Equality of Upper and Lower Tolerances in Combinatorial Optimization

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Abstract

For a given optimal solution to a combinatorial optimization problem, we show, under very natural conditions, the equality of the minimal values of upper and lower tolerances, where the upper tolerances are calculated for the given optimal solution and the lower tolerances outside the optimal solution.

1 Introduction; Libura's Theorem Generalized

A *combinatorial optimization problem* (COP)$(\mathcal{E}, C, \mathcal{S}, f_C)$ is the problem of finding

$$S^* \in \arg OPT\{f_C(S) \mid S \in \mathcal{S}\},$$

where $C : \mathcal{E} \to \Re$ is the given *instance* of the problem with a *ground set* $\mathcal{E}$ satisfying $|\mathcal{E}| = m$ $(m \geq 1)$, $\mathcal{S} \subseteq 2^{\mathcal{E}}$ is the *set of feasible solutions*, and $f_C : 2^{\mathcal{E}} \to \Re$ is the *objective function* of the problem. By $\mathcal{S}^* = \arg OPT\{f_C(S) \mid S \in \mathcal{S}\}$ the set of optimal solutions is denoted. It is assumed that $\mathcal{S}^* \neq \emptyset$, and that $S \neq \emptyset$ for some $S \in \mathcal{S}$. Let $g \in \mathcal{E}$, and $\alpha \geq 0$. By $C_{\alpha,g} : \mathcal{E} \to \Re$ we denote the instance defined as $c_{\alpha,g}(e) = c(e)$ for each $e \in \mathcal{E} \setminus \{g\}$, and $c_{\alpha,g}(g) = c(g) + \alpha$. Take any $S^* \in \mathcal{S}^*$. The *upper tolerance*, $u_{S^*}(e)$, of e with respect to S^* is defined as

$$u_{S^*}(e) = \max\{\alpha \geq 0 : S^* \in \arg\min\{f_{C_{\alpha,e}}(S) : S \in \mathcal{S}\}\},$$

and the *lower tolerance*, $l_{S^*}(e)$, with respect to S^* as

$$l_{S^*}(e) = \max\{\alpha \geq 0 : S^* \in \arg\min\{f_{C_{-\alpha,e}}(S) : S \in \mathcal{S}\}\}.$$

I.e. $u_{S^*}(e)$ is the maximal increase of $c(e)$ under which S^* stays optimal, and $l_{S^*}(e)$ is the maximal decrease of $c(e)$ under which S^* stays optimal. In Section 2 it will be shown that under very natural conditions, it holds that $\min\{u_{S^*}(e) : e \in S^*\} = \min\{l_{S^*}(e) : e \in \mathcal{E} \setminus S^*\}$. For an extensive account on properties of upper and lower tolerances in the context of sensitivity analysis, see among others, Gal [1] and Greenberg [2].

The following theorem can be seen as a generalization of Libura's well known theorem on tolerances (see, Libura [3], and Ramasvamy&Chakravarti [4]). We will use the following extra notations. Let $e \in \mathcal{E}$. Then $\mathcal{S}_+(e) = \{S \in \mathcal{S} : e \in S\}$, and $\mathcal{S}_-(e) = \{S \in \mathcal{S} : e \notin S\}$. Clearly, $\mathcal{S} = \mathcal{S}_-(e) \cup \mathcal{S}_+(e)$ and $\mathcal{S}_-(e) \cap \mathcal{S}_+(e) = \emptyset$ for all $e \in \mathcal{E}$. Similarly, $\mathcal{S}^*_+(e)$ and $\mathcal{S}^*_-(e)$ are the sets of optimal solutions containing e, and not containing e, respectively.

Theorem 1 *Consider a COP $(\mathcal{E}, C, \mathcal{S}, f_C)$. For each $S^* \in \mathcal{S}^*$ the following holds:*

1. $e \in \cap\mathcal{S}^*$ *iff* $u_{S^*}(e) = f_C(S) - f_C(S^*) > 0$ *for each* $S \in \mathcal{S}^*_-(e)$, $l_{S^*}(e) = \infty$;
2. $e \in \mathcal{E} \setminus \cup\mathcal{S}^*$ *iff* $u_{S^*}(e) = \infty$, $l_{S^*}(e) = f_C(S) - f_C(S^*) > 0$ *for each* $S \in \mathcal{S}^*_-(e)$;
3. $e \in S^* \setminus \cap\mathcal{S}^*$ *iff* $u_{S^*}(e) = 0$, $l_{S^*}(e) = \infty$;
4. $e \in \cup\mathcal{S}^* \setminus S^*$ *iff* $u_{S^*}(e) = \infty$, $l_{S^*}(e) = 0$.

The proof of Theorem 1 is left to the reader. Note that if $|\mathcal{S}^*| = 1$, then this theorem boils down to Libura's theorem on tolerances. Also note that if $\mathcal{S}_-(e) = \emptyset$, for some $e \in \mathcal{E}$, then $u_{S^*}(e) = \min\{f_C(T) : T \in \mathcal{S}_-(e)\} - f_C(S^*) = \min\{\emptyset\} = \infty$ (by definition). Similarly for $\mathcal{S}_+(e) = \emptyset$ we take $l_{S^*}(e) = \infty$.

2 Minimal Upper and Lower Tolerances

We call $\mathcal{S}^*$ *nested w.r.t.* S^* for $S^* \in \mathcal{S}^*$ if and only if either $\cup\mathcal{S}^* = S^*$ or $\cap\mathcal{S}^* = S^*$. We start with the rather trivial case of multi-optimal solutions.

Theorem 2 *Consider a COP$(\mathcal{E}, C, \mathcal{S}, f_C)$ with $|\mathcal{S}^*| \geq 2$. Then for any $S^* \in \mathcal{S}^*$ the following holds.*

1. *For $\mathcal{S}^*$ nested w.r.t. S^*:*
   - a. $S^* = \cap\mathcal{S}^*$ *implies that* $\min\{l_{S^*}(e) : e \in \mathcal{E} \setminus S^*\} = 0$, $\min\{u_{S^*}(e) : e \in S^*\} > 0$, *and* $< \infty$ *iff* $\mathcal{S}_-(e) \neq \emptyset$ *for some* $e \in S^*$;
   - b. $S^* = \cup\mathcal{S}^*$ *implies that* $\min\{u_{S^*}(e) : e \in S^*\} = 0$, $\min\{l_{S^*}(e) : e \in \mathcal{E} \setminus S^*\} > 0$, *and* $< \infty$ *iff* $\mathcal{S}_+(e) \neq \emptyset$ *for some* $e \in \mathcal{E} \setminus S^*$;
2. *For $\mathcal{S}^*$ nonnested w.r.t. S^*:* $\min\{u_{S^*}(e) : e \in S^*\} = 0 = \min\{l_{S^*}(e) : e \in \mathcal{E} \setminus S^*\}$.

Proof. 1a. Since $|\mathcal{S}^*| \geq 2$, it follows that $\cup\mathcal{S}^* \setminus S^* \neq \emptyset$. Hence, $l_{S^*}(e) = 0$ for each $e \in \cup\mathcal{S}^* \setminus S^*$ (see Theorem 1(4)). Therefore, $\min\{l_{S^*}(e) : e \in \mathcal{E} \setminus S^*\} = 0$. Moreover, since $S^* \subseteq T$ for all $T \in \mathcal{S}^*$, it follows that $u_{S^*}(e) > 0$ for each $e \in S^*$, while $u_{S^*}(e) < \infty$ iff $S_-(e) \neq \emptyset$ for some $e \in S^*$ (see Theorem 1(1)).
1b. The proof is similar to the one of 1a. Note that now $\cup(\mathcal{S}^* \setminus \{S^*\}) \neq \emptyset$.
2. $S^* \setminus \cap\mathcal{S}^* \neq \emptyset$ implies that $\min\{u_{S^*}(e) : e \in S^*\} = \min\{u_{S^*}(e) : e \in S^* \setminus \cap\mathcal{S}^*\} = 0$, and $\cup\mathcal{S}^* \setminus S^* \neq \emptyset$ implies that $\min\{l_{S^*}(e) : e \in \mathcal{E} \setminus S^*\} = \min\{l_{S^*}(e) : e \in \cup\mathcal{S}^* \setminus S^*\} = 0$. ∎

Since the situation for $|\mathcal{S}^*| = 1$ is somehow different from the situation with $|\mathcal{S}^*| \geq 2$, we have devoted a special theorem for the case with a unique optimal solution. Below we write $u_{S^*} = u$ and $l_{S^*} = l$. Moreover, we use the set

$$\mathcal{S}^2 = \{S \in \mathcal{S} : S \in \arg\min\{f_C(S) : S \in \mathcal{S} \setminus \mathcal{S}^*\}\},$$

i.e., the set of second-best solutions of the COP $(\mathcal{E}, C, \mathcal{S}, f_C)$ with $\mathcal{S}^* = \{S^*\}$ (see e.g. Van der Poort *et al.* [6]). We also assume that f_C is *monotone*, meaning that for each $S_1, S_2 \in 2^{\mathcal{E}}$, it holds that

$$\text{if} \quad S_1 \subseteq S_2 \quad \text{then} \quad f_C(S_1) \leq f_C(S_2).$$

Sum functions with $f_C(S) = \sum_{e \in S} c(e)$, *bottleneck functions* with $f_C(S) = \max_{e \in S} c(e)$, and *product functions* with $f_C(S) = \prod_{e \in S} c(e)$ and $c(e) \geq 1$ for each $e \in \mathcal{E}$ are all monotone functions.

Theorem 3 *Consider a COP $(\mathcal{E}, C, \mathcal{S}, f_C)$ with monotone f_C, and unique optimal solution S^*. Then the following holds.*

1. $\min\{u(e) : e \in S^*\} = \min\{l(e) : e \in \mathcal{E} \setminus S^*\} = \infty$ *iff* $\mathcal{S}_-(e) = \emptyset$ *for all* $e \in S^*$ *and* $\mathcal{S}_+(e) = \emptyset$ *for all* $e \in \mathcal{E} \setminus S^*$*;*

2. $0 < \min\{u(e) : e \in S^*\} < \min\{l(e) : e \in \mathcal{E} \setminus S^*\} = \infty$ *iff* $\mathcal{S}_-(e) \neq \emptyset$ *for all* $e \in S^*$ *and* $\mathcal{S}_+(e) = \emptyset$ *for all* $e \in \mathcal{E} \setminus S^*$*;*

3. $0 < \min\{l(e) : e \in \mathcal{E} \setminus S^*\} < \min\{u(e) : e \in S^*\} = \infty$ *iff* $\mathcal{S}_-(e) = \emptyset$ *for all* $e \in S^*$ *and* $\mathcal{S}_+(e) \neq \emptyset$ *for all* $e \in \mathcal{E} \setminus S^*$*;*

4. $0 < \min\{l(e) : e \in \mathcal{E} \setminus S*\} \leq \min\{u(e) : e \in S^*\} < \infty$ *iff* $\mathcal{S}_-(e) \neq \emptyset$ *for all* $e \in S^*$ *and* $\mathcal{S}_+(e) \neq \emptyset$ *for all* $e \in \mathcal{E} \setminus S^*$*;*
   moreover, $\mathcal{S}^2$ *is nonnested w.r.t.* S^* *iff* $0 < \min\{u(e) : e \in S^*\} = \min\{l(e) : e \in \mathcal{E} \setminus S*\} < \infty$.

Proof.
1. This is by definition.
2. If $\mathcal{S}_-(e) \neq \emptyset$ for some $e \in S^*$, then there is a set $S \in \mathcal{S}_-(e)$ with $f_C(S) - f_C(S^*) > 0$. Hence, $\min\{u(e) : e \in S^*\} = \min\{\min\{f_C(S) : S \in \mathcal{S}_-(e)\} : e \in S^*\} - f_C(S^*) > 0$.

3. The proof of this part is similar to the proof of (2).
4. Since $\mathcal{S}_-(e) \neq \emptyset$ for some $e \in S^*$, there exists an element $e_0 \in S^*$ and a set $S_0 \in \mathcal{S}_-(e)$ such that $u(e_0) = \min\{u(e) : e \in S^*\} = f_C(S_0) - f_C(S^*)$. Clearly $S_0 \setminus S^* \neq \emptyset$, because otherwise $S_0 \subseteq S^*$ and the fact that f_C is monotone would imply that $f_C(S_0) = f_C(S^*)$, contradicting the fact that S^* is unique. Now let $t \in S_0 \setminus S^* \subseteq \mathcal{E} \setminus S^*$. Then $S_0 \in \mathcal{S}_+(t)$. Hence,

$$\min\{l(e) : e \in \mathcal{E} \setminus S^*\} \leq l(t) = \min\{f_C(S) : S \in \mathcal{S}_+(t)\} - f_C(S^*) = u(e_0).$$

The "moreover" part can be seen as follows. Since $\mathcal{S}^2$ is nonnested w.r.t. S^*, there is a set $S^0 \in \mathcal{S}^2$ such that $S^0 \setminus S^* \neq \emptyset \neq S^* \setminus S^0$. Then for each $e^0 \in S^0 \setminus S^*$ and each $e \in S^* \setminus S^0$, it follows that:

$$\min\{u(e) : e \in S^*\} \leq u(e) = \min\{f_C(T) : T \in \mathcal{S}_-(e)\} - f_C(S^*) =$$
$$f_C(S^0) - f_C(S^*) = \min\{f_C(T) : T \in \mathcal{S}_+(e^0)\} - f_C(S^*) = \min\{l(e) : e \in \mathcal{E} \setminus S^*\}.$$

If $\mathcal{S}^2$ is nested w.r.t. S^*, then $\min\{l(e) : e \in \mathcal{E} \setminus S^*\} = l(e)$ for each $e \in S \setminus S^*$ with $S \in \mathcal{S}^2$, while $\min\{u(e) : e \in S^*\} = f_C(T) - f_C(S^*)$ for some T with $f_C(T) > f_C(S)$ for $S \in \mathcal{S}^2$. This completes the "iff" of the "moreover" part of 4. ∎

Theorem 4 *Consider a COP $(\mathcal{E}, C, \mathcal{S}, f_C)$ with monotone f_C. If for each $S_1, S_2 \in \mathcal{S}$ with $S_1 \neq S_2$, it holds that neither $S_1 \subset S_2$ nor $S_2 \subset S_1$, then*

$$\min\{u_{S^*}(e) : e \in S^*\} = \min\{l_{S^*}(e) : e \in \mathcal{E} \setminus S^*\},$$

for each $S^* \in \mathcal{S}^*$.

Proof. If $|\mathcal{S}^*| \geq 2$, then the assertion of this theorem follows from Theorem 2(2). If $|\mathcal{S}^*| = 1$ it follows from Theorem 3(4), because both $\mathcal{S}^*$ and $\mathcal{S}^2$ are nonnested w.r.t. S^* for each $S^* \in \mathcal{S}^*$. ∎

The conditions from Theorem 4 holds for "regular" COPs, such as the traveling salesman, quadratic assignment, linear ordering, assignment, max-flow, shortest path, minimum spanning tree, and matching problems. The more irregular situation of Theorem 3(4) with " $>$ ", occurs in plant location problems. The following example shows such a situation.

Let $\mathcal{E} = \{e_1, e_2, e_3\}$ with $c(e_1) = 1$, $c(e_2) = 2$, $c(e_3) = 4$, $S_1 = \{e_1\}$, $S_2 = \{e_1, e_2\}$, and $S_3 = \{e_3\}$. Then $\mathcal{S}^* = \{S_1\}$, $\mathcal{S}^2 = \{S_2\}$ with $u(e_1) = f_C(S_3) - f_C(S_1) = 3 > l(e_2) = f_C(S_2) - f_C(S_1) = 2$. Note that $l(e_3) = f_C(S_3) - f_C(S_1) = 3$.

The above example may suggest that there exist "easy" conditions for which $\max\{u_{S^*}(e) : e \in S^*\} = \max\{l_{S^*}(e) : e \in \mathcal{E} \setminus S^*\}$ for $S^* \in \mathcal{S}^*$. However, the following example points in the opposite direction. Let

$$C = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

be the cost matrix of an assignment problem with three jobs and three machines. One can easily check that $S^* = \{(1,1), (2,2), (3,3)\}$, and that $\max\{u(e) : e \in S^*\} = u(3,3) = 4 \neq 6 = l(3,1) = \max\{l(e) : e \in \mathcal{E} \setminus S^*\}$.

Finally, note that Theorem 3, and more specifically Theorem 4, allow us to compute the minimal upper and lower tolerances w.r.t. a given optimal solution S^*, within S^* and outside S^* respectively, by solving at most $\min\{O(|S^*|), O(|\mathcal{E} \setminus S^*|)\}$ (sub)COPs. This can be a considerable improvement, in comparison to the situation where only Theorem 1 (including Libura's theorem) was available. In the latter case, $O(|\mathcal{E}|)$ (sub)COPs needed to be solved. Compare in this respect the results from Van Hoesel& Wagelmans [5], and Ramasvamy&Chakravarti [4].

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