbb{N}$) if there is a constant $c > 0$ such that $|\lambda^n \hat{a}^{(n)}(\lambda)| \leq c\,|\hat{a}(\lambda)|$ for all $\operatorname{Re}\lambda > 0$, $0 \leq n \leq k$. Some important properties of 1-regular kernels are contained in the following lemma, cp. [14, Lemma 8.1].

Lemma 2.1 *Suppose $a \in L_{1,loc}(\mathbb{R}_+)$ is of subexponential growth and 1-regular. Then (i) $\hat{a}(i\rho) := \lim_{\lambda \to i\rho} \hat{a}(\lambda)$ exists for each $\rho \neq 0$; (ii) $\hat{a}(\lambda) \neq 0$ for each $\operatorname{Re}\lambda \geq 0$; (iii) $\hat{a}(i\cdot) \in W^{\infty}_{1,loc}(\mathbb{R} \setminus \{0\})$; (iv) $|\rho \hat{a}'(i\rho)| \leq c|\hat{a}(i\rho)|$ for a.a. $\rho \in \mathbb{R}$; and (v) there is a constant $c > 0$ such that $c|\hat{a}(|\lambda|)| \leq |\hat{a}(\lambda)| \leq c^{-1}|\hat{a}(|\lambda|)|$ for $Re\,\lambda \geq 0$, $\lambda \neq 0$.*

A kernel $a \in L_{1,loc}(\mathbb{R}_+)$ of subexponential growth satisfying $\hat{a}(\lambda) \neq 0$, $\operatorname{Re}\lambda > 0$, is called *$\theta$-sectorial* ($\theta > 0$) if $|\arg \hat{a}(\lambda)| \leq \theta$ for all $\operatorname{Re}\lambda > 0$ (cp. [14, Def. 3.2]).

With regard to Volterra operators in L_p (see Section 2.5), we introduce now the subsequent class of kernels.

Definition 2.1 *Let $a \in L_{1,loc}(\mathbb{R}_+)$ be of subexponential growth, and assume $r \in \mathbb{N}$, $\theta_a > 0$, and $\alpha \geq 0$. Then a is said to belong to the class $\mathcal{K}^r(\alpha, \theta_a)$ if a is r-regular and θ_a-sectorial, and satisfies*

$$\limsup_{\mu \to \infty} |\,\hat{a}(\mu)|\,\mu^{\alpha} < \infty,\ \liminf_{\mu \to \infty} |\,\hat{a}(\mu)|\,\mu^{\alpha} > 0,\ \liminf_{\mu \to 0} |\,\hat{a}(\mu)| > 0.$$

*Further, $\mathcal{K}^{\infty}(\alpha, \theta_a) := \{a \in L_{1,loc}(\mathbb{R}_+) : a \in \mathcal{K}^r(\alpha, \theta_a)$ for all $r \in \mathbb{N}\}$. The kernel a is called a $\mathcal{K}$-*kernel *if there exist $r \in \mathbb{N}$, $\theta_a > 0$, and $\alpha \geq 0$, such that $a \in \mathcal{K}^r(\alpha, \theta_a)$.*

An example for a $\mathcal{K}$-kernel is the so-called *standard kernel* $a(t) = t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$, with $\alpha > 0$. Its Laplace transform is given by $\hat{a}(\lambda) = \lambda^{-\alpha}$, $\operatorname{Re}\lambda > 0$. Hence $a \in \mathcal{K}^{\infty}(\alpha, \alpha\frac{\pi}{2})$. We note further that $\mathcal{K}^r(\alpha, \theta) \neq \emptyset$ entails the inequality $\theta \geq \alpha\frac{\pi}{2}$, see [13, p. 4793]. For more properties of $\mathcal{K}$-kernels we refer to [24, Lemma 2.6.2.].

2.4 Evolutionary integral equations

Let X be a Banach space, A a closed linear, but in general unbounded operator in X with dense domain $\mathcal{D}(A)$, and $a \in L_{1,loc}(\mathbb{R}_+)$ a scalar kernel of subexponential growth which is not identically zero. We consider the Volterra equation (1), where $f : \mathbb{R}_+ \to X$ is a given function, strongly measurable and locally integrable, at least. Following Prüss [17] (see also [14, Def. 3.1]), we call (1) *parabolic* if $\hat{a}(\lambda) \neq 0$ for $\operatorname{Re}\lambda > 0$, $-1/\hat{a}(\lambda) \in \rho(A)$, and there is a constant $M > 0$ s.t. $|(I + \hat{a}(\lambda)A)^{-1}| \leq M$ for $\operatorname{Re}\lambda > 0$. If A belongs to the class $\mathcal{S}(X)$ with spectral angle ϕ_A, and a is θ_a-sectorial, then (1) is parabolic provided that $\theta_a + \phi_A < \pi$, cf. [14, Prop. 3.1].

An important property of parabolic Volterra equations consists in that they admit bounded *resolvents* whenever the kernel a is 1-regular, see [14, Thm 3.1]. By a resolvent for (1) we mean a family $\{S(t)\}_{t \geq 0}$ of bounded linear operators in X which satisfy the following conditions: (i) $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$; (ii) $S(t)\mathcal{D}(A) \subset \mathcal{D}(A)$ and $AS(t)x = S(t)Ax$, $x \in \mathcal{D}(A)$, $t \geq 0$; (iii) $S(t)x + A(a * Sx)(t) = x$, $x \in X$, $t \geq 0$. (iii) is called *resolvent equation*, cf. [14, Def. 1.3, Prop. 1.1]. One can show that (1) admits at most one resolvent, and if it exists, then (1) has a unique mild solution u represented by the variation of parameters formula

$$u(t) = \tfrac{d}{dt} \int_0^t S(t-s) f(s)\, ds, \quad t \geq 0, \tag{3}$$

at least for such f for which (3) is meaningful, see [14, Section 1.1 and 1.2].

2.5 Volterra operators in L_p

This paragraph looks at convolution operators in L_p which are associated to a $\mathcal{K}$-kernel. We begin by repeating two fundamental theorems from the monograph Prüss [14] on the inversion of the convolution in $L_p(\mathbb{R}; X)$.

Theorem 2.3 ([14, Thm. 8.6]) *Let $X \in \mathcal{HT}$, and $p \in (1, \infty)$. Assume $a \in L_{1,loc}(\mathbb{R}_+)$ is of subexponential growth, and that a is 1-regular and θ-sectorial, where $\theta < \pi$. Then there is a unique operator $B \in \mathcal{S}(L_p(\mathbb{R}; X))$ such that $(Bf)\tilde{}(\rho) = \hat{a}(i\rho)^{-1} \tilde{f}(\rho)$, $\rho \in \mathbb{R}$, $\tilde{f} \in C_0^\infty(\mathbb{R} \setminus \{0\}; X)$. Moreover, B is causal, commutes with the group of translations, and belongs to the class $\mathcal{BIP}(L_p(\mathbb{R}; X))$ with $\theta_B = \phi_B = \sup\{|\arg \hat{a}(\lambda)| \ : \ \mathrm{Re}\,\lambda > 0\}$.*

Proposition 2.4 ([14, Cor. 8.1]) *Assume $X \in \mathcal{HT}$, $p \in (1, \infty)$, $a \in \mathcal{K}^1(\alpha, \theta)$ with $\theta < \pi$, and let B be the operator from Theorem 2.3. Then $\mathcal{D}(B) = H_p^\alpha(\mathbb{R}; X)$.*

Here $H_p^\alpha(\mathbb{R}; X) := \mathcal{D}(B_0^{\alpha/2})$, $\alpha \in \mathbb{R}_+$, where $B_0 = -(d^2/dt^2) \in \mathcal{BIP}(L_p(\mathbb{R}; X))$, cf. [14, p. 226].

Suppose the assumptions of Proposition 2.4 hold. Let $J = [0, T]$ or $J = \mathbb{R}_+$. We put $H_p^\alpha(J; X) = \{f|_J : f \in H_p^\alpha(\mathbb{R}; X)\}$ and endow this space with the norm $|f|_{H_p^\alpha(J;X)} = \inf\{|g|_{H_p^\alpha(\mathbb{R};X)} \ : \ g|_J = f\}$. We further introduce the subspace ${}_0H_p^\alpha(J; X)$ by means of ${}_0H_p^\alpha(J; X) = \{f|_J : f \in H_p^\alpha(\mathbb{R}; X) \text{ and supp } f \subseteq \mathbb{R}_+\}$. Define then the operator $\mathcal{B} \in \mathcal{S}(L_p(J; X))$ as the restriction of the operator B constructed in Theorem 2.3 to $L_p(J; X)$. This makes sense in virtue of causality. In fact, we have

$$\begin{aligned}
\mathcal{D}(\mathcal{B}) = \mathcal{D}(B|_{L_p(J;X)}) &= \{f \in L_p(J; X) \cap \mathcal{D}(B) : \ Bf \in L_p(J; X)\} \\
&= \{f \in L_p(J; X) \cap H_p^\alpha(\mathbb{R}; X) : Bf \in L_p(J; X)\} \\
&= \{f|_J : \ f \in H_p^\alpha(\mathbb{R}; X), \text{ supp } f \subseteq \mathbb{R}_+, \ Bf \in L_p(\mathbb{R}; X), \text{ and supp } Bf \subseteq \mathbb{R}_+\} \\
&= \{f|_J : \ f \in H_p^\alpha(\mathbb{R}; X) \text{ and supp } f \subseteq \mathbb{R}_+\} \ = \ {}_0H_p^\alpha(J; X),
\end{aligned}$$

the equals sign before the last following from the causality of B. Thus we obtain

Corollary 2.1 *Let $X \in \mathcal{HT}$, $p \in (1, \infty)$, and $J = [0, T]$ or $J = \mathbb{R}_+$. Suppose $a \in \mathcal{K}^1(\alpha, \theta)$ with $\theta < \pi$, and assume in addition $a \in L_1(\mathbb{R}_+)$ in case $J = \mathbb{R}_+$. Then the restriction $\mathcal{B} := B|_{L_p(J;X)}$ of the operator B constructed in Theorem 2.3 to $L_p(J; X)$ is well-defined. The operator $\mathcal{B}$ belongs to $\mathcal{BIP}(L_p(J; X))$ with $\theta_\mathcal{B} \leq \theta_B = \theta_a$ and is invertible satisfying $\mathcal{B}^{-1} w = a * w$ for all $w \in L_p(J; X)$. Moreover $\mathcal{D}(\mathcal{B}) = {}_0H_p^\alpha(J; X)$.*

Example 2.1 For $J = [0,T]$ and $a_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$, $\alpha \in (0,1)$, the operator $\mathcal{B}$ in Corollary 2.1 takes the form $\mathcal{B}u(t) = d/dt \int_0^t a_{1-\alpha}(t-s)u(s)\,ds$, $t > 0$, $u \in {}_0H_p^\alpha(J;X)$, thus coincides with $(d/dt)^\alpha$, the *derivation operator of order* α.

3 Abstract parabolic Volterra equations

In this section we study the abstract Volterra equation

$$u + a * Au = f, \qquad t \geq 0, \tag{4}$$

on a Banach space X. Here $A \in \mathcal{RS}(X)$, $a \in \mathcal{K}^1(\alpha, \theta_a)$ with $\alpha \in (0,2)$, and we assume the parabolicity condition $\theta_a + \phi_A^R < \pi$. Our aim is to find conditions on the given function f which are necessary and sufficient for the existence of a unique solution u of (4) in the space $H_p^{\alpha+\kappa}(J;X) \cap H_p^\kappa(J;D_A)$, where J is $\mathbb{R}_+$ or a compact time-interval $[0,T]$, $1 < p < \infty$, and $\kappa \in [0, 1/p)$.

We begin with the special case of vanishing traces at $t = 0$.

Theorem 3.1 *Let $X \in \mathcal{HT}$, $p \in (1,\infty)$, and $A \in \mathcal{RS}(X)$. Further let J be $\mathbb{R}_+$ or a compact time-interval $[0,T]$. Suppose that a belongs to $\mathcal{K}^1(\alpha,\theta_a)$ with $\alpha \in (0,2)$ and that in addition $a \in L_1(\mathbb{R}_+)$ in case $J = \mathbb{R}_+$. Further let $\kappa \in [0,1/p)$, $\alpha + \kappa \notin \{1/p, 1 + 1/p\}$, and suppose the* parabolicity condition $\theta_a + \phi_A^R < \pi$.

Then (4) has a unique solution in $Z := {}_0H_p^{\alpha+\kappa}(J;X) \cap H_p^\kappa(J;D_A)$ if and only if $f \in {}_0H_p^{\alpha+\kappa}(J;X)$.

Proof. Suppose that $u \in Z$ is a solution of (4). This clearly implies $Au \in H_p^\kappa(J;X) = {}_0H_p^\kappa(J;X)$. From Corollary 2.1 we then deduce that $a * Au \in {}_0H_p^{\alpha+\kappa}(J;X)$. This, together with $u \in {}_0H_p^{\alpha+\kappa}(J;X)$, entails $f \in {}_0H_p^{\alpha+\kappa}(J;X)$. Hence, the necessity part is established.

To prove the converse, we first consider the case $\kappa = 0$. Suppose $f \in {}_0H_p^\alpha(J;X)$ is given. From Section 2.4 we know that equation (4) is parabolic and admits a resolvent $S(\cdot)$, with the aid of which the mild solution u of (4) can be represented by the variation of parameters formula (3). Let $\mathcal{B} \in \mathcal{S}(L_p(J;X))$ be the operator $(a*)^{-1}$ from Corollary 2.1. Since $\mathcal{D}(\mathcal{B}) = {}_0H_p^\alpha(J;X)$, we see that $g := \mathcal{B}f \in L_p(J;X)$ and $u(t) = d/dt(S * (a * g))(t)$, $t \in J$.

Let $E_J : L_p(J;X) \to L_p(\mathbb{R};X)$ denote the operator of extension by 0: $(E_J h)(t) = h(t)$, $t \in J$, $(E_J h)(t) = 0$, $t \notin J$; let $P_J : L_p(\mathbb{R};X) \to L_p(J;X)$ be the restriction to J, and define the operator-valued kernel K by means of $K(t) = (S * a)(t)\chi_{[0,\,\infty)}(t)$, $t \in \mathbb{R}$. Then the solution u can be written in terms of a convolution operator on $L_p(\mathbb{R};X)$: $u = P_J d/dt(K * E_J g)$.

In order to show that $Au \in L_p(J;X)$, we study the symbol of the operator $(A\frac{d}{dt}K*)$, which reads

$$M(\rho) = A\left(\frac{1}{\hat{a}(i\rho)} + A\right)^{-1}, \quad \rho \in \mathbb{R},\ \rho \neq 0. \tag{5}$$

By Lemma 2.1, for each $\rho \neq 0$, $\hat{a}(i\rho) := \lim_{\lambda\to i\rho} \hat{a}(\lambda)$ exists and does not vanish. Besides, $\hat{a}(i\cdot) \in W^1_{\infty,\,loc}(\mathbb{R}\setminus\{0\})$, and the sectoriality of a implies $|\arg(\hat{a}(i\rho))| \leq \theta_a$ for all $\rho \neq 0$. The idea is to apply the Mikhlin multiplier theorem in the operator-valued version, cf. [22] or [5, Theorem 3.19], to the symbol M. But it is not clear that $M \in C^1(\mathbb{R}\setminus\{0\}; \mathcal{B}(L_p(J;X)))$, so we introduce the sequence of symbols

$$M_n(\rho) := A\left(\frac{1}{\hat{a}((i+\frac{1}{n})\rho)} + A\right)^{-1}, \quad \rho \in \mathbb{R},\ n \in \mathbb{N}.$$

Since A is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi_A^R < \pi - \theta_a$, we deduce that $\mathcal{R}(\{M_n(\rho) : \rho \in \mathbb{R}\setminus\{0\}\}) \leq \kappa_1 < \infty$ for all $n \in \mathbb{N}$ with κ_1 not depending on n. From

$$\rho M_n'(\rho) = \frac{(i+\frac{1}{n})\rho\hat{a}'((i+\frac{1}{n})\rho)}{\hat{a}((i+\frac{1}{n})\rho)^2}\left(\frac{1}{\hat{a}((i+\frac{1}{n})\rho)} + A\right)^{-1} M_n(\rho), \quad \rho \in \mathbb{R},$$

using 1-regularity of a, $\mathcal{R}$-sectoriality of A, and Kahane's contraction principle (see [5, Lemma 3.5]), we obtain $\mathcal{R}(\{\rho M_n'(\rho) : \rho \in \mathbb{R}\setminus\{0\}\}) \leq C\kappa_1(1+\kappa_1)$ for all n with C not depending on n. By the Mikhlin multiplier theorem, it follows that the operators T_n defined by $T_n\phi = \mathcal{F}^{-1}(M_n\mathcal{F}\phi)$, for all $\mathcal{F}\phi \in \mathcal{D}(\mathbb{R};X)$, are uniformly L_p-bounded, i.e. $|T_n|_{\mathcal{B}(L_p(\mathbb{R};X))} \leq \kappa_0$, $n \in \mathbb{N}$. Besides we have, for all $\rho \neq 0$, $\lim_{n\to\infty} M_n(\rho) = M(\rho)$ and $|M_n(\rho) - M(\rho)| \leq 2\kappa_1$. Thus, by Lebesgue's dominated convergence theorem, we conclude $M_n \to M$ in $L_{1,\,loc}(\mathbb{R}; \mathcal{B}(X))$ as $n \to \infty$. It follows now from an approximation result, cp. Clément and Prüss [4, p. 6] or [5, Proposition 3.18], that $(A\frac{d}{dt}K*)$, which corresponds to the symbol M, is a bounded linear operator on $L_p(\mathbb{R};X)$ with $|A\frac{d}{dt}K*)|_{\mathcal{B}(L_p(\mathbb{R};X))} \leq \kappa_0$. Since E_J and P_J are bounded as well, we see that $Au \in L_p(J;X)$. As in the necessity part, this implies $a * Au \in {}_0H_p^\alpha(J;X)$, i.e. $u = f - a * Au \in {}_0H_p^\alpha(J;X)$. Hence $u \in Z$.

We now consider the case $\kappa \in (0, 1/p)$. Suppose that $f \in {}_0H_p^{\alpha+\kappa}(J;X)$. Putting $b(t) = e^{-t}t^{\kappa-1}$, $t > 0$, yields $b \in \mathcal{K}^1(\kappa, \kappa\pi/2) \cap L_1(\mathbb{R}_+)$. Let $\mathcal{B}_\kappa = (b*)^{-1}$; then $\mathcal{B}_\kappa f \in {}_0H_p^\alpha(J;X)$. Sufficiency being already established for $\kappa = 0$, we may define $v \in {}_0H_p^\alpha(J;X) \cap L_p(J;D_A)$ as the solution of $v + a * Av = \mathcal{B}_\kappa f$, $t \geq 0$. Since $\mathcal{B}_\kappa$ commutes with both A and the Volterra operator corresponding to the kernel a, we see that $u := b * v$ solves (4) and lies in Z. $\square$

Remarks 3.1 (i) Although not explicitly stated in Theorem 3.1, we have an estimate of the form

$$C^{-1}|f|_{{}_0H_p^{\alpha+\kappa}(J;X)} \leq |u|_Z \leq C|f|_{{}_0H_p^{\alpha+\kappa}(J;X)}, \quad f \in {}_0H_p^{\alpha+\kappa}(J;X),$$

where C is a positive constant not depending on f. This follows immediately from the above proof. Note that the subsequent theorems have to be understood in the same sense: whenever necessary resp. sufficient conditions are stated in terms of regularity classes, this, by convention, means that the corresponding *a priori* estimates hold true.

(ii) Observe that the statement of Theorem 3.1 remains true if $\kappa \geq 1/p$ and Z is defined by ${}_0H^{\alpha+\kappa}_p(J;X) \cap {}_0H^{\kappa}_p(J;D_A)$.

(iii) If $\kappa = 0$, equation (4) is equivalent to $\mathcal{B}u + \mathcal{A}u = \mathcal{B}f$ in the space $Y := L_p(J;X)$, where $\mathcal{A}$ stands for the natural extension of A to Y. If one additionally assumes that $A \in \mathcal{BIP}(X)$ and $\theta_A + \theta_a < \pi$, the assertion of Theorem 3.1 can also be proved by means of the Dore-Venni theorem, Theorem 2.2. This approach has been used in Prüss [14, Thm. 8.7].

(iv) In the case $J = \mathbb{R}_+$ there is a variant of Theorem 3.1 which does not need the assumption $a \in L_1(\mathbb{R}_+)$. Instead one assumes that A is invertible and that f in (4) is of the form $f = a * g$. In this situation, existence of a unique solution of (4) in Z is equivalent to the condition $g \in {}_0H^{\kappa}_p(\mathbb{R}_+;X)$. In fact, if $g \in L_p(\mathbb{R}_+;X)$, then, as seen in the above proof, we have $Au \in L_p(\mathbb{R}_+;X)$, which by invertibility of A entails $u \in L_p(\mathbb{R}_+;X)$. From $\mathcal{B}u + \mathcal{A}u = g$, we then deduce that $\mathcal{B}u \in L_p(\mathbb{R}_+;X)$. So $u \in {}_0H^{\alpha}_p(\mathbb{R}_+;X) \cap L_p(\mathbb{R}_+;D_A)$. The converse direction is trivial, and the case $\kappa > 0$ can be reduced to the case $\kappa = 0$ as above.

(v) The idea to use the Mikhlin multiplier theorem to show that the linear operator corresponding to the symbol (5) is bounded in $L_p(\mathbb{R};X)$ goes back to Clément and Prüss [4]. However, they do not give a detailed proof including the approximation argument, by the aid of which one can surmount the technical difficulty consisting in the fact that the Mikhlin theorem cannot be applied directly to (5).

We now turn our attention to situations where the function f or its derivative $\dot{f}$, if it exists, has a non-vanishing trace at $t = 0$. If $f \in H^{\alpha+\kappa}_p(J;X)$ and $\alpha + \kappa > 1/p$, then $x := f(0) \in X$ exists and we are led to ask under what conditions on $x \in X$ the solution of the problem

$$u(t) + (a * Au)(t) = x, \quad t \geq 0, \tag{6}$$

lies in the space $Z = H^{\alpha+\kappa}_p(J;X) \cap H^{\kappa}_p(J;D_A)$. In case $\alpha + \kappa > 1 + 1/p$ we even have to take into account the trace $y := \dot{f}(0)$. Thus we also have to examine the problem

$$u(t) + (a * Au)(t) = ty, \quad t \geq 0. \tag{7}$$

Observe that the solution u of (6) is given by $u(t) = S(t)x$, $t \geq 0$, while that of (7) equals $(1 * S)(\cdot)y$. This follows immediately from (3).

Theorem 3.2 *Let $X \in \mathcal{HT}$, $J = [0,T]$ a compact time-interval, $p \in (1,\infty)$, and $A \in \mathcal{S}(X)$. Suppose that $\kappa \in [0,1/p)$, $a \in \mathcal{K}^1(\alpha,\theta_a)$ with $\alpha \in (1/p - \kappa, 2)$, $\alpha + \kappa \neq 1 + 1/p$. Further suppose that $\theta_a + \phi_A < \pi$. Then*

$$x \in D_A(1 + \tfrac{\kappa}{\alpha} - \tfrac{1}{p\alpha}, p) \Rightarrow S(\cdot)x \in Z = H^{\alpha+\kappa}_p(J;X) \cap H^{\kappa}_p(J;D_A), \tag{8}$$

and if $\alpha + \kappa > 1 + 1/p$,

$$y \in D_A(1 + \tfrac{\kappa}{\alpha} - \tfrac{1}{\alpha} - \tfrac{1}{p\alpha}, p) \Rightarrow (1 * S)(\cdot)y \in Z. \tag{9}$$

Proof. To prove (8), in case $0 < \kappa < 1/p$ let $\mathcal{B}_\kappa = (b*)^{-1} \in \mathcal{S}(L_p(J;X))$ with $b(t) = t^{\kappa-1}/\Gamma(\kappa)$, $t > 0$; if $\kappa = 0$, we set $\mathcal{B}_\kappa = I$. Suppose $x \in \mathcal{D}(A)$. Letting $u(\cdot) = S(\cdot)x$ our goal is to show that $|\mathcal{B}_\kappa Au|_{L_p(J;X)}$ is bounded above by the $D_A(1+\kappa/\alpha - 1/p\alpha, p)$-norm of x. The assertion then follows by approximation.

To establish the desired estimate we use the representation of the resolvent S via Laplace transform. Recall that $\hat{S}$ is given by $\hat{S}(\lambda) = \lambda^{-1}\left(1+\hat{a}(\lambda)A\right)^{-1}$, $\mathrm{Re}\lambda > 0$.

Let Γ denote a contour $\gamma + i(-\infty,\infty)$ with some $\gamma > 0$. We have

$$\begin{aligned}
\widehat{t\mathcal{B}_\kappa S}(\lambda) &= -\frac{d}{d\lambda}\widehat{\mathcal{B}_\kappa S}(\lambda) = -\frac{d}{d\lambda}\left(\lambda^\kappa \cdot \frac{1}{\lambda}\left(1+\hat{a}(\lambda)A\right)^{-1}\right) \\
&= \frac{1-\kappa}{\lambda^{2-\kappa}}\left(1+\hat{a}(\lambda)A\right)^{-1} + \frac{1}{\lambda^{1-\kappa}}\,\hat{a}'(\lambda)A\left(1+\hat{a}(\lambda)A\right)^{-2}.
\end{aligned}$$

With

$$\phi(\lambda) = 1-\kappa + \frac{\lambda\hat{a}'(\lambda)}{\hat{a}(\lambda)}\cdot \hat{a}(\lambda)A\left(1+\hat{a}(\lambda)A\right)^{-1}, \quad \mathrm{Re}\lambda > 0,$$

and

$$G_\kappa(\lambda) = \lambda^\kappa A(1+\hat{a}(\lambda)A)^{-1}, \quad \mathrm{Re}\lambda > 0,$$

we then obtain

$$(t\mathcal{B}_\kappa AS)\hat{}\,(\lambda) = \frac{\phi(\lambda)}{\lambda^2}\,G_\kappa(\lambda), \quad \mathrm{Re}\lambda > 0.$$

Inversion of the Laplace transform now yields for $t > 0$

$$\begin{aligned}
t\mathcal{B}_\kappa AS(t)x &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(t\mathcal{B}_\kappa AS)\hat{}\,(\lambda)x\,d\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\gamma+i\rho)t}\,\phi(\gamma+i\rho)\,G_\kappa(\gamma+i\rho)x\,\frac{d\rho}{(\gamma+i\rho)^2} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\gamma t+i\sigma}\,\phi(\gamma+i\frac{\sigma}{t})\,G_\kappa(\gamma+i\frac{\sigma}{t})x\,\frac{t\,d\sigma}{(\gamma t+i\sigma)^2},
\end{aligned}$$

where we used the change of variables $\sigma = t\rho$. By 1-regularity of a and parabolicity of (6) we get a bound $|\phi(\lambda)| \le C$, for all $\mathrm{Re}\,\lambda > 0$. Using this estimate and choosing $\gamma = \frac{1}{t}$ we obtain

$$|\mathcal{B}_\kappa AS(t)x|_X \le C\int_{-\infty}^{\infty} |G_\kappa((1+i\rho)/t)x|_X\,\frac{d\sigma}{1+\sigma^2}, \; t > 0.$$

Taking the L_p-norm on the interval $J = [0,T]$ and applying the continuous version of Minkowski's inequality yields

$$|\mathcal{B}_\kappa AS(\cdot)x|_{L_p(J;X)} \le C\int_{-\infty}^{\infty}\left(\int_0^T |G_\kappa((1+i\rho)/t)x|_X^p\,dt\right)^{1/p}\frac{d\sigma}{1+\sigma^2}.$$

Now we employ the change of variables $s = \sqrt{1+\sigma^2}/t$ for the inner integral and enlarge its interval of integration to get

$$\begin{aligned} |\mathcal{B}_\kappa AS(\cdot)x|_{L_p(J;X)} &\leq C\int_{-\infty}^{\infty}(\int_{\frac{1}{T}}^{\infty}(s^{-\frac{2}{p}}|G_\kappa(\tfrac{(1+i\sigma)s}{\sqrt{1+\sigma^2}})x|_X)^p\,ds)^{1/p}\frac{d\sigma}{(1+\sigma^2)^{1-\frac{1}{2p}}} \\ &\leq C\sup_{\sigma\in\mathbb{R}}\{(\int_{\frac{1}{T}}^{\infty}(s^{-\frac{1}{p}}|G_\kappa(\tfrac{(1+i\sigma)s}{\sqrt{1+\sigma^2}})x|_X)^p\,\frac{ds}{s})^{1/p}\}. \end{aligned}$$

This shows that $\mathcal{B}_\kappa AS(\cdot)x \in L_p(J;X)$ whenever

$$\eta := \sup_{\sigma\in\mathbb{R}}\{(\int_{\frac{1}{T}}^{\infty}(s^{\kappa-\frac{1}{p}}|\hat{a}(\tfrac{(1+i\sigma)s}{\sqrt{1+\sigma^2}})|^{-1}|A(1/\hat{a}(\tfrac{(1+i\sigma)s}{\sqrt{1+\sigma^2}})+A)^{-1}x|_X)^p\,\frac{ds}{s})^{1/p}\} < \infty. \tag{10}$$

Now we have by the resolvent equation

$$A\left(\frac{1}{\hat{a}(\lambda)}+A\right)^{-1} = A(|\lambda|^\alpha+A)^{-1}+\left(|\lambda|^\alpha-\frac{1}{\hat{a}(\lambda)}\right)\left(\frac{1}{\hat{a}(\lambda)}+A\right)^{-1}A(|\lambda|^\alpha+A)^{-1},$$

for $\operatorname{Re}\lambda > 0$, thus using the parabolicity of (6),

$$\begin{aligned} |A(1/\hat{a}(\lambda)+A)^{-1}x| &\leq |A(|\lambda|^\alpha+A)^{-1}x| + \\ &\quad +|\,|\lambda|^\alpha\hat{a}(\lambda)-1|\,|(1+\hat{a}(\lambda)A)^{-1}|\,|A(|\lambda|^\alpha+A)^{-1}x| \\ &\leq (C_1+C_2|\hat{a}(\lambda)|\,|\lambda|^\alpha)\,|A(|\lambda|^\alpha+A)^{-1}x|, \end{aligned}$$

with two positive constants C_1, C_2 not depending on λ. Therefore, we can estimate η as follows.

$$\begin{aligned} \eta &\leq C_1\sup_{\sigma\in\mathbb{R}}\{(\int_{\frac{1}{T}}^{\infty}(s^{\kappa-\frac{1}{p}}|\hat{a}(\tfrac{(1+i\sigma)s}{\sqrt{1+\sigma^2}})|^{-1}|A(s^\alpha+A)^{-1}x|_X)^p\,\frac{ds}{s})^{1/p}\} + \\ &\quad +C_2(\int_{\frac{1}{T}}^{\infty}(s^{\alpha+\kappa-\frac{1}{p}}|A(s^\alpha+A)^{-1}x|_X)^p\,\frac{ds}{s})^{1/p} =: \eta_1+\eta_2. \end{aligned}$$

As for η_2, by employing the change of variables $r = s^\alpha$ we get

$$\eta_2 \leq C_2(\int_{(\frac{1}{T})^\alpha}^{\infty}(r^{1+\frac{\kappa}{\alpha}-\frac{1}{p\alpha}}|A(r+A)^{-1}x|_X)^p\,\frac{dr}{r\alpha})^{1/p} \leq \tilde{C}_2|x|_{D_A(1+\frac{\kappa}{\alpha}-\frac{1}{p\alpha},\,p)}.$$

Concerning η_1, there exists $C_3 > 0$ independent of λ such that $|\hat{a}(|\lambda|)\,| \leq C_3|\hat{a}(\lambda)|$, for all $\operatorname{Re}\lambda \geq 0$, $\lambda \neq 0$, see Lemma 2.1. Thus

$$\eta_1 \leq C_1C_3(\int_{\frac{1}{T}}^{\infty}(s^{\kappa-\frac{1}{p}}|\hat{a}(s)|^{-1}|A(s^\alpha+A)^{-1}x|_X)^p\,\frac{ds}{s})^{1/p}.$$

Exploiting the assumption $\liminf_{\mu\to\infty}|\,\hat{a}(\mu)|\,\mu^\alpha > 0$ for a bound $|\hat{a}(s)|\,s^\alpha \geq C_4 > 0$, $1/T \leq s < \infty$, we deduce that $\eta_1 \leq C_1C_3/(C_2C_4)\eta_2$, hence the desired estimate follows. So if $x \in D_A(1+\kappa/\alpha-1/p\alpha, p)$, then $\mathcal{B}_\kappa Au \in L_p(J;X)$, which is equivalent to $Au \in H^\kappa_p(J;X) = {}_0H^\kappa_p(J;X)$, by Corollary 2.1. Applying this once more, we get $a * Au \in {}_0H^{\alpha+\kappa}_p(J;X)$, thus $u = x - a * Au \in H^{\alpha+\kappa}_p(J;X)$, and so $u \in Z$.

Assertion (9) follows by a similar argument, see [24, Thm. 3.1.2] for more details. □

Remarks 3.2 (i) (9) in Theorem 3.2 remains true for $\kappa \in [1/p, 1+1/p)$.

(ii) Let $\kappa = 0$, A be invertible, and assume that the kernel a admits in addition the estimate $C \leq s^{-\alpha}|\hat{a}(s)|$, $s > 0$, for some constant $C > 0$. Then by taking $J = \mathbb{R}_+$ in the above lines and by employing this additional estimate, one obtains

$$\begin{aligned} x \in D_A(1 - \tfrac{1}{p\alpha}, p) \ &\Rightarrow S(\cdot)x \in L_p(\mathbb{R}_+; D_A), \text{ if } \alpha > \tfrac{1}{p}; \\ y \in D_A(1 - \tfrac{1}{\alpha} - \tfrac{1}{p\alpha}, p) \ &\Rightarrow (1 * S)(\cdot)y \in L_p(\mathbb{R}_+; D_A), \text{ if } \alpha > 1 + \tfrac{1}{p}, \end{aligned}$$

see also Prüss [15, Theorem 6].

(iii) The proof of Theorem 3.2 is inspired by the estimates derived in the proof of Theorem 6 in [15].

We next want to show that the above conditions on the traces are also necessary.

Theorem 3.3 *Let $X \in \mathcal{HT}$, $p \in (1,\infty)$, and $A \in \mathcal{S}(X)$. Suppose that $\kappa \in [0, 1/p)$, $a \in \mathcal{K}^1(\alpha, \theta_a)$ with $\alpha \in (1/p - \kappa, 2)$, $\alpha + \kappa \neq 1 + 1/p$. Further let $\omega \geq 0$ and assume that $\theta_a + \phi_A < \pi$. Then*

$$e^{-\omega\cdot} AS(\cdot)x \in H^\kappa_p(\mathbb{R}_+; X) \Rightarrow x \in D_A(1 + \tfrac{\kappa}{\alpha} - \tfrac{1}{p\alpha}, p), \tag{11}$$

and if $\alpha + \kappa > 1 + 1/p$,

$$e^{-\omega\cdot} A(1 * S)(\cdot)y \in H^\kappa_p(\mathbb{R}_+; X) \Rightarrow y \in D_A(1 + \tfrac{\kappa}{\alpha} - \tfrac{1}{\alpha} - \tfrac{1}{p\alpha}, p). \tag{12}$$

Proof. The main idea of the proof is to use Proposition 1 in [15], which says that for every function g in $L_p(\mathbb{R}_+; X)$, the Laplace transform $\hat{g}(\lambda)$ exists for $\operatorname{Re}\lambda > 0$, and with $p^{-1} + q^{-1} = 1$ the following estimate holds true.

$$\int_0^\infty |\hat{g}(\lambda)|^p \lambda^{p-2}\, d\lambda \leq |g|^p_{L_p(\mathbb{R}_+;X)} \Gamma(1/q)^p. \tag{13}$$

To establish implication (11), let $\mathcal{B}_\kappa$ be defined as in the proof of Theorem 3.2 and suppose that $g(t) := \mathcal{B}_\kappa(e^{-\omega t} AS(t)x)$, $t \geq 0$, is contained in $L_p(\mathbb{R}_+; X)$. Then g is Laplace transformable, according to the proposition mentioned above. From $S(t)x + (a * AS)(t)x = x$, $t \geq 0$, and the convolution theorem it follows that

$$\hat{g}(\lambda) = \frac{\lambda^\kappa}{\lambda + \omega} A(1 + \hat{a}_\omega(\lambda)A)^{-1}x, \quad \operatorname{Re}\lambda > 0.$$

Here, $a_\omega(t) := a(t)e^{-\omega t}$, $t \geq 0$. Therefore, using (13) we obtain

$$\eta := \int_0^\infty \Big(\frac{\lambda^{1+\kappa-\frac{1}{p}}}{\lambda + \omega} |A(1 + \hat{a}_\omega(\lambda)A)^{-1}x|_X\Big)^p \frac{d\lambda}{\lambda} \leq C|g|^p_{L_p(\mathbb{R}_+;X)}. \tag{14}$$

Since $a \in \mathcal{K}^1(\alpha, \theta_a)$ and $\omega \geq 0$, we also have $a_\omega \in \mathcal{K}^1(\alpha, \theta_a)$. So, in similar manner as in the proof of Theorem 3.2, we can derive the following estimate from the resolvent equation for A and parabolicity of (6):

$$|A(|\lambda|^\alpha + A)^{-1}x| \leq (C_1 + C_2(|\hat{a}_\omega(\lambda)|\,|\lambda|^\alpha)^{-1})\,|A(1/\hat{a}_\omega(\lambda) + A)^{-1}x|, \ \mathrm{Re}\,\lambda > 0,$$

with two positive constants C_1 and C_2. Employing this estimate as well as the inequality $\lambda/(\lambda + \omega) \geq C_0 > 0$, $\lambda \geq 1$, we deduce that

$$\eta \geq C_0^p \int_1^\infty \Big(\frac{\lambda^{\alpha+\kappa-\frac{1}{p}}}{C_1|\hat{a}_\omega(\lambda)|\lambda^\alpha + C_2}|A(\lambda^\alpha + A)^{-1}x|\Big)^p \frac{d\lambda}{\lambda}. \tag{15}$$

Thanks to the assumption $\limsup_{\mu\to\infty} |\hat{a}(\mu)|\,\mu^\alpha < \infty$, we have an upper bound $|\hat{a}_\omega(\lambda)|\,\lambda^\alpha \leq C_3 < \infty$, $1 \leq \lambda < \infty$. Using this and the substitution $r = \lambda^\alpha$, it follows from (14) and (15) that

$$\tilde{C}|g|^p_{L_p(\mathbb{R}_+;X)} \geq \int_1^\infty (r^{1+\frac{\kappa}{\alpha}-\frac{1}{p\alpha}}|A(r + A)^{-1}x|_X)^p \,\frac{dr}{r\alpha},$$

i.e. $x \in D_A(1 + \kappa/\alpha - 1/p\alpha, p)$.

Assertion (12) follows by the same line of conclusions applied to the function $g(t) = \mathcal{B}_\kappa(e^{-\omega t}A(1 * S)(t)x)$, $t \geq 0$, see [24, Thm. 3.1.3] for more details. $\square$

We are now in position to prove the main theorem concerning (4).

Theorem 3.4 *Let $X \in \mathcal{HT}$, $p \in (1, \infty)$, and J a compact time-interval $[0, T]$ or $\mathbb{R}_+$. Suppose that $A \in \mathcal{RS}(X)$, $a \in \mathcal{K}^1(\alpha, \theta_a)$ with $\alpha \in (0, 2)$ and that in addition $a \in L_1(\mathbb{R}_+)$ in case $J = \mathbb{R}_+$. Further let $\kappa \in [0, 1/p)$ and $\alpha + \kappa \notin \{1/p, 1 + 1/p\}$. Assume the* parabolicity condition *$\theta_a + \phi_A^R < \pi$. Then (4) has a unique solution in $Z := H_p^{\alpha+\kappa}(J;X) \cap H_p^\kappa(J;D_A)$ if and only if the function f satisfies the subsequent conditions: (i) $f \in H_p^{\alpha+\kappa}(J;X)$; (ii) $f(0) \in D_A(1 + \frac{\kappa}{\alpha} - \frac{1}{p\alpha}, p)$, if $\alpha + \kappa > 1/p$; (iii) $\dot{f}(0) \in D_A(1 + \frac{\kappa}{\alpha} - \frac{1}{\alpha} - \frac{1}{p\alpha}, p)$, if $\alpha + \kappa > 1 + 1/p$.*

Proof. We consider three cases with respect to the sum $\alpha + \kappa$.

Case 1: $\alpha + \kappa < 1/p$. Because here $H_p^{\alpha+\kappa}(J;X) = {}_0H_p^{\alpha+\kappa}(J;X)$, we are in the situation of Theorem 3.1.

Case 2: $1/p < \alpha+\kappa < 1+1/p$. We begin with the necessity part. Suppose $u \in Z$ solves (4). Then $Au \in H_p^\kappa(J;X) = {}_0H_p^\kappa(J;X)$, and so $a * Au \in {}_0H_p^{\alpha+\kappa}(J;X)$, by Corollary 2.1. Thus $f = u + a * Au \in H_p^{\alpha+\kappa}(J;X)$, i.e. (i) is proved.

To show (ii), we extend u in the case $J = [0, T]$ to a function v on $\mathbb{R}_+$ such that $v \in Z_{\mathbb{R}_+} := H_p^{\alpha+\kappa}(\mathbb{R}_+;X) \cap H_p^\kappa(\mathbb{R}_+;D_A)$. If $J = \mathbb{R}_+$, we simply set $v := u$. From $a \in \mathcal{K}^1(\alpha, \theta_a)$, it follows that the kernel a_1 defined by $a_1(t) = a(t)e^{-t}$, $t \geq 0$, belongs to $\mathcal{K}^1(\alpha, \theta_a)$, too. Further, $a_1 \in L_1(\mathbb{R}_+)$. Thus $a_1 * Av \in {}_0H_p^{\alpha+\kappa}(\mathbb{R}_+;X)$, and therefore $g := v + a_1 * Av \in H_p^{\alpha+\kappa}(\mathbb{R}_+;X)$ as well as $g - g(0)e^{-\cdot} \in {}_0H_p^{\alpha+\kappa}(\mathbb{R}_+;X)$. Define now v_1 by means of $v_1 + a_1 * Av_1 = g - g(0)e^{-t}$, $t \in \mathbb{R}_+$. This makes sense owing to Theorem 3.1, which also yields $v_1 \in {}_0H_p^{\alpha+\kappa}(\mathbb{R}_+;X) \cap H_p^\kappa(\mathbb{R}_+;D_A)$. Then $v_2 := v - v_1 \in Z_{\mathbb{R}_+}$ solves $w + a_1 * Aw = g(0)e^{-t}$, $t \in \mathbb{R}_+$. Denoting the resolvent of (4) by $S(\cdot)$ one verifies that $v_2(t) = e^{-t}S(t)g(0)$, $t \ge 0$. Consequently, by Theorem 3.3, we get $g(0) = v(0) = u(0) = f(0) \in D_A(1 + \kappa/\alpha - 1/p\alpha, p)$. This shows (ii).

To prove the converse, suppose that the conditions (i) and (ii) are satisfied. Uniqueness is a direct consequence of Theorem 3.1. Concerning existence, we construct a solution of (4) in the following way. If $J = [0, T]$, we define $u_1, u_2 \in Z$ as the solutions of the problems $w_1 + a * Aw_1 = f - f(0)$, $t \in J$, and $w_2 + a * Aw_2 = f(0)$, $t \in J$, respectively. These solutions exist and lie in Z, by virtue of Theorem 3.1 and Theorem 3.2. Thus $u := u_1 + u_2$ has the desired regularity and solves (4). In case $J = \mathbb{R}_+$ let v_1 be the solution of $w_1 + a * Aw_1 = f(0)$, $t \in [0, 1]$. Condition (ii) implies $v_1 \in H_p^{\alpha+\kappa}([0,1];X) \cap H_p^\kappa([0,1];D_A)$, by Theorem 3.2. We extend v_1 to a function $u_1 \in Z$. Then $g := u_1 + a * Au_1 - f \in {}_0H_p^{\alpha+\kappa}(\mathbb{R}_+;X)$, and the solution u_2 of the problem $w_2 + a * Aw_2 = g$, $t \in \mathbb{R}_+$, lies in Z, thanks to Theorem 3.1. Hence, $u := u_1 + u_2 \in Z$ is a solution of (4).

Case 3: $\alpha + \kappa > 1 + 1/p$. We first prove the necessity part. Suppose $u \in Z$ solves (4). Then (i) can be derived as in the second case. Our next objective is to show (iii). The idea behind the following argument is a reduction to Case 2.

For this purpose we extend u to a function $v \in H_p^{\alpha+\kappa}(\mathbb{R};X) \cap H_p^\kappa(\mathbb{R};D_A)$. Define $\mathcal{A}$ as the natural extension of A to $Y := L_p(\mathbb{R};X)$ and let $\mathcal{G} := (I - D_t^2)^{\alpha/2}$ with domain $\mathcal{D}(\mathcal{G}) = H_p^\alpha(\mathbb{R};X)$. Then the operators $\mathcal{A}, \mathcal{G}$ are sectorial in Y with spectral angles $\phi_\mathcal{A} \le \phi_A^R$ and $\phi_\mathcal{G} = 0$. Thus, $\phi_\mathcal{A} + \phi_\mathcal{G} < \pi$. Furthermore the resolvents of $\mathcal{A}$ and $\mathcal{G}$ commute, and the pair $(\mathcal{G}, \mathcal{A})$ is coercively positive. This allows us to apply Proposition 2.3 to this pair of operators acting on Y to obtain

$$|\mathcal{A}^{1-\frac{1-\kappa}{\alpha}}\mathcal{G}^{\frac{1-\kappa}{\alpha}}y|_Y \le C|\mathcal{A}y + \mathcal{G}y|_Y, \text{ for all } y \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{G}), \tag{16}$$

where $C > 0$ is a constant not depending on y. By definition, $\mathcal{D}(\mathcal{G}^{\frac{1-\kappa}{\alpha}}) = H_p^{1-\kappa}(\mathbb{R};X)$. This, together with (16), implies

$$H_p^\alpha(\mathbb{R};X) \cap L_p(\mathbb{R};D_A) \hookrightarrow H_p^{1-\kappa}(\mathbb{R};D_{A^{1-\frac{1-\kappa}{\alpha}}}).$$

Thus, by the boundedness of $D_t\mathcal{G}^{-\frac{1}{\alpha}}$ in $L_p(\mathbb{R};D_{A^{1-\frac{1-\kappa}{\alpha}}})$,

$$\dot{v} = D_t\mathcal{G}^{-\frac{1}{\alpha}}\mathcal{G}^{\frac{1-\kappa}{\alpha}}\mathcal{G}^{\frac{\kappa}{\alpha}}v \in H_p^{\alpha+\kappa-1}(\mathbb{R};X) \cap L_p(\mathbb{R};D_{A^{1-\frac{1-\kappa}{\alpha}}}).$$

To determine the regularity of $\dot{v}(0)$, let $b(t) = t^{\alpha+\kappa-2}$, $t \ge 0$; clearly $b \in \mathcal{K}^1(\alpha + \kappa - 1, (\alpha + \kappa - 1)\pi/2)$. Since $A \in \mathcal{RS}(X)$, we have $\mathcal{C} := A^{1-(1-\kappa)/\alpha} \in \mathcal{RS}(X)$ and

$$\phi_\mathcal{C}^R \le (1 - \tfrac{1-\kappa}{\alpha})\phi_A^R \le (\tfrac{\alpha+\kappa-1}{\alpha})(\pi - \alpha\tfrac{\pi}{2}) = (\tfrac{\alpha+\kappa-1}{\alpha})\pi - (\tfrac{\alpha+\kappa-1}{2})\pi,$$

by Proposition 2.1, the parabolicity condition, and the fact that $\theta_a \ge \alpha\pi/2$ (cp. the remarks after Def. 2.1). So we see that $\theta_b + \phi_\mathcal{C}^R < \pi$. By the necessity part of Case 2, applied to the equation $\dot{v}(t) + (b * \mathcal{C}\dot{v})(t) = g(t)$, $t \in [0, 1]$, with some

$g \in H_p^{\alpha+\kappa-1}([0,1];X)$, we get $\dot f(0) = \dot v(0) \in D_C(1-1/p(\alpha+\kappa-1),p)$. It follows now, by Theorem 2.1, that $\dot f(0)$ lies in

$$\begin{aligned} D_{A^{1-\frac{1-\kappa}{\alpha}}}\left(1-\tfrac{1}{p(\alpha+\kappa-1)},p\right) &= D_A\left((1-\tfrac{1-\kappa}{\alpha})(1-\tfrac{1}{p(\alpha+\kappa-1)}),p\right)\\ &= D_A\left(1+\tfrac{\kappa}{\alpha}-\tfrac{1}{\alpha}-\tfrac{1}{p\alpha},p\right). \end{aligned}$$

Hence condition (iii) is satisfied.

It remains to show (ii). Put $J_1 := [0,1]$ in case $J = \mathbb{R}_+$ and $J_1 := J$, otherwise. Define w_1 by means of $w_1(t) + (a * Aw_1)(t) = t\dot f(0)$, $t \in J_1$. Then, due to Theorem 3.2, it follows from condition (iii) that $u|_{J_1} - w_1$ lies in $H_p^{\alpha+\kappa}(J_1;X) \cap H_p^{\kappa}(J_1;D_A)$. We extend $u|_{J_1} - w_1$ to a function $v \in Z_{\mathbb{R}_+}$ and put $h(t) := v(t) + (a_1 * Av)(t)$, $t \in \mathbb{R}_+$, where a_1 is defined as in Case 2. As above, we see that $h \in H_p^{\alpha+\kappa}(\mathbb{R}_+;X)$. By construction, we have $\dot v(0) = \dot h(0) = 0$, i.e. $h - \psi(\cdot)h(0) \in {}_0H_p^{\alpha}(\mathbb{R}_+;X)$, where $\psi(t) = (1+t)e^{-t}$, $t \geq 0$. In fact, $\psi \in H_p^{\alpha+\kappa}(\mathbb{R}_+)$, $\psi(0) = 1$, $\dot\psi(t) = -te^{-t}$, $t \geq 0$, in particular $\dot\psi(0) = 0$. Define now the function v_1 by means of $v_1 + a_1 * Av_1 = h - \psi h(0)$, $t \in \mathbb{R}_+$. This is possible in view of Theorem 3.1, which also gives $v_1 \in {}_0H_p^{\alpha+\kappa}(\mathbb{R}_+;X) \cap H_p^{\kappa}(\mathbb{R}_+;D_A)$. Consequently, $v_2 := v - v_1 \in Z_{\mathbb{R}_+}$, and v_2 solves the equation $w + a_1 * Aw = \psi h(0)$, $t \in \mathbb{R}_+$. If $S(\cdot)$ denotes the resolvent for (4), then one verifies that $v_2(t) = e^{-t}S(t)h(0) + e^{-t}(1 * S)(t)h(0)$, $t \geq 0$. Letting $k(t) = e^{-t}$ and $\xi(t) = e^{-t}S(t)h(0)$, $t \geq 0$, we thus have

$$v_2(t) = \xi(t) + (k * \xi)(t), \quad t \geq 0. \tag{17}$$

Define now $r \in L_{1,loc}(\mathbb{R}_+)$ by means of $r(t) + (r * k)(t) = k(t)$, $t \geq 0$. Then (17) implies $\xi(t) = v_2(t) - (r * v_2)(t)$, $t \geq 0$, in particular $A\xi(t) = Av_2(t) - (r * Av_2)(t)$, $t \geq 0$. Since $k \in L_1(\mathbb{R}_+)$ and $1 + \hat k(\lambda) = 1 + (\lambda+1)^{-1} \neq 0$, $\operatorname{Re}\lambda \geq 0$, it follows by the halfline Paley-Wiener theorem (see [8, Chapter 2, Theorem 4.1, p. 45]) that $r \in L_1(\mathbb{R}_+)$. So, letting $b(t) = t^{\kappa-1}e^{-t}$, $t \geq 0$, and $\mathcal{B}_\kappa = (b*)^{-1} \in \mathcal{S}(L_p(\mathbb{R}_+;X))$, we see with the aid of Young's inequality and Corollary 2.1 that

$$r * Av_2 = r * b * \mathcal{B}_\kappa Av_2 = b * r * \mathcal{B}_\kappa Av_2 \in H_p^{\kappa}(\mathbb{R}_+;X).$$

Therefore $A\xi \in H_p^{\kappa}(\mathbb{R}_+;X)$, and so, by Theorem 3.3, we get $h(0) = v(0) = u(0) = f(0) \in D_A(1+\kappa/\alpha - 1/p\alpha, p)$. This proves condition (ii).

To prove the converse direction, suppose that the conditions (i),(ii) and (iii) are fulfilled. Uniqueness is an immediate consequence of Theorem 3.1. As for existence, put $x = f(0)$ and $y = \dot f(0)$. If $J = [0,T]$, let u_1 be the solution of $w(t) + (a * Aw)(t) = f(t) - x - ty$, $t \in J$. This solution exists and lies in ${}_0H_p^{\alpha+\kappa}(J;X) \cap H_p^{\kappa}(J;D_A)$, thanks to Theorem 3.1. By Theorem 3.2, we also have $S(\cdot)x$, $(1 * S)(\cdot)y \in Z$, so that $u := u_1 + Sx + 1 * Sy \in Z$. It is easy to see that u solves (4). In case of $J = \mathbb{R}_+$, we extend $v_1 \in H_p^{\alpha+\kappa}([0,1];X) \cap L_p([0,1];D_A)$ defined by $v_1(t) = S(t)x + (1 * S)(t)y$, $t \in [0,1]$, to a function $u_1 \in Z$ and set $f_1 = f - (u_1 + a * Au_1)$. By construction, $f_1 \in {}_0H_p^{\alpha+\kappa}(\mathbb{R}_+;X)$. So, due to Theorem 3.1, we can define $u_2 \in Z$ as the solution of $v + a * Av = f_1$, $t \in \mathbb{R}_+$. Clearly, $u := u_1 + u_2 \in Z$ solves (4). $\square$

Remarks 3.3 In the case of a compact interval J, one can weaken the assumptions on both a and A. Due to the transformation property of (4) discussed in [24, Section 2.8], it suffices to assume that $\mu + A \in \mathcal{RS}(X)$ with $\theta_a + \phi^R_{\mu+A} < \pi$ for some $\mu \geq 0$. As for the kernel, the theorem is also true, if a is of the form $a = b + dk * b$, where b is like a in the statement of Theorem 3.4 and $k \in BV_{loc}(\mathbb{R}_+)$ with $k(0) = k(0+) = 0$. This follows from a straightforward perturbation argument, cp. [14, Section 8.5].

There is a companion result to Theorem 3.4 in the case where $\kappa \in (1/p, 1 + 1/p)$ and $\alpha + \kappa < 2 + 1/p$.

Theorem 3.5 *Let $X \in \mathcal{HT}$, $p \in (1, \infty)$, $J = [0, T]$. Suppose that $A \in \mathcal{RS}(X)$ and $a \in \mathcal{K}^1(\alpha, \theta_a)$ with $\alpha \in (0, 2)$. Further let $\kappa \in (1/p, 1 + 1/p)$, $\alpha + \kappa < 2 + 1/p$ and $\alpha + \kappa \neq 1 + 1/p$. Assume $\theta_a + \phi^R_A < \pi$. Then (4) has a unique solution in $H^{\alpha+\kappa}_p(J; X) \cap H^{\kappa}_p(J; D_A)$ if and only if (i) $f(0) \in \mathcal{D}(A)$; and (ii) $f = h + (1 * a)Af(0)$, with $h \in H^{\alpha+\kappa}_p(J; X)$, and $\dot{h}(0) \in D_A(1 + \frac{\kappa}{\alpha} - \frac{1}{\alpha} - \frac{1}{p\alpha}, p)$ in case $\alpha + \kappa > 1 + 1/p$.*

The proof of this result is similar to that of Theorem 3.4, see [24, Section 3.3].

With the aid of Theorem 3.4, Proposition 2.3, Theorem 2.1, and real interpolation one can derive the following general trace theorem, see [24, Section 3.2].

Theorem 3.6 *Let $X \in \mathcal{HT}$, $p \in (1, \infty)$, $\gamma \in [0, 1/p)$, and $s + \gamma > n + 1/p$ with $n \in \mathbb{N}_0$. Let further $J = [0, T]$ or $\mathbb{R}_+$, and $A \in \mathcal{RS}(X)$ with $\phi^R_A < \pi/s$. Then for all $0 \leq k \leq n$, the subsequent embeddings are valid.*

$$\begin{aligned} H^{s+\gamma}_p(J; X) \cap H^{\gamma}_p(J; D_{A^s}) &\hookrightarrow BUC^k(J; D_A(s + \gamma - k - \tfrac{1}{p}, p)), \\ B^{s+\gamma}_{pp}(J; X) \cap H^{\gamma}_p(J; D_A(s, p)) &\hookrightarrow BUC^k(J; D_A(s + \gamma - k - \tfrac{1}{p}, p)). \end{aligned}$$

This result is an extension of [16, Proposition 3], where $\gamma = 0$ and A is assumed to belong to $\mathcal{BIP}(X)$.

4 Abstract equations of first and second order on the halfline

In this paragraph we collect some known results on maximal L_p-regularity of abstract problems on the *halfline.*

The first theorem, which is due to Weis [23], concerns the abstract Cauchy problem

$$\dot{u} + Au = f, \qquad t > 0, \qquad u(0) = u_0, \tag{18}$$

in a Banach space X.

Theorem 4.1 *Let $X \in \mathcal{HT}$, $p \in (1, \infty)$, $A \in \mathcal{RS}(X)$, $\phi^R_A < \pi/2$, and $0 \in \rho(A)$. Then (18) has a unique solution in $H^1_p(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; D_A)$ if and only if*

$$(i) \quad f \in L_p(\mathbb{R}_+; X); \qquad (ii) \quad u_0 \in D_A(1 - 1/p, p).$$

Proof. The assertion follows immediately from Remark 3.1(iv), Remark 3.2(ii) and Theorem 3.4 with $a \equiv 1$. □

The following theorem concerns the abstract second order problem with Dirichlet condition

$$\begin{cases} -u''(y) + F^2 u(y) = f(y), \quad y > 0, \\ u(0) = \phi, \end{cases} \tag{19}$$

in $L_p(\mathbb{R}_+; X)$.

Theorem 4.2 *Let $X \in \mathcal{HT}$, $p \in (1, \infty)$, $F \in \mathcal{BIP}(X)$, $\theta_F < \pi/2$, and $0 \in \rho(F)$. Then (19) has a unique solution u in $H^2_p(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; D_{F^2})$ if and only if*

$$(i) \quad f \in L_p(\mathbb{R}_+; X); \qquad (ii) \quad \phi \in D_F(2 - \tfrac{1}{p}, p).$$

This result has been obtained by Prüss, cf. [16, Theorem 3]. Recall the relation $D_F(2 - 1/p, p) = \{g \in \mathcal{D}(F) : Fg \in D_F(1 - 1/p, p)\}$.

There is a corresponding result for the abstract second order problem with abstract Robin condition

$$\begin{cases} -u''(y) + F^2 u(y) = f(y), \quad y > 0, \\ -u'(0) + Du(0) = \psi, \end{cases} \tag{20}$$

in $L_p(\mathbb{R}_+; X)$. For $D = 0$, the Robin condition becomes the Neumann condition.

Theorem 4.3 *Let $X \in \mathcal{HT}$, $p \in (1, \infty)$, $F \in \mathcal{BIP}(X)$, $\theta_F < \pi/2$, and $0 \in \rho(F)$. Suppose that D is pseudo-sectorial in X, belongs to $\mathcal{BIP}(\overline{\mathcal{R}(D)})$, commutes with F, and is such that $\theta_F + \theta_D < \pi$.*

Then (20) has a unique solution u in $H^2_p(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; D_{F^2})$ with $u(0) \in \mathcal{D}(D)$ and $Du(0) \in D_F(1 - 1/p, p)$ if and only if

$$(i) \quad f \in L_p(\mathbb{R}_+; X); \qquad (ii) \quad \psi \in D_F(1 - \tfrac{1}{p}, p).$$

This result is also due to Prüss, see [16, Theorem 4].

5 Abstract problems with inhomogeneous boundary data

We now study the vector-valued problem

$$\begin{cases} u - a * \partial_y^2 u + a * Au = f, \quad t \in J, \ y > 0, \\ u(t, 0) = \phi(t), \quad t \in J, \end{cases} \tag{21}$$

in $L_p(J; L_p(\mathbb{R}_+; X))$. Here $J = [0, T]$ is a compact time-interval, $A \in \mathcal{RS}(X)$, and $a \in \mathcal{K}^1(\alpha, \theta_a)$ with $\alpha \in (0, 2)$. Our aim is to characterize unique existence of a solution u in the maximal regularity class of type L_p, i.e.

$$u \in Z := H^\alpha_p(J; L_p(\mathbb{R}_+; X)) \cap L_p(J; H^2_p(\mathbb{R}_+; X)) \cap L_p(J; L_p(\mathbb{R}_+; D_A)) \tag{22}$$

in terms of regularity classes for the data f and ϕ.

Theorem 5.1 *Let $X \in \mathcal{HT}$, and $p \in (1,\infty)$. Suppose $A \in \mathcal{BIP}(X)$ and $a \in \mathcal{K}^1(\alpha,\theta_a)$ with $\alpha \in (0,2) \setminus \left\{\frac{1}{p},\ \frac{2}{2p-1},\ 1+\frac{1}{p},\ 1+\frac{3}{2p-1}\right\}$, and assume $\theta_a + \theta_A < \pi$. Then the problem (21) has a unique solution in Z if and only if the data f and ϕ satisfy the following conditions.*

$$
\begin{array}{ll}
(i) & f \in H^\alpha_p(J; L_p(\mathbb{R}_+; X));\\
(ii) & \phi \in B^{\alpha(1-\frac{1}{2p})}_{pp}(J;X) \cap L_p(J; D_A(1-\frac{1}{2p},p));\\
(iii) & f|_{t=0} \in B^{2-\frac{2}{p\alpha}}_{pp}(\mathbb{R}_+;X) \cap L_p(\mathbb{R}_+; D_A(1-\frac{1}{p\alpha},p)),\ \textit{if}\ \alpha > \frac{1}{p};\\
(iv) & f|_{t=0,\,y=0} = \phi|_{t=0},\ \textit{if}\ \alpha > \frac{2}{2p-1};\\
(v) & \partial_t f|_{t=0} \in B^{2(1-\frac{1}{\alpha}-\frac{1}{p\alpha})}_{pp}(\mathbb{R}_+;X) \cap L_p(\mathbb{R}_+; D_A(1-\frac{1}{\alpha}-\frac{1}{p\alpha},p)),\ \textit{if}\ \alpha > 1+\frac{1}{p};\\
(vi) & \partial_t f|_{t=0,\,y=0} = \dot\phi|_{t=0},\ \textit{if}\ \alpha > 1+\frac{3}{2p-1}.
\end{array}
$$

If this is the case, then additionally

$$
\begin{array}{ll}
(vii) & f|_{t=0,\,y=0},\ \phi|_{t=0} \in D_A(1-\frac{1}{2p}-\frac{1}{p\alpha},p),\ \textit{if}\ \alpha > \frac{2}{2p-1};\\
(viii) & \partial_t f|_{t=0,\,y=0},\ \dot\phi|_{t=0} \in D_A(1-\frac{1}{2p}-\frac{1}{\alpha}-\frac{1}{p\alpha},p),\ \textit{if}\ \alpha > 1+\frac{3}{2p-1}.
\end{array}
$$

Proof. We begin with the necessity part. Suppose $u \in Z$ is a solution of (21). Then clearly $f = u - a * \partial_y^2 u + a * Au \in H^\alpha_p(J; L_p(\mathbb{R}_+;X))$, i.e. (i) is satisfied.

Next we extend u w.r.t. y to all of $\mathbb{R}$ by $u(t,y) = 3u(t,-y) - 2u(t,-2y), y < 0$, resulting in a function (again denoted by u) that belongs to $H^\alpha_p(J; L_p(\mathbb{R};X)) \cap L_p(J; H^2_p(\mathbb{R};X)) \cap L_p(J; L_p(\mathbb{R}; D_A))$. Define $\mathcal{A}$ as the natural extension of A to $Y := L_p(\mathbb{R};X)$ and let $G = -\partial_y^2$ with domain $\mathcal{D}(G) = H^2_p(\mathbb{R};X)$. Then $G \in \mathcal{BIP}(Y)$ with $\theta_G = 0$. Since $\mathcal{A}$ and G commute, Theorem 2.2 yields that $\Lambda := \mathcal{A} + G$ with domain $\mathcal{D}(\Lambda) = \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(G)$ belongs to $\mathcal{BIP}(Y)$ with $\theta_\Lambda \le \theta_A$, in particular $\Lambda \in \mathcal{RS}(Y)$ and $\phi^R_\Lambda \le \theta_A$. It follows now from Theorem 3.4 that $u|_{t=0} \in D_\Lambda(1-1/p\alpha,p)$, if $\alpha > 1/p$. Proposition 2.2 gives

$$
\begin{aligned}
D_\Lambda(1-\tfrac{1}{p\alpha},p) &= D_G(1-\tfrac{1}{p\alpha},p) \cap D_A(1-\tfrac{1}{p\alpha},p)\\
&= B^{2-\frac{2}{p\alpha}}_{pp}(\mathbb{R};X) \cap L_p(\mathbb{R}; D_A(1-\tfrac{1}{p\alpha},p)),
\end{aligned} \tag{23}
$$

which entails (iii), by restriction to $y \in \mathbb{R}_+$. If $\alpha > 1 + 1/p$, then Theorem 3.4 further yields $\partial_t u|_{t=0} \in D_\Lambda(1-1/\alpha-1/p\alpha,p)$. By Proposition 2.2, we get

$$
D_\Lambda(1-\tfrac{1}{\alpha}-\tfrac{1}{p\alpha},p) = B^{2(1-\frac{1}{\alpha}-\frac{1}{p\alpha})}_{pp}(\mathbb{R};X) \cap L_p(\mathbb{R}; D_A(1-\tfrac{1}{\alpha}-\tfrac{1}{p\alpha},p)). \tag{24}
$$

Thus, after restriction to $y \in \mathbb{R}_+$ we arrive at condition (v).

To show necessity of (ii), we choose an extension of the solution $u \in Z$ w.r.t. t to all of $\mathbb{R}$ (again denoted by u) which lies in $Z_1 := H^\alpha_p(\mathbb{R}; L_p(\mathbb{R}_+;X)) \cap$

$L_p(\mathbb{R}; H^2_p(\mathbb{R}_+; X)) \cap L_p(\mathbb{R}; L_p(\mathbb{R}_+; D_A))$. Set $A_1 = I + A$ with domain $\mathcal{D}(A_1) = \mathcal{D}(A)$ and let $\mathcal{A}_1$ be the natural extension of A_1 to $Y := L_p(\mathbb{R}; X)$. Then A_1 is invertible, and $A_1 \in \mathcal{BIP}(X)$ with $\theta_{A_1} \le \theta_A$, by Theorem 2.2; thus $0 \in \rho(\mathcal{A}_1)$ and $\mathcal{A}_1 \in \mathcal{BIP}(Y)$ with $\theta_{\mathcal{A}_1} \le \theta_A$. Further let $B \in \mathcal{BIP}(Y)$ be the inverse Volterra operator from Theorem 2.3. Then $\mathcal{A}_1$ and B commute, and we have $\theta_B + \theta_{\mathcal{A}_1} \le \theta_a + \theta_A < \pi$. It follows from Theorem 2.2 that $B + \mathcal{A}_1$ with $\mathcal{D}(B) \cap \mathcal{D}(\mathcal{A}_1) = H^\alpha_p(\mathbb{R}; X) \cap L_p(\mathbb{R}; D_A)$ is invertible and contained in $\mathcal{BIP}(Y)$ with $\theta_{B+\mathcal{A}_1} \le \max\{\theta_a, \theta_A\}$. The function $u \in Z_1$ now satisfies a problem of the form $Bu - \partial_y^2 u + A_1 u = g$, $y > 0$, $t \in \mathbb{R}$, $u(t,0) = \varphi(t)$, $t \in \mathbb{R}$, with $g \in L_p(\mathbb{R}; Y)$ and some φ, which is an extension of ϕ to all of $\mathbb{R}$. To determine the regularity of φ, we apply Theorem 4.2 to the invertible operator $F := (B + \mathcal{A}_1)^{1/2}$, which belongs again to $\mathcal{BIP}(Y)$ with $\theta_F \le \max\{\theta_a, \theta_A\}/2 < \pi/2$. This results in $\varphi \in D_F(2 - 1/p, p)$. Theorem 2.1 and Proposition 2.2 yield

$$D_F(\gamma, p) = D_{(B+\mathcal{A}_1)^{1/2}}(\gamma, p) = D_{B+\mathcal{A}_1}(\tfrac{\gamma}{2}, p) = D_B(\tfrac{\gamma}{2}, p) \cap D_{\mathcal{A}_1}(\tfrac{\gamma}{2}, p), \quad \gamma \in (0,1).$$

Therefore

$$D_F(1 - \tfrac{1}{p}, p) = B^{\alpha(\frac{1}{2} - \frac{1}{2p})}_{pp}(\mathbb{R}; X) \cap L_p(\mathbb{R}; D_A(\tfrac{1}{2} - \tfrac{1}{2p}, p)), \tag{25}$$

which, by Proposition 2.3 and real interpolation, implies

$$D_F(2 - \tfrac{1}{p}, p) = B^{\alpha(1 - \frac{1}{2p})}_{pp}(\mathbb{R}; X) \cap L_p(\mathbb{R}; D_A(1 - \tfrac{1}{2p}, p)). \tag{26}$$

Hence condition (ii) in Theorem 5.1 is necessary, by restriction to J.

To derive (iv), note that by Proposition 2.3, $u \in Z$ entails

$$u \in H^{\alpha s}_p(J; H^{2(1-s)}_p(\mathbb{R}_+; X)), \quad s \in [0,1]. \tag{27}$$

The space in (27) embeds into $BUC(J \times \mathbb{R}_+; X)$ if $1/p < s\alpha$ and $1/p < 2 - 2s$, i.e. if $1/p\alpha < s < 1 - 1/2p$. This shows (iv).

We next determine the regularity of $\phi|_{t=0}$ in case $\alpha > 2/(2p-1)$, which is equivalent to $1/p < \alpha(1 - 1/2p)$. In light of Theorem 3.6, we have the embedding

$$B^{s\gamma}_{pp}(J; X) \cap L_p(J; D_A(\gamma, p)) \hookrightarrow C(J; D_A(\gamma - \tfrac{1}{p\,s}, p)), \tag{28}$$

if $\gamma, \gamma - 1/ps \in (0,1)$ and $s\gamma \in (1/p, 2)$. Thus, by taking $s = \alpha$ and $\gamma = 1 - 1/2p$, we see that $\phi|_{t=0} \in D_A(1 - 1/2p - 1/p\alpha, p)$. Hence property (vii) is established.

Let now $\alpha > 1 + 3/(2p-1)$, which in particular means that $\alpha > 1 + 1/p$. Proposition 2.3 and (27) with $s = 1/\alpha$ yield that $\partial_t u$ lies in

$$H^{\alpha-1}_p(J; L_p(\mathbb{R}_+; X)) \cap L_p(\mathbb{R}; H^{2(1-\frac{1}{\alpha})}_p(\mathbb{R}_+; X)) \hookrightarrow H^{(\alpha-1)s}_p(J; H^{2(1-\frac{1}{\alpha})(1-s)}_p(\mathbb{R}_+; X)),$$

for each $s \in [0,1]$. The space on the right is embedded into $BUC(J \times \mathbb{R}_+; X)$ whenever $(\alpha - 1)s > 1/p$ and $2(1 - 1/\alpha)(1 - s) > 1/p$, i.e. if $1/p(\alpha - 1) < s <$

$1-\alpha/2p(\alpha-1)$. An easy calculation shows that existence of such an s is equivalent to $\alpha > 1+3/(2p-1)$. Therefore $\partial_t u(0,0)$ exists and so (vi) is necessary. Condition (viii) follows from (v) and (28) with $J=\mathbb{R}_+$ applied to $\gamma = 1-1/\alpha-1/p\alpha$, and $s=2$. Hence the necessity part as well as (vii) and (viii) are established.

We turn now to sufficiency. Suppose the given data f and ϕ satisfy (i)-(vi). We extend the function f w.r.t. y to all of $\mathbb{R}$ in such a way that (i),(iii),(v) are fulfilled with $\mathbb{R}_+$ replaced by $\mathbb{R}$. Let $Y = L_p(\mathbb{R};X)$ and $\Lambda \in \mathcal{BIP}(Y)$ be as above. Then $f|_{t=0} \in D_\Lambda(1-1/p\alpha, p)$, if $\alpha > 1/p$, and $\partial_t f|_{t=0} \in D_\Lambda(1-1/\alpha-1/p\alpha, p)$ in case $\alpha > 1+1/p$, owing to (23) and (24). Thus, by Theorem 3.4, the Volterra equation $v + a*\Lambda v = f$, $t\in J$, admits a unique solution v_1 in $H_p^\alpha(J;L_p(\mathbb{R};X)) \cap L_p(J;H_p^2(\mathbb{R};X)) \cap L_p(J;L_p(\mathbb{R};D_A))$. Let u_1 be the restriction of v_1 to $y\in\mathbb{R}_+$. Then $u_1 \in Z$, and u_1 solves the Volterra equation in (21).

We next put $\phi_1 := u_1|_{y=0}$. Due to the necessity part of Theorem 5.1 (conditions (ii),(iv),(vi)), we see that $\phi_1 \in B_{pp}^{\alpha(1-1/2p)}(J;X) \cap L_p(J;D_A(1-1/2p,p))$, and $f|_{t=0,\,y=0} = \phi_1|_{t=0}$, if $\alpha > 2/2p-1$, as well as $\partial_t f|_{t=0,\,y=0} = \dot{\phi}_1|_{t=0}$, in case $\alpha > 1+3/(2p-1)$. Hence $\phi-\phi_1 \in {}_0B_{pp}^{\alpha(1-1/2p)}(J;X) \cap L_p(J;D_A(1-1/2p,p))$. Consider now the problem

$$\begin{cases} v - a*\partial_y^2 v + a*Av = 0, \quad t\in J,\ y>0,\\ \qquad\qquad v(t,0) = \phi-\phi_1, \quad t\in J. \end{cases} \tag{29}$$

Define this time $\mathcal{A}$ as the natural extension of A to $Y_1 := L_p(J;X)$. Let further $\mathcal{B} = (a*)^{-1} \in \mathcal{BIP}(L_p(J;X))$. Then $\mathcal{B}$ is invertible, $\mathcal{A}$, $\mathcal{B} \in \mathcal{BIP}(Y_1)$ commute, and $\theta_\mathcal{A} + \theta_\mathcal{B} \le \theta_A + \theta_a < \pi$. Thus, by Theorem 2.2, the operator $\mathcal{B}+\mathcal{A}$ with $\mathcal{D}(\mathcal{B}) \cap \mathcal{D}(\mathcal{A}) = {}_0H_p^\alpha(J;X) \cap L_p(J;D_A)$ is invertible and contained in $\mathcal{BIP}(Y_1)$ with $\theta_{\mathcal{B}+\mathcal{A}} \le \max\{\theta_a,\theta_A\}$. Hence $F_1 := (\mathcal{B}+\mathcal{A})^{1/2} \in \mathcal{BIP}(Y_1)$ with $\theta_{F_1} \le \max\{\theta_a,\theta_A\}/2 < \pi/2$ and $0 \in \rho(F_1)$. We can now rewrite (29) as

$$-v''(y) + F_1^2 v(y) = 0,\ y>0, \quad v(0) = \phi-\phi_1.$$

Similarly as above for the operator F, one computes the spaces $D_{F_1}(1-1/p,p)$ and $D_{F_1}(2-1/p,p)$ (replace in (25) and (26) F by F_1, $\mathbb{R}$ by J and B_{pp}^s by ${}_0B_{pp}^s$) to see that $\phi-\phi_1 \in D_{F_1}(2-1/p,p)$, which implies existence of a solution u_2 of (29) in ${}_0H_p^\alpha(J;L_p(\mathbb{R}_+;X)) \cap L_p(J;H_p^2(\mathbb{R}_+;X)) \cap L_p(J;L_p(\mathbb{R}_+;D_A))$, by Theorem 4.2.

Finally $u := u_1+u_2 \in Z$ solves (21). Uniqueness follows from Theorem 4.2. □

There is a corresponding result for the problem

$$\begin{cases} u - a*\partial_y^2 u + a*Au = f, \quad t\in J,\ y>0,\\ -\partial_y u(t,0) + Du(t,0) = \phi(t), \quad t\in J. \end{cases} \tag{30}$$

Here the operator D is pseudo-sectorial in X, and we assume that $D_{A^{1/2}} \hookrightarrow D_D$.

Theorem 5.2 *Let $X \in \mathcal{HT}$, and $p \in (1,\infty)$. Suppose $A \in \mathcal{BIP}(X)$ and $a \in \mathcal{K}^1(\alpha,\theta_a)$ with $\alpha \in (0,2) \setminus \left\{\frac{1}{p},\ \frac{2}{p-1},\ 1+\frac{1}{p}\right\}$. Let further D be a pseudo-sectorial operator in X such that $D \in \mathcal{BIP}(\overline{\mathcal{R}(D)})$. Suppose that A and D commute, and that $D_{A^{1/2}} \hookrightarrow D_D$. Assume further $\theta_a + \theta_A < \pi$ and $\theta_D < \pi - \max\{\theta_a,\theta_A\}/2$. Then the problem (30) has a unique solution in Z (see (22)) if and only if*

(i) $\quad f \in H^\alpha_p(J; L_p(\mathbb{R}_+; X))$;

(ii) $\quad \phi \in B^{\alpha(\frac{1}{2}-\frac{1}{2p})}_{pp}(J;X) \cap L_p(J; D_A(\frac{1}{2}-\frac{1}{2p}, p))$;

(iii) $\quad f|_{t=0} \in B^{2-\frac{2}{p\alpha}}_{pp}(\mathbb{R}_+;X) \cap L_p(\mathbb{R}_+; D_A(1-\frac{1}{p\alpha}, p))$, *if* $\alpha > \frac{1}{p}$;

(iv) $\quad f|_{t=0,\,y=0} \in \mathcal{D}(D)$ *and* $-\partial_y f|_{t=0,\,y=0} + Df|_{t=0,\,y=0} = \phi|_{t=0}$, *if* $\alpha > \frac{2}{p-1}$;

(v) $\quad \partial_t f|_{t=0} \in B^{2(1-\frac{1}{\alpha}-\frac{1}{p\alpha})}_{pp}(\mathbb{R}_+;X) \cap L_p(\mathbb{R}_+; D_A(1-\frac{1}{\alpha}-\frac{1}{p\alpha}, p))$, *if* $\alpha > 1+\frac{1}{p}$.

If this is the case, then additionally

(vi) $\quad f|_{t=0,\,y=0} \in D_A(1-\frac{1}{2p}-\frac{1}{p\alpha}, p)$, *if* $\alpha > \frac{2}{2p-1}$;

(vii) $\quad Df|_{t=0,\,y=0},\ \phi|_{t=0} \in D_A(\frac{1}{2}-\frac{1}{2p}-\frac{1}{p\alpha}, p)$, *if* $\alpha > \frac{2}{p-1}$.

This result can be proved similarly as Theorem 5.1, the key tool being Theorem 4.3 to determine the regularity of ϕ. We remark that the assumption $D_{A^{1/2}} \hookrightarrow D_D$ ensures that any function $v \in Z$ automatically satisfies $v|_{y=0} \in L_p(J; D_D)$ and $Dv(\cdot,0) \in B^{\alpha(\frac{1}{2}-\frac{1}{2p})}_{pp}(J;X) \cap L_p(J; D_A(\frac{1}{2}-\frac{1}{2p}, p))$. For more details, see [24, Thm. 3.5.2].

Remarks 5.1 (i) Theorem 5.1 and Theorem 5.2 are also true, if the kernel a is of the form $a = b + dk * b$, where b is like a above and $k \in BV_{loc}(\mathbb{R}_+)$ with $k(0) = k(0+) = 0$.

(ii) In the case $a \equiv 1$, Theorem 5.1 is equivalent with a version of [16, Thm. 5] for compact J, while Theorem 5.2 can be regarded as an extension of [16, Thm. 6], at least in the case $D_{A^{1/2}} \hookrightarrow D_D$.

To conclude with some applications of Theorem 5.1 and Theorem 5.2, let $X = L_p(\mathbb{R}^n)$, $1 < p < \infty$, A be the negative Laplacian on $\mathbb{R}^n$, and $D = 0$. Then (21) becomes the heat equation of memory type on $\mathbb{R}^{n+1}_+$ with Dirichlet condition while (30) is the corresponding problem with Neumann condition, cp. the remarks on (2). For D we may also choose an oblique derivative operator of the form $Du = b \cdot \nabla u + cu$, where $b \in \mathbb{R}^n$ and $c \geq 0$. It is even possible to take $Du = -\Delta u + b \cdot \nabla u + cu$.

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