# The domain of the Schrödinger operator $-\Delta + x^2y^2$

G. Metafune* R. Schnaubelt †

**Abstract**

We compute the domain of the Schrödinger operator $-\Delta + x^2y^2$ in $L^2(\mathbb{R}^2)$.

Mathematics subject classification (2000): 35J10.

## 1 Introduction

Let $V$ be a nonnegative potential in $\mathbb{R}^d$ which belongs to $L^2_{loc}(\mathbb{R}^d)$. Then the quadratic form

$$a(u,v) = \int_{\mathbb{R}^d} (\nabla u \cdot \nabla \bar{v} + V u \bar{v})\, dx, \qquad u, v \in H = \{u \in H^1(\mathbb{R}^d) : V^{1/2}u \in L^2(\mathbb{R}^d)\}.$$

is closed, symmetric and nonnegative in $L^2(\mathbb{R}^d)$. Therefore $a$ defines a self-adjoint operator $(A, D(A))$ in $L^2(\mathbb{R}^d)$ formally given by $A = -\Delta + V$, see e.g. [2, Chapter 8]. Moreover, $A$ can be described by

$$D(A) = \{u \in H : \exists f \in L^2(\mathbb{R}^d) \text{ s.t. } a(u,v) = \int_{\mathbb{R}^d} f\bar{v}\, dx \quad \forall\, v \in H\}, \qquad Au = f. \tag{1.1}$$

The test function space $C_0^\infty(\mathbb{R}^d)$ is a core for $A$ since $V \in L^2_{loc}(\mathbb{R}^d)$, due to [3, Corollary VII.2.7]. Thus the question arises whether $D(A)$ coincides with the intersection $H^2(\mathbb{R}^d) \cap D(V)$. Here $H^k(\mathbb{R}^d)$ is the usual Sobolev space and $D(V) = \{u \in L^2(\mathbb{R}^d) : Vu \in L^2(\mathbb{R}^d)\}$ is the domain of the multiplication operator $V : u \mapsto Vu$. The equality $D(A) = H^2(\mathbb{R}^d) \cap D(V)$ holds if $V$ satisfies the oscillation condition

$$|\nabla V(x)| \le \varepsilon V(x)^{3/2} + C_\varepsilon \tag{1.2}$$

for $x \in \mathbb{R}^d$ and a sufficiently small $\varepsilon$; see [1], [7], [8], where also the $L^p$-case, $p \neq 2$, is treated. An example in [7] shows that $D(A)$ can be strictly larger than $H^2(\mathbb{R}^d) \cap D(V)$ even for $p = 2$ if (1.2) does not hold. Surprisingly enough the situation is much better in $L^1(\mathbb{R}^d)$ where the domain of $-\Delta + V$ is always the intersection of the domains of $-\Delta$ and of the potential $V$, [4].

In this note we prove that $D(A) = H^2(\mathbb{R}^2) \cap D(V)$ for the potential $V(x,y) = x^2y^2$ which, as is easy to see, does not satisfy (1.2). The same potential was studied in detail in [9] where the compactness of the resolvent was proved. (See also [6] for a characterization of the discreteness of the spectrum for polynomial potentials.)

**Notation.** The norm of $L^p(\mathbb{R}^d)$ is denoted by $\|\cdot\|_p$. $H^k(\mathbb{R}^d)$ is the Sobolev space of all functions in $L^2(\mathbb{R}^d)$ having weak derivatives in $L^2(\mathbb{R}^d)$ up to the order $k$. $C_0^\infty(\mathbb{R}^d)$ is the space of test functions.

*Dipartimento di Matematica "Ennio De Giorgi", Università di Lecce, C.P.193, 73100, Lecce, Italy. E-mail: giorgio.metafune@unile.it

†Fachbereich Mathematik und Informatik, Institut für Analysis, Martin-Luther-Universität Halle-Wittenberg, 06099 Halle (Saale), Germany. E-mail: schnaubelt@mathematik.uni-halle.de

## 2 The result

We begin with the following elementary lemma.

**Lemma 2.1** *Let $0 \leq V \in L^2_{loc}(\mathbb{R})$. Assume that there exists a constant $C > 0$ such that*

$$\|Vu\|_2 \leq C \, \| - u'' + Vu\|_2 \tag{2.1}$$

*for every $u \in C_0^\infty(\mathbb{R})$. Then the potential $V_\lambda(x) = \lambda^{-2}V(x/\lambda)$ satisfies (2.1) with the same constant $C$ for every $\lambda > 0$.*

PROOF. Applying (2.1) to the function $v(x) = u(\lambda x)$, we obtain

$$\int_{\mathbb{R}} |V(x)u(\lambda x)|^2 \, dx \leq C^2 \int_{\mathbb{R}} | - \lambda^2 u''(\lambda x) + V(x)u(\lambda x)|^2 \, dx.$$

Setting $y = \lambda x$, this inequality leads to

$$\int_{\mathbb{R}} |V(y/\lambda)u(y)|^2 \, dy \leq C^2 \int_{\mathbb{R}} | - \lambda^2 u''(y) + V(y/\lambda)u(y)|^2 \, dy,$$

which implies the assertion. □

In order to compute the domain of $-\Delta + x^2y^2$ we have to estimate the constant $C$ in (2.1) for the potential $V(x) = x^2$.

**Proposition 2.2** *The estimate*

$$\|x^2u\|_2 \leq C \, \| - D^2u + x^2u\|_2$$

*holds for every $u \in C_0^\infty(\mathbb{R})$ and a constant $C > 0$ satisfying $C^2 < 2$.*

Before proving this proposition, we show how the announced domain characterization follows from Proposition 2.2 and Lemma 2.1.

**Theorem 2.3** *The domain of $-\Delta + x^2y^2$ in $L^2(\mathbb{R}^2)$ coincides with $H^2(\mathbb{R}^2) \cap D(V)$.*

PROOF. The representation (1.1) of $A$ implies that $H^2(\mathbb{R}^2) \cap D(V)$ is contained in $D(A)$ and that $Au = -\Delta u + x^2y^2u$ for $u \in H^2(\mathbb{R}^2) \cap D(V)$. Since $C_0^\infty(\mathbb{R}^2)$ is a core for $D(A)$, see [3, Corollary VII.2.7], it suffices to prove that the graph norm and the canonical norm of $H^2(\mathbb{R}^2) \cap D(V)$ are equivalent on $C_0^\infty(\mathbb{R}^2)$. Clearly, $\|u\|_2 + \|Au\|_2 \leq \|u\|_{H^2} + \|x^2y^2u\|_2$ for $u \in C_0^\infty(\mathbb{R}^2)$.

Thus it remains to establish the converse inequality. To estimate the $H^1$-norm of $u \in C_0^\infty(\mathbb{R}^2)$, we note that

$$\int_{\mathbb{R}^2} (u + Au)\bar{u} \, dx \, dy = \int_{\mathbb{R}^2} (|u|^2 + |\nabla u|^2 + x^2y^2|u|^2) \, dx \, dy,$$

Hence, $\|u\|_{H^1} \leq c\,(\|u\|_2 + \|Au\|_2)$ for a suitable $c > 0$. We next treat the $L^2$-norms of the functions $x^2y^2u$ and $D^2u$. We set $f = -\Delta u + x^2y^2u$ for $u \in C_0^\infty(\mathbb{R}^2)$. Then $-u_{xx} + x^2y^2u = f + u_{yy}$. Fix $y \in \mathbb{R} \setminus \{0\}$. Proposition 2.2 and Lemma 2.1 with $\lambda^4 = y^{-2}$ show that

$$\int_{\mathbb{R}} x^4y^4u(x,y)^2 \, dx \leq C^2 \int_{\mathbb{R}} |f(x,y) + u_{yy}(x,y)|^2 \, dx,$$

where $C$ is the constant from Proposition 2.2. Integrating this estimate with respect to $y$, we obtain

$$\int_{\mathbb{R}^2} x^4y^4u^2 \, dx \, dy \leq C^2 \int_{\mathbb{R}^2} |f + u_{yy}|^2 \, dx \, dy.$$

In the same way one deduces that

$$\int_{\mathbb{R}^2} x^4y^4u^2\,dx\,dy \le C^2 \int_{\mathbb{R}^2} |f+u_{xx}|^2\,dx\,dy.$$

Summing the last two inequalities and using $f = -\Delta u + x^2y^2u$, we conclude

$$\begin{aligned}\int_{\mathbb{R}^2} x^4y^4u^2\,dx\,dy &\le C^2 \int_{\mathbb{R}^2} (f^2 + f\Delta u + \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2)\,dx\,dy \\ &= C^2 \int_{\mathbb{R}^2} (f^2 - |\Delta u|^2 + x^2y^2u\Delta u + \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2)\,dx\,dy.\end{aligned}$$

On the other hand, we compute

$$\int_{\mathbb{R}^2} |\Delta u|^2\,dx\,dy = \int_{\mathbb{R}^2} (u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2)\,dx\,dy \tag{2.2}$$

integrating by parts twice, which leads to

$$\int_{\mathbb{R}^2} x^4y^4u^2\,dx\,dy \le C^2 \int_{\mathbb{R}^2} (f^2 - \frac{1}{2}\,|\Delta u|^2 + x^2y^2u\Delta u)\,dx\,dy.$$

Young's inequality then implies

$$\begin{gathered}\int_{\mathbb{R}^2} x^4y^4u^2\,dx\,dy \le C^2 \int_{\mathbb{R}^2} (f^2 + \frac{1}{2}x^4y^4u^2)\,dxdy, \\ \|x^2y^2u\|_2^2 \le \frac{C^2}{1-C^2/2}\|f\|_2^2\,,\end{gathered}$$

since $1 - C^2/2 > 0$ by Proposition 2.2. This estimate and (2.2) further yield

$$\|D^2u\|_2^2 \le C_1\,\|\Delta u\|_2^2 = C_1\,\|x^2y^2u - f\|_2^2 \le C_2\,\|f\|_2^2\,.$$

As a result, $\|u\|_{H^2} + \|x^2y^2u\|_2 \le C_3\,(\|u\|_2 + \|Au\|_2)$ for some constants $C_k$. $\square$

In order to prove Proposition 2.2 we need some elementary properties of the Hermite functions

$$H_n(x) = \frac{(-1)^n}{\sqrt{2^n n!\sqrt{\pi}}}\,e^{x^2/2}\,\frac{d^n}{dx^n}e^{-x^2} =: \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\,\psi_n(x), \qquad n \in \mathbb{N}_0\,,$$

for which we refer to [5, §5.6.2]. The Hermite functions are an orthonormal basis of $L^2(\mathbb{R})$ and $(-D^2 + x^2)H_n = (2n+1)H_n$. The functions $\psi_n$ satisfy the identity $\psi_{n+1} = 2x\psi_n - 2n\psi_{n-1}$ for $n \in \mathbb{N}_0$, where $\psi_{-1} = 0$. Using this recursion formula, one easily computes the integrals

$$c_{n,m} = \int_{\mathbb{R}} x^2 H_n(x)H_m(x)\,dx, \qquad n,m \in \mathbb{N}_0\,,$$

obtaining

$$\begin{aligned} &c_{n,n-2} = \frac{1}{2}\sqrt{n(n-1)} \quad (n \ge 2), \quad c_{n,n} = \frac{1}{2}(2n+1), \quad c_{n,n+2} = \frac{1}{2}\sqrt{(n+2)(n+1)} \\ &c_{n,m} = 0, \qquad \text{if } \ m \ne n, n-2, n+2.\end{aligned} \tag{2.3}$$

Proof of Proposition 2.2. Let $u \in C_0^\infty(\mathbb{R})$ and expand $f = -D^2u + x^2u$ with respect to the orthonormal basis $(H_n)$, i.e.,

$$f = \sum_{m=0}^{\infty} \langle f, H_m \rangle H_m = \sum_{m=0}^{\infty} f_m H_m$$

where the brackets denote the inner product of $L^2(\mathbb{R})$ and $f_m = \langle f, H_m \rangle$. Then we obtain

$$u = \sum_{m=0}^{\infty} (2m+1)^{-1} f_m H_m \qquad \text{and} \qquad x^2 u = \sum_{m=0}^{\infty} (2m+1)^{-1} f_m x^2 H_m.$$

From the identities (2.3) it follows that

$$\langle x^2 u, H_n \rangle = \alpha_n f_{n-2} + \frac{1}{2} f_n + \beta_n f_{n+2}$$

for $n \in \mathbb{N}_0$, where

$$\alpha_n = \frac{\sqrt{n(n-1)}}{2(2n-3)}, \qquad \beta_n = \frac{\sqrt{(n+2)(n+1)}}{2(2n+5)}, \qquad f_{-2} = f_{-1} = 0.$$

These equalities yield

$$x^2 u = \frac{1}{2} f + \sum_{n=0}^{\infty} (\alpha_n f_{n-2} + \beta_n f_{n+2}) H_n =: \frac{1}{2} f + g. \tag{2.4}$$

We further estimate

$$\begin{aligned} \|g\|_2^2 &= \sum_{n=0}^{\infty} (\alpha_n f_{n-2} + \beta_n f_{n+2})^2 \\ &= \alpha_2^2 f_0^2 + \alpha_3^2 f_1^2 + 2\alpha_2\beta_2 f_0 f_4 + 2\alpha_3\beta_3 f_1 f_5 + \sum_{n=2}^{\infty} (\alpha_{n+2}^2 + \beta_{n-2}^2) f_n^2 \\ &\quad + 2 \sum_{n=4}^{\infty} \alpha_n f_{n-2} \beta_n f_{n+2} \\ &\le \alpha_2^2 f_0^2 + \alpha_3^2 f_1^2 + 2\alpha_2\beta_2 f_0 f_4 + 2\alpha_3\beta_3 f_1 f_5 + 2 \sum_{n=2}^{\infty} (\alpha_{n+2}^2 + \beta_{n-2}^2) f_n^2 \end{aligned}$$

using Hölder's and Young's inequalities. Observe that $\alpha_{n+2}^2 + \beta_{n-2}^2 \le 7/50$ for $n \ge 2$. Hence,

$$\begin{aligned} \|g\|_2^2 &\le \frac{1}{2} f_0^2 + \frac{1}{6} f_1^2 + \frac{\sqrt{6}}{9} f_0 f_4 + \frac{\sqrt{30}}{33} f_1 f_5 + \frac{14}{50} \sum_{n=2}^{\infty} f_n^2 \\ &\le \left( \frac{1}{2} + \frac{\sqrt{6}}{18} \right) f_0^2 + \left( \frac{1}{6} + \frac{\sqrt{30}}{66} \right) f_1^2 + \frac{14}{50} f_2^2 + \frac{14}{50} f_3^2 + \left( \frac{\sqrt{6}}{18} + \frac{14}{50} \right) f_4^2 \\ &\quad + \left( \frac{\sqrt{30}}{66} + \frac{14}{50} \right) f_5^2 + \frac{14}{50} \sum_{n=6}^{\infty} f_n^2 \\ &\le \left( \frac{1}{2} + \frac{\sqrt{6}}{18} \right) \|f\|_2^2 . \end{aligned}$$

Together with (2.4), we conclude

$$\|x^2 u\|_2 \leq \left( \frac{1}{2} + \sqrt{\frac{1}{2} + \frac{\sqrt{6}}{18}} \right) \|f\|_2 =: C\, \|f\|_2 \,.$$

The assertion is established since $C^2 < 2$. $\square$

**Remark 2.4** As in the proof of Theorem 2.3 one can establish that $D(-\Delta + bV) = H^2(\mathbb{R}^2) \cap D(V)$ for $b > 0$ and $V(x,y) = x^2 y^2$.

**Remark 2.5** Let $V(x,y) = x^2y^2$ and $A = -\Delta + V$. The domain characterization in Theorem 2.3 is equivalent to the boundedness of the operator $V(I+A)^{-1}$ in $L^2(\mathbb{R}^2)$. On the other hand, the operator $V(I+A)^{-1}$ is bounded in $L^1(\mathbb{R}^2)$ due to [4]. By interpolation, $V(I+A)^{-1}$ is bounded in $L^p(\mathbb{R}^2)$ for $1 < p < 2$, and thus the domain of $A$ in $L^p(\mathbb{R}^2)$, $1 < p < 2$, is equal to $W^{2,p}(\mathbb{R}^2) \cap D_p(V)$, where $D_p(V) = \{u \in L^p(\mathbb{R}^2) : Vu \in L^p(\mathbb{R}^2)\}$. We do not know whether a similar result is valid for $p > 2$.

# References


[1] P. Cannarsa, V. Vespri: Generation of analytic semigroups in the $L^p$ topology by elliptic operators in $\mathbb{R}^n$, *Israel J. Math.* **61** (1988), 235–255.

[2] E. B. Davies: *Spectral Theory and Differential Operators,* Cambridge U. P., 1995.

[3] D. E. Edmunds, W. D. Evans: *Spectral Theory and Differential Operators,* Oxford U.P., 1990.

[4] T. Kato: $L^p$-theory of Schrödinger operators with a singular potential, in: R. Nagel, U. Schlotterbeck, M.P.H. Wolff (Eds.), *Aspects of Positivity in Functional Analysis*, North-Holland, 1986.

[5] W. Magnus, F. Oberhettinger, R.P. Soni: *Formulas and Theorems for the Special Functions in Mathematical Physics,* Springer–Verlag, 1977.

[6] G. Metafune, D. Pallara: Discreteness of the spectrum for some differential operators with unbounded coefficients in $\mathbb{R}^n$, *Rend. Mat. Acc. Lincei, s.9* **11** (2000), 9-19.

[7] G. Metafune, J. Prüss, A. Rhandi, R. Schnaubelt: $L^p$-regularity for elliptic operators with unbounded coefficients, *Preprint of the Institute of Analysis, Martin-Luther University, Halle-Wittenberg* **21** (2002).

[8] N. Okazawa: An $L^p$ theory for Schrödinger operators with nonnegative potentials, *J. Math. Soc. Japan* **36** (1984), 675-688.

[9] B. Simon: Some quantum operators with discrete spectrum but classically continuous spectrum, *Annals of Physics* **146** (1983), 209-220.