Finite Automata Encoding Geometric Figures*

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Abstract

Finite automata are used for the encoding and compression of images. For black-and-white images, for instance, using the quad-tree representation, the black points correspond to ω-words defining the corresponding paths in the tree that lead to them. If the ω-language consisting of the set of all these words is accepted by a deterministic

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finite automaton then the image is said to be encodable as a finite automaton. For grey-level images and colour images similar representations by automata are in use.

In this paper we address the question of which images can be encoded as finite automata with full infinite precision. In applications, of course, the image would be given and rendered at some finite resolution – this amounts to considering a set of finite prefixes of the ω-language – and the features in the image would be approximations of the features in the infinite precision rendering.

We focus on the case of black-and-white images – geometrical figures, to be precise – but treat this case in a d-dimensional setting, where d is any positive integer. We show that among all polygons and convex polyhedra in d-dimensional space exactly those with rational corner points are encodable as finite automata.

In the course of proving this we show that the set of images encodable as finite automata is closed under rational affine transformations.

Several properties of images encodable as finite automata are consequences of this result. Finally we show that many simple geometric figures such as circles and parabolas are not encodable as finite automata.

Contents

1 Introduction

Finite automata are widely used as a means for describing certain fractals (see [BM, CD, MS]). Usually, the investigation of automaton-generated fractals starts from the underlying automaton and aims at a description of the image or the calculation of some of its parameters like density, dimension or measure (see [JS1]). Less is known about the converse direction, that is, starting from a class of images to ask whether they are generated by automata or, if so, to describe these automata. Some structural properties of images generated by finite automata can be derived from the structure of the ω-languages accepted by the automata. Finite-automaton generated images turn out to have specific shapes (see e. g. [BM, MS]).

In this paper we focus on d-dimensional black-and-white images. Using their representation as infinite trees with a branching of up to 2^{d} – in the case of $d = 2$ these are quad-trees – the black points correspond to the infinite branches in these trees. Hence an image would be represented by the ω-language describing these branches. An image is *encodable as (or definable by) a finite automaton* if its ω-language is accepted by such an automaton, that is, if that ω-language is regular (see [S2]). The cases of grey-level or colour images require additional parameters.

The encoding of an image as an automaton represents the image at infinite resolution. Sampling or rendering the image at a bounded resolution corresponds to running the automaton for a bounded time only. These connections are exploited, for example, in an automaton-based image compression procedure (see [CK] or [E]).

In this paper we address the question of which images are encodable as finite automata. In particular, we consider polygons and convex polyhedra in d-dimensional Euclidean space, that is, convex hulls of finite sets of points.

Our main results are that a d-dimensional convex polyhedron is definable by a finite automaton if and only if it is the convex hull of a finite set of points with rational coordinates, and that a polygon is definable by a finite automaton if and only if its corner points are rational. This result is independent of the base chosen for the number representation. The set of images definable by finite automata being closed under union, projection, inverse projection and, essentially, also difference,[1] the class of geometrical figures definable by finite automata turns out to be quite rich.

One of the main tools for proving these results is the following property of images encodable as finite automata: The set of these images is closed under rational affine transformations, that is, transformations of the form $\mathfrak{y} = A\mathfrak{x} + \mathfrak{b}$ with only rational numbers as entries of the transformation matrix A and the translation vector $\mathfrak{b}$.

From closure properties of the set of regular ω-languages and these results, one can determine further interesting classes of simple geometrical figures encodable as finite automata. On the other hand, some very simple geometrical figures like circles or parabolas cannot be encoded as finite automata. For image compression by automata this implies that such figures will, of necessity, be approximated by convex polyhedra sampled at some bounded resolution.

2 Notation

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the sets of non-negative integers, integers, rational and real numbers, respectively. An alphabet is a finite and non-empty set. For an alphabet X, X^* and X^ω denote the sets of finite and right-infinite words over X, respectively. For a word $w \in X^*$, $|w|$ is its length. Right-infinite words are referred to as ω-words in the sequel. An ω-language is a set of ω-words. The fact that $w \in X^*$ is a prefix of $\eta \in X^* \cup X^\omega$ is denoted by $w \sqsubseteq \eta$.

For any alphabet Y and any positive integer d, let $[Y, d\,]$ denote the d-fold Cartesian product

$$[Y, d\,] = \underbrace{Y \times \cdots \times Y}_{d \text{ times}}.$$

For $y = (y_1, \ldots, y_d) \in [Y, d\,]$ and an integer i with $1 \le i \le d$, the i-th projection of y is $\mathrm{proj}_i\, y = y_i$.

For the representation of real numbers, we fix a base $r \in \mathbb{N}$ with $r \ge 2$. Then the set $Y = \{0, 1, \ldots, r-1\}$ is considered as the set of r-ary number symbols. Every real number in the closed interval $[0, 1] = \{p \mid 0 \le p \le 1\}$

[1] We consider figures that are bounded and closed in Euclidean space. Therefore, difference here means the closure of the set theoretical difference.

has a base-r representation of the form $0.\eta$ where $\eta \in Y^\omega$. In particular, a finite representation of a rational number can be padded by an infinite sequence of the symbol 0. Conversely, every ω-word η over Y denotes a unique real number $\nu_r(\eta)$ in the interval $[0,1]$, represented by $0.\eta$. It is well-known that the mapping from representations of numbers to their values is not injective.

Let d be a positive integer. To specify points in the closed d-dimensional unit cube $[0,1]^d$ we use ω-words over the alphabet $X = [Y, d]$. For $\xi = x_1 x_2 \cdots \in X^\omega$ and an integer i with $1 \le i \le d$, the i-th projection of ξ is the ω-word

$$\mathrm{proj}_i\, \xi := \mathrm{proj}_i\, x_1\, \mathrm{proj}_i\, x_2 \cdots$$

obtained from the i-th projections of the symbols of x. The point $\nu_r(\xi)$ in $[0,1]^d$ defined by ξ has, as coordinates, the values the numbers represented by the projections of ξ, $0.\mathrm{proj}_i\, \xi$.

We generalise this concept of projection to multiple co-ordinates by letting $\mathrm{proj}_{\mathbf{i}}(y_1, \ldots, y_d) := (y_{i_1}, \ldots, y_{i_k})$ for $\mathbf{i} = (i_1, \ldots, i_k)$ and $i_\ell \in \{1, \ldots, d\}$. Then $\mathrm{proj}_{\mathbf{i}}\, \xi \in [Y, k]^\omega$ and its value $\nu_r(\mathrm{proj}_{\mathbf{i}}\, \xi)$ is a point in the cube $[0,1]^k$. Observe that $\mathbf{i} = (i_1, \ldots, i_k)$ may contain the same co-ordinate several times, that is, $i_l = i_m$ for some $l, m \in \{1, \ldots, d\}$; in particular, it may happen that $(i_1, \ldots, i_d)$ is a permutation of $(1, \ldots, d)$, or $k > d$.

The following diagram is commutative.

$$\begin{array}{ccc} [Y, d]^\omega & \xrightarrow{\mathrm{proj}_{\mathbf{i}}} & [Y, k]^\omega \\ \downarrow \nu_r & & \downarrow \nu_r \\ [0,1]^d & \xrightarrow{\mathrm{proj}_{\mathbf{i}}} & [0,1]^k \end{array}$$

By slight abuse of notation we write $(\xi_1, \ldots, \xi_d)$ to denote the element $\xi \in [Y, d]^\omega$ which has projections $\mathrm{proj}_i\, \xi = \xi_i,\ i = 1, \ldots, d$.

On X^ω one defines an ultra-metric ρ by

$$\rho(\zeta, \xi) = \inf\{r^{-|w|} \mid w \text{ is a common prefix of } \zeta \text{ and } \xi\}.$$

Since X is finite, the space (X^ω, ρ) is a compact metric space. Moreover, the mapping ν_r of X^ω onto $[0,1]^d$ is continuous.

If we denote by $\mathcal{C}(F)$ the smallest closed subset of $[Y, d]^\omega$ containing $F \subseteq [Y, d]^\omega$, and likewise by $\mathrm{cl}(M)$ the smallest closed subset of $[0,1]^d$ containing $M \subseteq [0,1]^d$ then we have the identity

$$\nu_r(\mathcal{C}(F)) = \mathrm{cl}(\nu_r(F))\,. \tag{1}$$

For more detailed properties of ν_r see [S3].

3 Regular ω-languages

In this section we consider the sets of infinite words defined by finite automata, the so-called *regular* ω-languages. For the purpose of our paper, it is convenient to introduce regular ω-languages in the following way.

First we introduce ω-languages definable in a simple way by finite automata: Let $\mathfrak{A} = (X, S, s_0, \delta)$ be a finite automaton with input alphabet X, set of states S, initial state s_0 and transition function $\delta : S \times X \to S \cup \{\perp\}$ where $\delta(s, x) = \perp$ means that $\delta(x, s)$ is undefined. We extend δ in the usual way to a function mapping $S \times X^*$ to $S \cup \{\perp\}$ with $\delta(s, w) = \perp$ if and only if $\delta(s, w') = \perp$ for some prefix $w' \sqsubseteq w$.

We say that $\xi \in \mathcal{T}_\omega(\mathfrak{A})$ provided $\delta(s_0, w) \neq \perp$ for all $w \sqsubset \xi$, and we call $\mathcal{T}_\omega(\mathfrak{A})$ *the ω-language defined by the finite automaton* $\mathfrak{A}$. In other words, $\mathcal{T}_\omega(\mathfrak{A})$ is the set of all infinite words on which the automaton $\mathfrak{A}$ does not get stuck.

The subsequent definition of regular ω-languages follows the line of [TB, Chapter III, §6].

An ω-language $F \subseteq [Y, d]^\omega$ is referred to as *regular* provided it can be obtained from ω-languages $\mathcal{T}_\omega(\mathfrak{A}_i)$ contained in possibly different spaces $[Y, d_i]^\omega$, $i = 1, \ldots, m$, by applications of union[2], intersection, set-theoretical difference and projections proj_i and their inverse mappings proj_i^{-1}.

Then it holds the following (see [TB]).

Lemma 1 *The family of all regular ω-languages $F \subseteq [Y, d]^\omega$ is closed under Boolean operations.*

The next lemma is an easy consequence of the definition.

Lemma 2 *For $F \subseteq [Y, d]^\omega$ and $E \subseteq [Y, k]^\omega$ and $i = (i_1, \ldots, i_k)$, $i_1, \ldots, i_k \in \{1, \ldots, d\}$ the ω-languages $\mathrm{proj}_i F$ and $\mathrm{proj}_i^{-1} E$ are regular ω-languages provided F and E are regular.*

For ω-languages being closed in the topology defined by ρ we have the following (see [S2, TB]).

Proposition 3 *Let $F \subseteq [Y, d]^\omega$ be regular. Then the closure of F, $\mathcal{C}(F)$, is also regular.*

[2]Here we need to consider only unions of subsets of the same space $[Y, k]^\omega$.

Theorem 4 *An ω-language $F \subseteq [Y, d]^\omega$ is closed and regular if and only if there is a finite automaton $\mathfrak{A}$ such that $F = \mathcal{T}_\omega(\mathfrak{A})$.*

We mention still the following property of regular ω-languages.

Lemma 5 *Every nonempty regular ω-language $F \subseteq X^\omega$ contains an ultimately periodic ω-word, that is, an ω-word of the form $v \cdot u^\omega$ where $v, u \in X^*$, and every at most countable regular ω-language $F \subseteq X^\omega$ consists entirely of ultimately periodic ω-words.*

4 Rational Affine Transformations

Simple geometric figures are (convex) polyhedra. In this section we derive basic tools for the investigation of polyhedra encodable as finite automata.

Recall that an affine transformation of $\mathbb{R}^d$ into $\mathbb{R}^k$ is given by an equation of the form $\mathfrak{y} = A\mathfrak{x} + \mathfrak{b}$ where $\mathfrak{y}$ and $\mathfrak{b}$ are $k \times 1$-vectors, $\mathfrak{x}$ is a $d \times 1$-vector and A is a $k \times d$-matrix. An affine transformation is said to be *rational* if the entries of A and $\mathfrak{b}$ are rational. Likewise, a system of linear inequalities $\mathfrak{b}_1 \leq A\mathfrak{x} \leq \mathfrak{b}_2$ is called rational, if the entries of $A, \mathfrak{b}_1$ and $\mathfrak{b}_2$ are rational. Here for column vectors $\mathfrak{b}_1$ and $\mathfrak{b}_2$ we say that $\mathfrak{b}_1 \leq \mathfrak{b}_2$ if each component of $\mathfrak{b}_1$ is less than or equal to the corresponding component of $\mathfrak{b}_2$.

The following theorem plays a fundamental rôle:

Theorem 6 *Let A be a rational $k \times d$-matrix, and let $\mathfrak{b}_1$ and $\mathfrak{b}_2$ be rational $k \times 1$-vectors. Then*

$$F := \{(\xi_1, \ldots, \xi_d) \mid \mathfrak{b}_1 \leq A \cdot (\nu_r(\xi_1), \ldots, \nu_r(\xi_d))^\top \leq \mathfrak{b}_2\}$$

is a regular and closed ω-language.

Here $\mathfrak{x}^\top$ denotes the transpose of the row vector $\mathfrak{x}$.

Lemmata 1 and 2 show that it suffices to prove the theorem for the case of a single inequality. In the following lemma, we split, for technical reasons, the coefficients into positive and negative ones, arranging them in a convenient order.

Lemma 7 *Let $d, d' \in \mathbb{N}$, $c_i, c'_{i'}, c \in \mathbb{Q}$, $c_i, c'_{i'} \geq 0$ for $i = 1, \ldots, d$ and $i' = 1, \ldots, d'$ and let*

$$F := \{(\xi_1, \ldots, \xi_d, \xi'_1, \ldots, \xi'_{d'}) \mid \textstyle\sum_{i=1}^{d} c_i \cdot \nu_r(\xi_i) - \sum_{i'=1}^{d'} c'_{i'} \cdot \nu_r(\xi'_{i'}) \leq c\}.$$

Then F is a regular and closed ω-language.

While the addition of real numbers cannot be carried out by a finite automaton as the carries can travel unbounded distances, the proof of Lemma 7 exploits the fact that the correctness of an addition of real numbers can be checked by a finite automaton. For the purpose of the lemma this is sufficient as the elements of F are to be recognised rather than computed.

In the proof we give a construction of a finite automaton accepting F. Here we make explicit the somewhat sketchy proofs of related facts as given in [BRW, Section 4.3] and our preceding paper [JS2, Section 3]. Before proceeding with the construction we need some preparatory considerations on truncations of expansions of real numbers.

First, observe that $\nu_r(\xi) = \sum_{l=1}^{\infty} x_l \cdot r^{-l}$ for $\xi = x_1 x_2 \cdots x_j \cdots \in Y^\omega$. From this identity we have the following.

$$\nu_r(x_1 \cdots x_j) \;=\; r^{-j} \cdot \textstyle\sum_{l=1}^{j} x_l \cdot r^{j-l} = r^{-j} \cdot \lfloor \nu_r(\xi) \cdot r^j \rfloor \tag{2}$$

Let $p \in [0,1]$. Then $p = \nu_r(\xi)$ for some $\xi \in Y^\omega$ and the following facts are consequences of Eq. (2).

$$0 \;\le\; p \cdot r^j - \lfloor p \cdot r^j \rfloor \;<\; 1 \tag{3}$$

$$\lim_{j\to\infty} r^{-j} \cdot \lfloor p \cdot r^j \rfloor \;=\; p \tag{4}$$

$$0 \;\le\; \lfloor p \cdot r^{j+1} \rfloor - r \cdot \lfloor p \cdot r^j \rfloor \;\le\; r - 1 \tag{5}$$

Next, for the linear inequality $\sum_{i=1}^{d} c_i \cdot p_i - \sum_{i=1}^{d'} c'_{i'} \cdot p'_{i'} \le c$ where $c_i, c'_{i'}, c \in \mathbb{R}$, $c_i, c'_{i'} \ge 0$ and $p_i, p'_{i'} \in [0,1]$ for $1 \le i \le d$ and $1 \le i' \le d'$ we consider j-*approximations*

$$\Delta_j := \begin{cases} 0, & \text{if } j = 0, \text{ and} \\ \sum_{i=1}^{d} c_i \cdot \lfloor p_i \cdot r^j \rfloor - \sum_{i'=1}^{d'} c'_{i'} \cdot \lfloor p'_{i'} \cdot r^j \rfloor, & \text{for } j \ge 1\,. \end{cases} \tag{6}$$

These j-approximations yield integers if the coefficients $c_i, c'_{i'}$ have integer values. Thus they appear to be convenient for the construction of finite automata. We have the following connection between j-approximations and linear combinations $\sum_{i=1}^{d} c_i \cdot p_i - \sum_{i'=1}^{d'} c'_{i'} \cdot p'_{i'}$.

Lemma 8 *Let* $c_i, c'_{i'}, c \in \mathbb{R}$, $c_i, c'_{i'} \ge 0$ *and* $p_i, p'_{i'} \in [0,1]$ *for* $1 \le i \le d$ *and* $1 \le i' \le d'$. *Then*

$$\sum_{i=1}^{d} c_i \cdot p_i - \sum_{i'=1}^{d'} c'_{i'} \cdot p'_{i'} \le c \quad \textit{if and only if} \quad \forall j \Big(j \in \mathbb{N} \to \Delta_j \le c \cdot r^j + \sum_{i'=1}^{d'} c'_{i'} \Big)\,.$$

Proof. Eq. (4) shows the implication from right to left.

To prove the converse, we observe that in view of $\lfloor p'_{i'} \cdot r^j \rfloor \geq p'_{i'} \cdot r^j - 1$ we have $r^{-j} \cdot \Delta_j \leq \sum_{i=1}^{d} c_i \cdot p_i - \sum_{i'=1}^{d'} (c'_{i'} \cdot p'_{i'} - r^{-j})$, whence the assertion follows for $j \geq 1$. In view of $0 \leq p_i, p'_i \leq 1$ the case of $j = 0$ is obvious. ❑

The next lemma deals with the case, when the linear combination $\sum_{i=1}^{d} c_i \cdot p_i - \sum_{i'=1}^{d'} c'_{i'} \cdot p'_{i'}$ is much smaller than the bound c.

Lemma 9 *If* $j \geq 1$ *and* $\Delta_j \leq c \cdot r^j - \sum_{i=1}^{d} c_i$ *then* $\Delta_l \leq c \cdot r^l - \sum_{i=1}^{d} c_i$ *for all* $l \geq j$.

Proof. Using Eq. (5) we obtain $\Delta_{j+1} \leq r \cdot \Delta_j + (r-1) \cdot \sum_{i=1}^{d} c_i$ for $j \geq 1$. The proof proceeds by induction on l:
If $\Delta_l \leq c \cdot r^l - \sum_{i=1}^{d} c_i$ then $\Delta_{l+1} \leq r \cdot \Delta_l + (r-1) \cdot \sum_{i=1}^{d} c_i \leq c \cdot r^{l+1} - r \cdot \sum_{i=1}^{d} c_i + (r-1) \sum_{i=1}^{d} c_i = c \cdot r^{l+1} - \sum_{i=1}^{d} c_i$. ❑

Now we give the construction as announced.

Proof of Lemma 7. It suffices to prove the lemma for integers $c_i, c'_{i'} \in \mathbb{N}$ and $c \in \mathbb{Z}$. Consider an input $(\xi_1, \ldots, \xi_d, \xi'_1, \ldots, \xi'_{d'})$ and define $\Delta_j := \sum_{i=1}^{d} c_i \cdot \lfloor \nu_r(\xi_i) \cdot r^j \rfloor - \sum_{i'=1}^{d'} c'_{i'} \cdot \lfloor \nu_r(\xi'_{i'}) \cdot r^j \rfloor$.

Our automaton $\mathcal{A}$ successively checks whether

$$-\sum_{i=1}^{d} c_i < \Delta_j - c \cdot r^j \leq \sum_{i'=1}^{d'} c'_{i'} \text{ for all } j \geq 1 .$$

As soon as the left hand side inequality is violated, according to Lemmata 8 and 9, the automaton accepts the input without considering further input letters.

If the right hand side inequality is violated, according to Lemma 8 the input is rejected at once.

In the remaining case, the inequality to be checked is satisfied for all $j \geq 1$, hence the input is accepted.

The states of our automaton $\mathcal{A} = (X, S, s_0, \delta)$ are the initial state s_0 and integers $\{-\sum_{i=1}^{d} c_i, \ldots, \sum_{i'=1}^{d'} c'_{i'}\}$.

As explained above, an input ω-word $\xi = (\xi_1, \ldots, \xi_d, \xi'_1, \ldots, \xi'_{d'})$ is accepted if and only if the automaton does not get stuck.

The transition function is defined according to the equation

$$\Delta_{j+1} - c \cdot r^{j+1} = r(\Delta_j - c \cdot r^j) + \sum_{i=1}^{d} c_i \cdot x_{i,j+1} - \sum_{i'=1}^{d'} c'_{i'} \cdot x'_{i',j+1} \qquad (7)$$

where $\xi_i = x_{i,1} \cdots x_{i,j} \cdots$ and $\xi'_{i'} = x'_{i',1} \cdots x'_{i',j} \cdots$.

$$\text{Define } f(m,\vec{y}) := \begin{cases} \sum_{i=1}^{d} c_i y_i - \sum_{i'=1}^{d'} c'_{i'} y'_{i'} - c \cdot r, & \text{if } \ m = s_0 \\ & \text{and} \\ r(m - cr^j) + \sum_{i=1}^{d} c_i y_i - \sum_{i'=1}^{d'} c'_{i'} y_{i'}, & \text{if } \ m \in \mathbb{Z} \end{cases}$$

for $m \in \mathbb{Z} \cup \{s_0\}$ and $\vec{y} := (y_1, \ldots, y_d, y'_1, \ldots, y'_{d'}) \in X$.

Consequently, $\vec{y} = (x_{1,j}, \ldots, x_{d,j}, x'_{1,j}, \ldots, x'_{d',j})$ and $m = \Delta_{j-1} - c \cdot r^{j-1}$ imply $f(m,\vec{y}) = \Delta_j - c \cdot r^j$ in case $j > 1$ and $f(m,\vec{y}) = \Delta_1 - c \cdot r$ when $m = s_0$ and $j = 1$.

Thus the following definition of the transition function satisfies the above announced behaviour.

$$\delta(m,\vec{y}) \ := \ \begin{cases} f(m,\vec{y}), & \text{if } \ m \in S \setminus \{-\sum_{i=1}^{d} c_i\} \wedge \\ & \qquad -\sum_{i=1}^{d} c_i < f(m,\vec{y}) \le \sum_{i'=1}^{d'} c'_{i'}, \text{ and} \\ -\sum_{i=1}^{d} c_i, & \text{if } \ m = -\sum_{i=1}^{d} c_i \vee f(m,\vec{y}) \le -\sum_{i=1}^{d} c_i \,. \end{cases}$$

Observe that $\mathcal{A}$ gets stuck if and only if $f(m,\vec{y}) > \sum_{i=1}^{d'} c'_i$. ❑

We list a few immediate consequences. As is well-known, every rational number of the form k/r^l has two base-r representations. Thus a point in d-dimensional space $\mathbb{R}^d$ may have up to 2^d representations. A typical complication arises from the fact that, due to those multiple representations, for $F, F' \subseteq X^\omega$, the sets $\nu_r(F) \cap \nu_r(F')$ and $\nu_r(F \cap F')$ might not be equal. For example, with $d = 1$, $r = 2$, $F = \{1000\cdots\}$ and $F' = \{01111\cdots\}$ one has $\nu_r(F) = \nu_r(F') = \{\frac{1}{2}\}$ whereas $\nu_r(F \cap F') = \emptyset$. However, for any $F, F' \subseteq X^\omega$, one has

$$\nu_r(F) \cap \nu_r(F') = \nu_r\left(\nu_r^{-1}(\nu_r(F)) \cap F'\right) \,. \tag{8}$$

One is, therefore, led to work with *full representations*, that is, with ω-languages F satisfying $F = \nu_r^{-1}(\nu_r(F))$. Fortunately, the ω-languages F defined in Theorem 6 have already, by definition, full representation.

As a consequence, moving from a regular representation to the corresponding full representation preserves regularity.

Proposition 10 *Let F be an ω-language over $X = [Y, d]$. If F is regular then also $\nu_r^{-1}(\nu_r(F))$ is regular.*

Proof. Using Theorem 6 it is easy to see that the set

$$E_{=}^{(2d)} := \{(\xi_1, \ldots, \xi_d, \xi'_1, \ldots, \xi'_d) \mid \nu_r(\xi_i) = \nu_r(\xi'_i) \text{ for } i = 1, \ldots, d\}$$

is regular and closed. Then the assertion follows from

$\nu_r^{-1}(\nu_r(F)) = \mathrm{proj}_{(d+1,\ldots,2d)}(\mathrm{proj}_{(1,\ldots,d)}^{-1} F \cap E_{=}^{(2d)})$. ❑

In connection with Proposition 10 it should be mentioned that the result of the general base transformation, $\nu_r^{-1}(\nu_b(F))$, when $r \neq b$, need not be regular if F is regular. For more detailed information see [S3, Section 5.2].

Similarly to Proposition 10 one derives the following.

Theorem 11 *Let $\Psi : \mathbb{R}^d \to \mathbb{R}^k$ be a rational affine transformation and let $\Gamma(\Psi) \subseteq \mathbb{R}^{d+k}$ be its graph. Then the ω-language $F_\Psi := \nu_r^{-1}(\Gamma(\Psi) \cap [0,1]^{d+k})$ is regular.*

Proof. One easily verifies that for $\Psi(\mathfrak{x}) = A \cdot \mathfrak{x} + \mathfrak{b}$ we have

$$F_\Psi = \{(\xi_1, \ldots, \xi_d, \xi_1', \ldots, \xi_k') \mid A \cdot (\nu_r(\xi_1), \ldots, \nu_r(\xi_d))^\top + \mathfrak{b} = (\nu_r(\xi_1'), \ldots, \nu_r(\xi_k'))^\top\}$$

which is definable by a finite automaton by virtue of Theorem 6. ❑

From Theorem 11 one can conclude that affine transformations and their inverses preserve regularity.

Theorem 12 *Let $\Psi : \mathbb{R}^d \to \mathbb{R}^k$ and $\Phi : \mathbb{R}^k \to \mathbb{R}^d$ be rational affine transformations and let $F \subseteq X^\omega$ be regular.*

Then both $\nu_r^{-1}(\Psi(\nu_r(F)))$ and $\nu_r^{-1}\left(\Phi^{-1}(\nu_r(F))\right)$ are regular ω-languages.

Proof. We consider the space $[Y, d+k]^\omega$ and the projections $\mathrm{proj}_{(1,\ldots,d)}$ and $\mathrm{proj}_{(k+1,\ldots,k+d)}$. Then $\nu_r^{-1}(\Psi(\nu_r(F))) = \nu_r^{-1}\left(\nu_r(F_\Psi \cap \mathrm{proj}_{(1,\ldots,d)}^{-1} F)\right)$, and $\nu_r^{-1}\left(\Phi^{-1}(\nu_r(F))\right) = \nu_r^{-1}\left(\nu_r(F_\Phi \cap \mathrm{proj}_{(k+1,\ldots,k+d)}^{-1} F)\right)$. ❑

5 Simple Geometric Figures

In this section we are going to investigate subsets of the d-dimensional unit cube which can be represented by a finite automaton. We say that $M \subseteq [0,1]^d$ is r-*encodable* as a finite automaton provided there is a regular ω-language $F \subseteq [\{0, \ldots, r-1\}, d]^\omega$ such that $M = \nu_r(F)$.

In view of Proposition 10 this is equivalent to the condition that $\nu_r^{-1}(M)$ is a regular ω-language. As it was mentioned above whether $\nu_r^{-1}(M)$ is regular or not may depend on the choice of the base r. Therefore, we say that M is *encodable* as a finite automaton provided there is an $r \in \mathbb{N}$ such that M is r-encodable as a finite automaton.

5.1 Closure Properties

Next we investigate operations under which the class of sets r-encodable as finite automata is closed. The results obtained are closely related to the closure properties if the set of regular ω-languages as presented in Section 3. A first closure property concerns arbitrary rational affine mappings and was presented in Theorem 12. First we deal with Boolean operations.

Lemma 13 *The set of all images $M \subseteq [0,1]^d$ r-encodable as finite automata is closed under Boolean operations.*

Proof. Closure under union is obvious. Let $M = \nu_r(F)$ where F is a regular ω-language. Then, using Proposition 10 and Lemma 1, the complement $[0,1]^d \setminus M = \nu_r(X^\omega \setminus \nu_r^{-1}(\nu_r(F)))$ is also encodable as a finite automaton. ❑

Since the closure and the boundary of regular ω-languages are again regular, we obtain the following closure property of the family of images definable by finite automata.

Proposition 14 *Let $M \subseteq [0,1]^d$ r-encodable as a finite automaton. Then both the closure of M, $\mathrm{cl}(M)$, and the boundary of M, ∂M, are encodable as finite automata.*

Proof. Assume $M = \nu_r(F)$ for some regular $F \subseteq X^\omega$. Then in view of the identity $\mathrm{cl}(\nu_r(F)) = \nu_r(\mathcal{C}(F))$ (cf. [S3, Section 4]), we have $\mathrm{cl}(M) = \nu_r(\mathcal{C}(F))$ which is, in view of Theorem 4, encodable as a finite automaton.

The boundary of M, ∂M, is defined as $\mathrm{cl}(M) \cap \mathrm{cl}([0,1]^d \setminus M)$. Thus the assertion follows from the first part and Lemma 13. ❑

5.2 Convex Polyhedra

A point in $[0,1]^d$ is said to be rational if all its coordinates are rational. As we shall see in the subsequent section rational points play a crucial rôle for images encodable by finite automata. A convex polyhedron in $[0,1]^d$ is the convex hull of a finite set of points in $[0,1]^d$. A convex polyhedron in

$[0,1]^d$ is said to be *rational* if it is the convex hull of finitely many rational points.

Using Theorem 12, we show that rational convex polyhedra are encodable as finite automata. To this end observe that a convex polyhedron having d corner points $\mathfrak{p}_0, \ldots, \mathfrak{p}_{d-1}$ is the image of the $(d-1)$-dimensional simplex spanned by the all zero vector $\mathfrak{o}$ and the $d-1$ unit vectors $\mathfrak{e}_i$, $i = 1, \ldots, d-1$ under the affine mapping $\Psi(\mathfrak{x}) = A\mathfrak{x} + \mathfrak{p}_0$ where A is the matrix whose columns are the vectors $\mathfrak{p}_i - \mathfrak{p}_0$, $i = 1, \ldots, d-1$. Having all vectors $\mathfrak{p}_i$ as rational points, the affine transformation Ψ is rational, too. Thus Theorem 12 yields the following.

Lemma 15 *Every rational convex polyhedron* $M \subseteq [0,1]^k$ *is* r*-encodable as a finite automaton, for arbitrary* $r \in \mathbb{N}$, $r \geq 2$.

From rational convex polyhedra we obtain, via Boolean and topological operations, new simple geometric figures which are encodable as finite automata.

6 Images that Are not Encodable as Finite Automata

There are many images that are not encodable as finite automata. Proposition 16 states a necessary condition for an image to be encodable as a finite automaton. Here we state and apply other necessary conditions.

6.1 Rational Points

We derive some criteria for the automaton encodability of sets based on the correspondence between rational numbers and ultimately periodic ω-words. To this end we translate the fact about nonempty regular ω-languages stated in Lemma 5.

Proposition 16 *Let* $M \subseteq [0,1]^d$ *be encodable as a finite automaton. If* M *is non-empty then it contains a rational point. If* M *is at most countable, then all points in* M *are rational.*

As an immediate consequence we obtain a proposition about the rationality of isolated points.

Corollary 17 *If $M \subseteq [0,1]^d$ is encodable as a finite automaton then every isolated point of M is rational.*

We get a result on the endpoints of intervals in the line.

Lemma 18 *Let $M = \bigcup_{i \in I}(a_i, b_i) \subseteq [0,1]$ be an at most countable union of pairwise disjoint open intervals and let M be encodable as a finite automaton. Then $a_i, b_i \in \mathbb{Q}$ for all $i \in I$.*

Proof. According to Proposition 14 the boundary of M, ∂M, is also encodable as a finite automaton. Since the intervals are mutually disjoint, we have that $\partial M = \{a_i, b_i \mid i \in I\}$ is an at most countable set of points, and the assertion follows from Proposition 16. ❏

A remark is in order here. Although we formulated Lemma 18 only for open intervals (a_i, b_i) the proof remains valid also in case of closed mutually disjoint intervals $[a_i, b_i]$ and, if $a_i \neq b_j$ holds for $i, j \in I$, then also in case of semi-closed mutually disjoint intervals $[a_i, b_i)$ or $(a_i, b_i]$.

Lemma 19 *Let $M \subseteq [0,1]$ be encodable as a finite automaton. Then* $\inf M$ *and* $\sup M$ *are rational numbers.*

Proof. As $\inf M = \min\ \mathrm{cl}(M)$ and $\sup M = \max\ \mathrm{cl}(M)$, in view of Proposition 14 it suffices to prove the assertion for closed subsets of $M \subseteq [0,1]$.

The case $\inf M = 0$ is trivial. Let $0 < \inf M$. Since M is closed, the difference $(0,1) \setminus M$ is an at most countable union of pairwise disjoint open intervals, and one of these intervals is $(0, \inf M)$. The assertion follows from Lemma 18.

The proof for $\sup M$ is similar. ❏

Next we deal with smooth non-constant curves. These are particularly interesting examples of the application of the intersection-and-isolated-points method described below.

Example 1 The graph of the parabola $f(a) = a^2$ with $0 \leq a \leq 1$ is not encodable as a finite automaton.

Assume $\Gamma(f)$ to be r-encodable as a finite automaton. Since the line $\ell[y = \frac{1}{2}] := \left\{(x, \frac{1}{2}) \mid 0 \leq x \leq 1\right\}$ is r-encodable as a finite automaton, the intersection $\Gamma(f) \cap \ell[y = \frac{1}{2}] = \left\{(1/\sqrt{2}, 1/2)\right\}$ is also r-encodable as a finite automaton.

This contradicts Proposition 16.

The next example uses also Theorem 12 in order to prove the nonencodability.[3]

Example 2 Consider the hyperbola $g(x) = \frac{1}{1+x}$. Here every point on $\Gamma(g)$ with one rational coordinate is rational. So the simple intersection with a line $\ell[y = \alpha]$, $\alpha \in \mathbb{Q}$ yields only rational points.

Now transform $\Gamma(g)$ via the rational affine mapping defined by $A = \binom{1\,0}{1\,1}$ and $\mathfrak{b} = \binom{0}{-1}$. The image is $\Gamma(g')$ where $g'(x) = \frac{x^2}{1+x}$ for which the intersection $\Gamma(g') \cap \ell[y = \frac{1}{3}]$ contains the single point $(\frac{1+\sqrt{13}}{6}, \frac{1}{3})$ with exactly one rational coordinate.

Alternatively, one could intersect $\Gamma(g)$ with the line $\ell[y = \frac{4}{3} - x]$ and obtain $\Gamma(g) \cap \ell[y = \frac{4}{3} - x] = \left\{(\frac{1+\sqrt{13}}{6}, \frac{7-\sqrt{13}}{36})\right\}$.

6.2 The Zoom-in-Theorem

In this section we use a further property of regular ω-languages to establish necessary conditions for the encodability of images. To this end we introduce the notion of a *state* (or *left derivative*) of an ω-language $F \subseteq X^\omega$ derived by a word $w \in X^*$.

$$F/w := \{\xi \mid w \cdot \xi \in F\} \tag{9}$$

Then we have the following property.

Property 20 *If* $F \subseteq X^\omega$ *is regular then the set of all states* $\{F/w : w \in X^*\}$ *is finite.*

For a more detailed investigation of ω-languages having a finite set of states see [S1]. From Property 20 we immediately obtain the translation to the unit cube.

Lemma 21 *Let* $M \subseteq [0,1]^d$ *be* r*-encodable as a finite automaton then the set* $\{\nu_r(\nu_r^{-1}(M)/w) \mid w \in [\{0,\ldots,r-1\},d]^*\}$ *is finite.*

We call the set $\nu_r(\nu_r^{-1}(M)/w)$ the *zoom-in* of M defined by the word w. This is justified by the following observation.

[3] We are grateful to one of the referees of [JS2] for providing us with this simple instructive example.

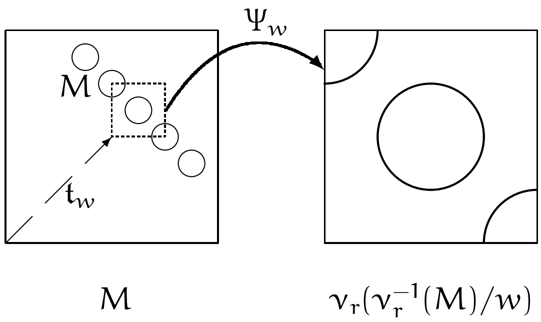

Figure 1: Zoom-in of the set M consisting of five circles

Let $t_w := (\nu_r(\mathrm{proj}_1\, w), \ldots, \nu_r(\mathrm{proj}_d\, w)) \in [0,1]^d$ the transition vector whose coordinates are defined by the coordinate words of w. Then $t_w + [0, r^{-|w|}]^d \subseteq [0,1]^d$ is a sub-cube of edge length $r^{-|w|}$ translated by the vector t_w.

Now the set $\nu_r(\nu_r^{-1}(M)/w)$ is nothing else than the image of the intersection $M \cap (t_w + [0, r^{-|w|}]^d)$ under the rational affine mapping Ψ_w defined by the identity $\Psi_w(\mathfrak{x}) = r^{|w|} \cdot \mathfrak{x} - t_w$. It is obvious that $\Psi_w(t_w + [0, r^{-|w|}]^d) = [0,1]^d$, thus $\nu_r(\nu_r^{-1}(M)/w) = \Psi_w(M \cap (t_w + [0, r^{-|w|}]^d))$ is the $r^{|w|}$-fold magnification of the part of M contained in the sub-cube $t_w + [0, r^{-|w|}]^d$ (see Fig. 1).

Hence the number of different images obtainable as zoom-ins depicted above is finite if only the image itself is encoded as a finite automaton. With every function $h : 2^{[0,1]^d} \to P$, where P is a suitably chosen set, we associate a family of h-*zoom-ins* $(h_w)_{w \in X^*}$ in the following way.

$$h_w(M) := h(\nu_r^{-1}(M)/w) \text{ for } M \subseteq [0,1]^d$$

Theorem 22 *If $M \subseteq [0,1]^d$ is encodable as a finite automaton, P is a set and $h : 2^{[0,1]^d} \to P$ then the family $\{h_w(M) \mid w \in X^*\}$ is finite.*

Proof. If M is r-encodable as a finite automaton then $\nu_r^{-1}(M)$ is a regular ω-language, whence $\{\nu_r^{-1}(M)/w \mid w \in X^*\}$ and also $\{h_w(M) \mid w \in X^*\}$ are finite. ❏

As in the previous part of this section with rational points and intersections, we use the zoom-in theorem to show that certain natural images are not encodable as finite automata by suitably choosing the set P and the function h. Observe that, unlike the intersection case, here one need not prove that $h(M)$ is encodable as a finite automaton. We apply our theorem to polyhedra in $[0,1]^d$ and smooth curves in $[0,1]^2$.

First we obtain a proposition converse, in some sense, to Lemma 15.

Lemma 23 *Let* $r \in \mathbb{N}$, $r \geq 2$. *A finite union of polyhedra* $M_0 \subseteq [0,1]^d$ *is* r*-encodable as a finite automaton only if all its corner points are rational.*

Proof. Assume M_0 to be r-encodable as a finite automaton and to have a corner point $(p_1, \ldots, p_d)$ with an irrational coordinate p_i. Define

$$h(M) := \begin{cases} \{p_j \mid 1 \leq j \leq d \wedge (p_1, \ldots, p_d) \text{ is a corner point of } M\}\ , \\ \qquad \text{if } M \text{ is a finite union of polyhedra} \\ \emptyset\ , \quad \text{otherwise.} \end{cases}$$

If M is a finite union of polyhedra and $w \in X^*$ then $M \cap (t_w + [0, r^{-|w|}]^d)$ and hence also every zoom-in $\nu_r(\nu_r^{-1}(M)/w) = \Psi_w(M \cap (t_w + [0, r^{-|w|}]^d))$ is a finite union of polyhedra. Since M_0 is r-encodable as a finite automaton, $\{h_w(M_0) \mid w \in X^*\}$ is a finite family of finite sets, and, consequently, $\bigcup_{w \in X^*} h_w(M_0)$ is a finite set.

As $(p_1, \ldots, p_d)$ has the irrational coordinate p_i, $(p_1, \ldots, p_d) = \nu_r(\xi)$ where $\xi \in X^\omega$ is not ultimately periodic.

Consider the sets $h_w(M_0)$ for $w \sqsubset \xi$. Among the corner points of $h_w(M_0)$ we have the points $\nu_r(\xi/w)$ with the coordinates $\nu_r(\mathrm{proj}_j\, \xi/w)$, $1 \leq j \leq d$.

As p_i is irrational, $\mathrm{proj}_i\, \xi/w$ is not ultimately periodic, and, consequently, the numbers $\nu_r(\mathrm{proj}_i\, \xi/w)$, $w \sqsubset \xi$ are pairwise different. Thus $\bigcup_{w \sqsubset \xi} h_w(M_0) \supseteq \{\nu_r(\mathrm{proj}_i\, \xi/w) \mid w \sqsubset \xi\}$ is an infinite set, contradicting the fact that $\bigcup_{w \in X^*} h_w(M_0)$ is finite. ❑

Combining Lemmata 15 and 23, the following characterisation is obtained.

Theorem 24 *A convex polyhedron* $M_0 \subseteq [0,1]^d$ *is* r*-encodable as a finite automaton if and only if all its corner points are rational.*

The next lemma deals with graphs of differentiable functions.

Lemma 25 *Let* $f : [0,1] \to [0,1]$ *be a continuous function differentiable at a point* $a_0 \in [0,1]$ *for which* $f'(a_0)$ *is irrational. Then the graph* $\Gamma(f)$ *is not encodable as a finite automaton.*

Proof. Assume the graph of f, $\Gamma(f)$, be encodable as a finite automaton over the alphabet $X = [Y, 2]$ where $Y = \{0, 1, \ldots, r-1\}$.

Without loss of generality we may assume that $f(a_0)$ is not of the form $m \cdot r^{-k}$. Otherwise consider the function $\bar{f}(a) := \frac{1}{2} \cdot f(a) + \frac{1}{r+1}$ whose graph is the image of $\Gamma(f)$ under a suitable rational affine transformation (a vertical shrinking and a subsequent vertical shift) to $\Gamma(f)$. The value of $\bar{f}'(a_0) = \frac{1}{2} \cdot f'(a_0)$ is also irrational.

In the sequel, let $f'(a_0) < 0$. In the case of $f'(a_0) > 0$ the proof is similar.

We choose our function $h : 2^{[0,1]^2} \to 2^{\mathbb{R}\cup\{-\infty,\infty\}}$ as follows: For $M \subseteq [0,1]^2$ define the following values (provided they exist):

$$\begin{aligned} x_0 &:= \inf M \cap \ell[y=1]\ , \quad & y_0 &:= \sup M \cap \ell[x=0] \\ x_1 &:= \sup M \cap \ell[y=0]\ , \quad & y_1 &:= \inf M \cap \ell[x=1]\ , \text{ and} \\ h(M) &:= \left\{ y_1 - y_0, \frac{-y_0}{x_1}, \frac{y_1}{-x_0}, \frac{1}{x_1 - x_0} \right\} \end{aligned}$$

The four possible values in $h(M)$ are the slopes of the lines connecting the points $(0, y_0)$ and $(x_0, 1)$ with $(1, y_1)$ and $(x_1, 0)$. Thus $h(M)$ has, depending on M, at most four elements.

If M is encodable as a finite automaton then according to Proposition 16 and Lemma 19 all four points $(x_0, 1)$, $(0, y_0)$, $(1, y_1)$ and $(x_1, 0)$, provided they exist, are rational, whence $h(M) \subseteq \mathbb{Q} \cup \{-\infty, \infty\}$.

Since f is continuous and $f'(a_0) < 0$, for sufficiently small $\varepsilon > 0$ we have $f(a) > f(a_0)$ for $a_0 - \varepsilon < a < a_0$ and $f(a) < f(a_0)$ for $a_0 < a < a_0 + \varepsilon$. Consider a sufficiently small cube $\nu_r(w \cdot X^\omega)$ containing the point $(a_0, f(a_0))$. Let $(\underline{a}_w, \overline{a}_w)$ be its lower left corner. Then $\underline{a}_w \leq a_0 \leq \underline{a}_w + r^{-|w|}$ and $\overline{a}_w < f(a_0) < \overline{a}_w + r^{-|w|}$. The behaviour of f in this small cube shows that for $M = \Psi_w(\Gamma(f) \cap \nu_r(w \cdot X^\omega))$ at least one of the points $(0, y_0)$ or $(x_0, 1)$ and at least one of the points $(1, y_1)$ or $(x_1, 0)$ exist. These points correspond to points $(a_w, f(a_w)), (b_w, f(b_w)) \in \Gamma(f) \cap \nu_r(w \cdot X^\omega)$ such that $\underline{a}_w \leq a_w \leq a_0 \leq b_w \leq \underline{a}_w + r^{-|w|}$ and $\overline{a}_w \leq f(a_w) \leq f(a_0) \leq f(b_w) \leq \overline{a}_w + r^{-|w|}$.

As in the proof of Lemma 23 we conclude that, if $\Gamma(f)$ is encodable as a finite automaton, $\bigcup_{v \in X^*} h_v(\Gamma(f))$ is a finite subset of $\mathbb{Q} \cup \{-\infty, \infty\}$.

On the other hand, there is a sequence of words $(w_i)_{i \in \mathbb{N}}$ such that $|w_i| \geq i$, $(a_0, f(a_0)) \in \nu_r(w_i \cdot X^\omega)$ and $f'(a_0) = \lim\limits_{i \to \infty} \frac{f(b_{w_i}) - f(a_{w_i})}{b_{w_i} - a_{w_i}}$. Since $\frac{f(b_{w_i}) - f(a_{w_i})}{b_{w_i} - a_{w_i}} \in h_{w_i}(\Gamma(f)) \subseteq \mathbb{Q} \cup \{-\infty, \infty\}$, the irrationality of $f'(a_0)$ requires $\bigcup\limits_{i \in \mathbb{N}} h_{w_i}(\Gamma(f))$ to be infinite, contradicting the finiteness of $\bigcup\limits_{v \in X^*} h_v(\Gamma(f))$. ❑

As an immediate consequence we obtain.

Corollary 26 *Let* $f : [0,1] \to [0,1]$ *be a continuously differentiable function with non-constant derivative. Then the graph* $\Gamma(f)$ *is not encodable as a finite automaton.*

This corollary explains Examples 1 and 2 and also the following one.

Example 3 No circle is encodable as a finite automaton.

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