How Large is the Set of Disjunctive Sequences ?*†

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Abstract

We consider disjunctive sequences, that is, infinite sequences (ω-words) having all finite words as infixes. It is shown that the set of all disjunctive sequences can be described in an easy way using recursive languages and, besides being a set of measure one, is a residual set in Cantor space.

Moreover, we consider the subword complexity of sequences: here disjunctive sequences are shown to be sequences of maximal complexity.

Along with disjunctive sequences we consider the set of real numbers having disjunctive expansions with respect to some bases and to all bases. The latter are called absolutely disjunctive real numbers. We show that the set of absolutely disjunctive reals is also a residual set and has representations in terms of recursive languages similar to the ones in case of disjunctive sequences. To this end we derive some fundamental properties of the functions translating a base r-expansion of a real $\alpha \in [0,1]$ into α.

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Following Jürgensen, Shyr and Thierrin [Jürgensen et al. 83, Jürgensen and Thierrin 83] we say that an infinite sequence is *disjunctive* if it contains any (finite) word, or, equivalently, if any word appears in the sequence infinitely many times. “Disjunctivity” is a natural qualitative property; it is weaker, than the property of “normality” (introduced by Borel; see, for instance, [Calude 94, Hertling 96]).

In this paper we derive some properties of the set of all disjunctive sequences (ω-words). Here we focus on the properties in relation to the Chomsky and arithmetical hierarchies of sets of ω-words (ω-languages) (see e.g. [Thomas 90, Staiger 97]) and also on topological and information theoretic properties.

As is well known (see [Oxtoby 71]) a set is large in topological sense if is of second Baire category, and it is large in measure theoretic sense if it has nonzero measure. The latter implies also largeness in information theoretic sense.

Using a characterization of the set of disjunctive ω-words by means of so-called regular ω-languages, that is, ω-languages definable by finite automata, we show that the set of disjunctive ω-words is large as well in sense of category as in sense of measure.

Disjunctivity can be carried over to real numbers interpreting an ω-word ξ as an expansion of the number $0.\xi$ in a certain positional system. It appears that, under this interpretation, a property (e.g. disjunctivity, Borel normality etc.) may depend on the particular base chosen. It was shown in [Cassels 59, Schmidt 60] that Borel normality and, implicitly, disjunctivity are not invariant under changes of the base r. For detailed information see [Hertling 96]. In contrast to the preceding cases, randomness and Kolmogorov complexity of real numbers, which are defined also via expansions, are base invariant properties (see [Calude and Jürgensen 94, Hertling and Weihrauch 98, Staiger 02]).

Real numbers disjunctive with respect to all bases are called *absolutely disjunctive*. Utilizing a specific translation technique based on considerations in [Staiger 02] we prove in a constructive way that the set of absolutely disjunctive reals is large in the sense of Baire category.

The paper is organized as follows. After presenting the necessary background on ω-words and ω-languages in Section 1, we derive in recursion theoretic properties of the ω-language of disjunctive sequences, D. Then, Section 3 is devoted to topological and information theoretic properties of disjunctive sequences. Her we show a close relationship between the subword complexity of ω-words and the entropy of finite-state ω-languages, and we prove that the set D is large with respect to category and measure.

In the fourth section we turn to the consideration of real numbers. We investigate in detail topological properties of the canonical mapping $\nu_r(\xi) := 0.\xi$ describing reals in terms of r-ary expansions. These properties allow us to translate the results of the previous section to the case of real numbers.

The final section, on the one hand, deals with constructive results in base r to base b conversion yielding a description of the set of absolutely disjunctive real numbers in terms of recursive languages, and, on the other hand, presents an example showing that the class of finite-state ω-languages is not invariant under base conversion.

1 Notation and Preliminaries

By $\mathbb{N} = \{0,1,2,\ldots\}$ we denote the set of natural numbers. In order to treat arbitrary finite alphabets we let $X_r := \{0,\ldots,r-1\}$ be our alphabet of cardinality $\#X_r = r$, $r \in \mathbb{N}, r \geq 2$. If there is no danger of confusion we will omit the subscript and simply write X for alphabets.

By X^* we denote the set of finite strings (words) on X, including the *empty* word e. We consider also the space X^ω of infinite sequences (ω-words) over X. For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$.

We extend the operations * and $^\omega$ to arbitrary subsets $W \subseteq X^*$ in the usual way

:

$$W^* := \bigcup_{n\in\mathbb{N}} W^n \quad \text{where} \quad W^0 := \{e\}\text{ , and}$$
$$W^\omega := \left\{w_0\cdot w_1\cdots w_i\cdot\ldots : i\in\mathbb{N}\wedge w_i\in W\setminus\{e\}\right\}$$

is the set of ω-words in X^ω formed by concatenating members of W.
We will refer to subsets of X^* and X^ω as languages or ω-languages, respectively.

By "$\sqsubseteq$" we denote the prefix relation, that is, $w\sqsubseteq\eta$ if and only if there is an η' such that $w\cdot\eta'=\eta$, and $\mathbf{A}(\eta):=\{w : w\in X^*\wedge w\sqsubseteq\eta\}$ and $\mathbf{A}(B):=\bigcup_{\eta\in B}\mathbf{A}(B)$ are the languages of finite prefixes of η and B, respectively.

The set of subwords (infixes) of $\eta\in X^*\cup X^\omega$ will be denoted by $\mathbf{T}(\eta):=\{w : w\in X^*\wedge\exists v(vw\sqsubseteq\eta)\}$.

In the study of ω-languages it is useful to consider X^ω as a metric space (Cantor space) with the following metric.

$$\rho(\eta,\xi)=\inf\ \{(\#X)^{-|w|} : w\sqsubset\eta\wedge w\sqsubset\xi\} \tag{1}$$

It is easily verified that ρ is indeed a metric which, in addition it satisfies the ultrametric inequality.

$$\rho(\zeta,\xi)\le\max\{\rho(\zeta,\eta),\rho(\xi,\eta)\} \tag{2}$$

Open (in view of Eq. (2) they are simultaneously closed) balls in this space (X^ω,ρ) are the sets $w\cdot X^\omega$. Then open sets in X^ω are of the form $W\cdot X^\omega$ where $W\subseteq X^*$. From this it follows that a subset $F\subseteq X^\omega$ is *closed* if and only if $\mathbf{A}(\xi)\subseteq\mathbf{A}(F)$ implies $\xi\in F$.

The *closure* of a subset $F\subseteq X^\omega$ in Cantor space, that is, the smallest closed subset of X^ω containing F is denoted by $\mathscr{C}(F)$. One has $\mathscr{C}(F)=\{\xi : \mathbf{A}(\xi)\subseteq\mathbf{A}(F)\}$.

Having defined open and closed sets in X^ω, we proceed to the next classes of the Borel hierarchy (see [Kuratowski 66]):

$\mathbf{F}_\sigma$ is the set of countable unions of closed subsets of X^ω,

$\mathbf{G}_\delta$ is the set of countable intersections of open subsets of X^ω.

$\mathbf{F}_{\sigma\delta}$ is the set of countable intersections of $\mathbf{F}_\sigma$-subsets of X^ω,

$\mathbf{G}_{\delta\sigma}$ is the set of countable unions of $\mathbf{G}_\delta$-subsets of X^ω, and so on.[1]

[1] Borel classes are also defined for larger countable ordinals than natural numbers, but since we will not need higher level Borel classes, we refer the interested reader to some textbook on topology, as e.g. [Kuratowski 66].

For $W \subseteq X^*$ the *δ-limit* of W, W^δ, consists of all infinite sequences of X^ω that contain infinitely many prefixes in W,

$$W^\delta = \{\xi \in X^\omega : \#(\mathbf{A}(\xi) \cap W) = \infty\}.$$

For $\mathbf{G}_\delta$-sets we have the following characterization via languages (see [Thomas 90, Staiger 97]).

Theorem 1 *In Cantor space, a subset $F \subseteq X^\omega$ is a $\mathbf{G}_\delta$-set if and only if there is a language $W \subseteq X^*$ such that $F = W^\delta$.*

The preceding theorem explains also why W^δ is called the δ-limit of the language W.

For $B \subseteq X^* \cup X^\omega$ we define the *state B/w* of B generated by the word $w \in X^*$ as $B/w = \{b : wb \in B\}$. A set B is called *finite-state* if its set of states $\{B/w : w \in X^*\}$ is finite.

A finite-state language $W \subseteq X^*$ is also called *regular*.[2] An ω-language F is called *regular* provided there is an $n \in \mathbb{N}$ and regular languages W_i, V_i $(1 \leq i \leq n)$ such that

$$F = \bigcup_{i=1}^{n} W_i V_i^\omega. \tag{3}$$

Along with the Cantor spaces (X_r^ω, ρ), $r \in \mathbb{N}$, $r \geq 2$, we consider the unit interval $[0,1]$ with the usual metric. For $\eta \in X_r^* \cup X_r^\omega$ we denote by $\nu_r(\eta) := 0.\eta$ the real number with (finite or infinite) base r expansion η. The surjective mapping $\nu_r : X_r^\omega \to [0,1]$ is continuous and nearly one-to-one[3]. In particular, all mappings ν_b are one-to-one outside the set of all ultimately periodic ω-words, $Ult := \{w \cdot v^\omega : w, v \in X_b^*\}$.

2 The ω-Language of Disjunctive Sequences

In this section we will present a few simple properties of the ω-language of all disjunctive sequences over X, $D = \{\xi : \mathbf{T}(\xi) = X^*\}$. Some of the results in this section are reported in [Calude et al. 97].

2.1 Basic Properties

From the very definition of disjunctive sequences we obtain

$$D = \bigcap_{w \in X^*} X^* w X^\omega. \tag{4}$$

[2]In fact, regularity of $W \subseteq X^*$ is usually defined in a different way, but it is well known that a language W is regular if and only if it is finite state.

[3]Only real numbers of the form $i \cdot r^{-j}$, $1 \leq i < r^j$ have two base r expansions.

Our next lemma shows that D is an example of a finite-state ω-language which is not a regular one.

Lemma 2 ([Jürgensen and Thierrin 83]) *The ω-language D is finite-state but not regular.*

Proof. Since $w\xi \in D$ if and only if $\xi \in D$, the ω-language D satisfies $D/w = D$, for all $w \in X^*$. Thus D has only a single state. Next, D is nonempty and does not contain an ultimately periodic sequence wv^ω. Following Eq. (3) the ω-language D cannot be regular. ❑

The representation of Eq. (4) verifies that D is a $\mathbf{G}_\delta$-set in Cantor space. Next we are going to show that its topological complexity cannot be decreased. To this end we quote Theorem 21 from [Staiger 83].

Theorem 3 *If $F \subseteq X^\omega$ is finite-state and simultaneously an $\mathbf{F}_\sigma$- and a $\mathbf{G}_\delta$-set, then F is regular.*

Combining Theorem 3 with Lemma 2 and Eq. (4) we get:

Proposition 4 *In Cantor space, D is not an $\mathbf{F}_\sigma$-set.*

2.2 Recursion Theoretic Properties of D

We turn our attention to recursion theoretic properties of D. To this end we introduce the first classes of the arithmetical hierarchy of ω-languages. As usual we say that an ω-language $E \subseteq X^\omega$ is Π_1*-definable* provided E is representable in the form

$$E = \{\xi \in X^\omega : \forall w (w \sqsubseteq \xi \Rightarrow w \in W_E)\}, \tag{5}$$

where $W_E \subseteq X^*$ is a recursive language, and we say that an ω-language $F \subseteq X^\omega$ is Π_2*-definable* provided F is representable in the form

$$F = \{\xi \in X^\omega : \forall w (w \in X^* \to \exists u (u \sqsubseteq \xi \wedge (w,u) \in M_F))\}, \tag{6}$$

where M_F is a recursive subset of $X^* \times X^*$.

It is well-known that in Cantor space, Π_1-definable ω-languages are closed sets and Π_2-definable ω-languages are $\mathbf{G}_\delta$-sets.

Lemma 5 *The ω-language of all disjunctive sequences D is Π_2-definable.*

Proof. We have $D = \{\xi \in X^\omega : \forall w \exists v (vw \sqsubseteq \xi)\}$. So it suffices to put $M_D = \{(w, vw) : w, v \in X^*\}$ in Eq. (6). ❑

In [Staiger 97], Lemma 2.12, it is shown that an ω-language $F \subseteq X^\omega$ is Π_2-definable if and only if there is a recursive language $W \subseteq X^*$ such that $F = W^\delta$. In case of D we construct W_D explicitly.

Proposition 6 *Let*

$$W_D = \{wx : w \in X^* \wedge x \in X \wedge \exists n(n \leq |w| + 1 \wedge \mathbf{T}(wx) \supseteq X^n \wedge \mathbf{T}(w) \not\supseteq X^n)\}.$$

Then W_D is a recursive language and $D = W_D^{\delta}$.

Proof. It is obvious that W_D is recursive. Let ξ be a sequence such that $\mathbf{T}(\xi) = X^*$. Then for every $n \geq 1$ there is a shortest prefix $w_n \sqsubseteq \xi$ such that $\mathbf{T}(w_n) \supseteq X^n$. Thus $\{w_n : n \geq 1\}$ is an infinite subset of W_D. The converse implication follows from the observation that if $u, v \in W_D$ and $u \sqsubset v$, then $X^m \subseteq \mathbf{T}(u)$ implies $X^m \subseteq \mathbf{T}(v)$, and there is an $n \in \mathbb{N}$ satisfying $\mathbf{T}(u) \not\supseteq X^n \subseteq \mathbf{T}(v)$. ❑

3 Complexity and Density

In this section we relate disjunctivity to an information theoretic size measure called entropy and to (topological) density in Cantor space.

3.1 Density and Baire Category

We first introduce the concept of topological density and Baire category for complete metric spaces $(\mathscr{X}, \rho)$ such as the Cantor space X^{ω} or the unit interval $[0,1]$.

A subset $M \subseteq \mathscr{X}$ is called *dense* in $\mathscr{X}$ provided its closure $cl(M)$ is the whole space $\mathscr{X}$. A set $M \subseteq \mathscr{X}$ is *nowhere dense* in provided its closure $cl(M)$ does not contain a nonempty open subset.

As for any nonempty open subset $\mathscr{O} \subseteq \mathscr{X}$ such that $\mathscr{O} \not\subseteq cl(M_2)$ the inclusion $cl(M_1) \cup cl(M_2) = cl(M_1 \cup M_2) \supseteq \mathscr{O}$ implies $cl(M_1) \supseteq \mathscr{O} \setminus cl(M_2)$, where $\mathscr{O} \setminus cl(M_2) \neq \emptyset$ is open, the family of nowhere dense sets is closed under finite union.

A set M is of *first Baire category* iff it is a countable union of nowhere dense sets, otherwise it is of *second Baire category*. The complements of sets of first Baire category are called *residual*.

It holds the Baire category theorem.

Theorem 7 (Baire category theorem) *If a subset M of $(\mathscr{X}, \rho)$ is of first category then $\mathscr{X} \setminus M$ is dense.*

This theorem has several consequences (see [Kuratowski 66, Oxtoby 71]).

Property 8 *If $\mathscr{O}$ is a nonempty open subset of $(\mathscr{X}, \rho)$ and $M \subseteq \mathscr{X}$ is of first category then*

$$cl(\mathscr{O} \setminus M) \supseteq \mathscr{O}. \tag{7}$$

Particular properties hold also for $\mathbf{G}_{\delta}$-sets.

Property 9 *1. A $\mathbf{G}_\delta$-set $M \subseteq \mathscr{X}$ of first Baire category is already nowhere dense.*

2. A subset $M \subseteq \mathscr{X}$ is residual iff it contains a dense $\mathbf{G}_\delta$-set.

3.2 Subword Complexity

Next we investigate a concept of complexity of infinite sequences ξ which is intimately related to disjunctive ω-words. This concept is based solely on the sets of subwords $\mathbf{T}(\xi)$. It turns out that the subword complexity $\tau(\xi)$ of a word $\xi \in X^\omega$ is also closely related to the entropy and density of the ω-languages containing ξ.

For a language $W \subseteq X^*$ let

$$\mathsf{s}_W(n) = \# W \cap X^n$$

be its *structure function* (cf. [Kuich 70]), and

$$\mathsf{H}_W = \limsup_{n\to\infty} \frac{\log_{\# X}(1+\mathsf{s}_W(n))}{n}$$

be its *entropy*. Define $\mathsf{s}_F = \mathsf{s}_{\mathbf{A}(F)}$ and $\mathsf{H}_F = \mathsf{H}_{\mathbf{A}(F)}$, for $F \subseteq X^\omega$.

The entropy of languages is monotone with respect to "$\subseteq$". Moreover, it has the following properties.

$$\begin{aligned} \mathsf{H}_{W\cup V} &= \mathsf{H}_{W\cdot V} = \max\left\{\mathsf{H}_W, \mathsf{H}_V\right\} \text{ whenever } W\cdot V \neq \emptyset\,, && (8)\\ \mathsf{H}_{W/w} &\leq \mathsf{H}_W, \text{ and} && (9)\\ \mathsf{H}_{\mathscr{C}(F)} &= \mathsf{H}_F\,. && (10) \end{aligned}$$

We call $\tau(\xi) = \mathsf{H}_{\mathbf{T}(\xi)}$ the *subword complexity* of the word $\xi \in X^\omega$. From the obvious relation $\#(\mathbf{T}(\xi) \cap X^{n+m}) \leq \#(\mathbf{T}(\xi)\cap X^n)\cdot\#(\mathbf{T}(\xi)\cap X^m)$ we obtain the following property of $\tau(\xi)$.

$$\tau(\xi) = \lim_{n\to\infty} \frac{\log_{\# X} \mathbf{s}_{\mathbf{T}(\xi)}(n)}{n} = \inf\left\{\frac{\log_{\# X} \mathbf{s}_{\mathbf{T}(\xi)}(n)}{n} : n \in \mathbb{N} \wedge n \geq 1\right\} \qquad (11)$$

The subword complexity of an ω-word ξ is closely connected to the entropy of finite-state ω-languages containing ξ.

Proposition 10 *Let F be a finite-state ω-language. Then $\tau(\xi) \leq \mathsf{H}_F$, for every $\xi \in F$.*

Proof. Since $\bigcup_{w\in\mathbf{A}(F)} F/w$ is a finite union and $\mathbf{T}(\xi) = \bigcup_{w\sqsubseteq\xi} \mathbf{A}(\{\xi\}/w) \subseteq \bigcup_{w\in\mathbf{A}(F)} F/w$ for $\xi \in F$, in view of Eqs. (8) and (9) we have $\tau(\xi) \leq \max_{w\in\mathbf{A}(F)} \mathsf{H}_{F/w}$. The assertion follows from $\mathsf{H}_{F/w} \leq \mathsf{H}_F$. ❑

Consequently, if $\xi \in F$, $\mathsf{H}_F < 1$ and F is finite-state then ξ is not disjunctive. We are going to prove that the converse is also true.

Theorem 11 *An ω-word $\xi \in X^\omega$ is disjunctive iff $\xi \in F$ implies $\mathsf{H}_F = 1$ for every finite-state ω-language $F \subseteq X^\omega$*

Proof. One direction is explained above. Let $\tau(\xi) < 1$. Then there is a word $w \notin \mathbf{T}(\xi)$. Consequently, $\xi \in X^\omega \setminus X^* \cdot w \cdot X^\omega \subseteq (X^{|w|} \setminus \{w\})^\omega$. Now the assertion follows from $\mathsf{H}_{(X^{|w|}\setminus\{w\})^\omega} = \frac{\log_{\#X}(\#X^{|w|}-1)}{\#X^{|w|}} < 1$. ❑

3.3 Entropy and Density in X^ω

The final part of this section brings together all three introduced concepts, density, entropy and subword complexity.

We start with special properties of ω-languages nowhere dense in X^ω. Here, in particular, a nowhere dense set contains no subset of the form wX^ω. This condition can be reformulated as follows.

Property 12 *A set $F \subseteq X^\omega$ is nowhere dense if and only if for every $w \in X^*$ there is a $v_w \in X^*$ such that $wv_wX^\omega \cap F = \emptyset$.*

Remark 1 *If the ω-language $F \subseteq X^\omega$ satisfies the condition $\forall w(F/w \subseteq F)$ then, apparently, we may choose all words v_w to coincide with $v := v_e$.*

For finite-state ω-languages we obtain the following connection between entropy an density.

Lemma 13 ([Staiger 85]) *A finite-state ω-language $F \subseteq X^\omega$ is nowhere dense iff $\mathsf{H}_F < 1$.*

Proof. Clearly, as $\mathsf{H}_{wX^\omega} = 1$, Eq. (10) shows that $\mathsf{H}_F = 1$ if F is not nowhere dense.

Conversely, let F be nowhere dense and finite-state. Then all states F/w and also $F' := \bigcup_{w\in X^*} F/w$ (as a finite union of nowhere dense sets) are nowhere dense. Since $F' \supseteq F'/w$ is satisfied for all $w \in X^*$, according to Remark 1 there is a word $v \in X^*$ such that $F \subseteq F' \subseteq X^\omega \setminus X^*vX^\omega \subseteq (X^{|v|} \setminus \{v\})^\omega$. Now, as in the proof of Theorem 11 one obtains $\mathsf{H}_F < 1$. ❑

The final theorem of this section summarizes properties relating (topological) density, entropy and disjunctivity. For a more detailed exposition see [Staiger 98].

Theorem 14 *Let $F \subseteq X^{\omega}$ be closed and finite-state. Then the following properties are equivalent.*

1. $\mathsf{H}_F < 1$
2. *F is nowhere dense.*
3. *$\tau(\xi) < 1$ for all $\xi \in F$.*

Proof. The equivalence of 1 and 2 is Lemma 13, and Proposition 10 shows that 1 implies 3.

In order to prove that 3 implies 2, we observe that in view of Property 9 a closed set is nowhere dense iff it is of first Baire category. Then this implication is a part of Theorem 3 in [Staiger 98]. ❏

Our theorem shows that $X^{\omega} \setminus D$ is the union of all finite-state nowhere dense ω-languages[4]. Utilizing Eq. (4) and the proof of Theorem 11 we obtain the following representation of D in terms of special regular nowhere dense ω-languages.

$$X^{\omega} \setminus D = \bigcup_{w \in X^*} (X^{\omega} \setminus X^* w X^{\omega}) = \bigcup_{w \in X^*} (X^{|w|} \setminus \{w\})^{\omega} . \tag{12}$$

3.4 Measure

We add a short consideration of the measure of D. The representation of Eq. (12) yields the following short proof that D has measure one for all non-vanishing product measures, thus establishing that D is also a large set in sense of measure.

Here, as usual, we refer to a measure $\overline{\mu}$ on X^{ω} as a *non-vanishing product measure* derived from a measure $\mu : X \to (0,1)$, where $\sum_{a \in X} \mu(a) = 1$, provided $\overline{\mu}(waX^{\omega}) = \mu(a) \cdot \overline{\mu}(wX^{\omega})$ for all $w \in X^*$ and $a \in X$. We obtain immediately

$$\overline{\mu}((X^{|w|} \setminus \{w\})^{\omega}) = 0 \tag{13}$$

for all non-vanishing product measures. This yields the announced result via Eq (12)

Lemma 15 *Let $\overline{\mu}$ be a non-vanishing product measure on X^{ω}. Then $\overline{\mu}(D) = 1$.*

[4] Since the closure of a nowhere dense ω-language F, $\mathscr{C}(F)$, is again nowhere dense, we can drop the requirement "closed".

4 Disjunctive Real Numbers

So far we considered only disjunctive ω-words. In this section we consider the real numbers which have in a positional system a notation which is disjunctive. In particular, we are interested in the set of absolutely disjunctive reals,

From the considerations in the preceding sections we know that for a particular base r the set of disjunctive ω-words D_r is residual, moreover, its complement $X_{r^\omega} \setminus D_r$ is the countable union of all (closed) nowhere dense finite-state ω-languages.

We translate these results by the natural interpretation of ω-words in X_r^ω as the r-ary positional notation of real numbers to the unit interval $[0,1]$. As a result we obtain that the set $\mathscr{D}$ of real numbers having disjunctive expansions with respect to all bases $b \in \mathbb{N}, b \geq 2$ is also a large set in the sense of category, although its complement has not the nice characterization as developed in Eq. (12) for D_r. Similar results were obtained in [Calude and Zamfirescu 95, Calude et al. 97] but without using the results on disjunctive ω-words and the translation results derived below.

4.1 ω-Words as Expansions of Real Numbers

First we investigate in more detail some fundamental properties of the mapping $\nu_r : X_r^\omega \to [0,1]$. A simple property is

$$|\nu_r(\xi) - \nu_r(\eta)| \leq \rho(\xi, \eta) \tag{14}$$

Thus ν_r is a continuous mapping. Since the spaces X_r^ω and $[0,1]$ are compact, we have also the following.

Property 16 *The mapping ν_r satisfies the identity*

$$\nu_r(\mathscr{C}(F)) = cl(\nu_r(F)), \tag{15}$$

where $F \subseteq X_r^\omega$ and $cl(M)$ denotes the closure of the set $M \subseteq [0,1]$.

Next, consider the *ambiguity set* of ν_r, $A_{\nu_r} := \{\xi : \exists \eta(\eta \neq \xi \wedge \nu_r(\xi) = \nu_r(\eta))\}$. It holds

Lemma 17 *A_{ν_r} is of first Baire category.*

Another property deals with the images of balls wX^ω in X^ω. To this end let $\mathscr{I}(F)$ be the interior (largest open subset) of $F \subseteq X^\omega$ and let $int(M)$ be the interior of $M \subseteq [0,1]$. Here the identity $cl(int(\nu_r(wX^\omega))) = \nu_r(wX^\omega)$ is obvious. Moreover, we have the following.

Lemma 18 *If $F \subseteq X_r^\omega$ then $\nu_r(\mathscr{I}(F)) \subseteq cl(int(\nu_r(F)))$.*

Proof. We have $\mathscr{I}(F) = W \cdot X_r^\omega$ for $W := \{w : wX_r^\omega \subseteq F\}$. Then $\nu_r(\mathscr{I}(F)) = \nu_r\left(\bigcup_{w\in W} wX_r^\omega\right) = \bigcup_{w\in W} \nu_r(wX_r^\omega)$, whence, in view of $cl(int(\nu_r(wX_r^\omega))) = \nu_r(wX_r^\omega)$, the inclusion

$$\nu_r(\mathscr{I}(F)) = \Big(\bigcup_{w\in W} cl(int(\nu_r(wX_r^\omega)))\Big) \subseteq cl\Big(int\Big(\nu_r\Big(\bigcup_{w\in W} wX_r^\omega\Big)\Big)\Big) \subseteq cl(int(F)))$$

follows. ❏

It should be mentioned that Property 16, Lemma 17 and Lemma 18 hold likewise for the d-dimensional version of ν_r mapping the space $(\underbrace{X_r \times \cdots \times X_r}_{d\ times})^\omega$ to the d-dimensional unit cube $[0,1]^d$ considered in [Jürgensen and Staiger 01].

Observe that for $w \in (\underbrace{X_r \times \cdots \times X_r}_{d\ times})^*$ the set $\nu_r(w \cdot (\underbrace{X_r \times \cdots \times X_r}_{d\ times})^\omega)$ is a so-called mesh cube in an $r^{-|w|}$-coordinate mesh of the unit cube $[0,1]^d$ (see Section 3.1 in [Falconer 90]). An $r^{-|w|}$-coordinate mesh in the unit interval is simply the collection of all intervals $\nu_r(wX_r^\omega)$ where $|w| = n$. These observations will turn out to be useful in Section 5.2.

4.2 Translation Results for General Metric Spaces

Next, we consider the relative density of a subset $F \subseteq X_r^\omega$ or $M \subseteq [0,1]$. Our aim is to show that F and $\nu_r(F)$ are either both nowhere dense or both not. In contrast to the preliminary version [Staiger 01] we are going to show this in the more general context of complete metric spaces $(\mathscr{X}, \rho)$.

To this end let $f : (\mathscr{X}, \rho) \to (\mathscr{X}', \rho')$ be a mapping between the spaces $\mathscr{X}$ and $\mathscr{X}'$, and let $\mathscr{A}_f := \{\mathbf{x} : \mathbf{x} \in \mathscr{X} \wedge \exists \mathbf{y}(\mathbf{x} \neq \mathbf{y} \wedge f(\mathbf{x}) = f(\mathbf{y}))\}$ be the ambiguity set of f.

Lemma 19 *Let $(\mathscr{X}, \rho)$ be a complete metric space, $f : \mathscr{X} \to \mathscr{X}'$ and let $\mathscr{A}_f$ be of first Baire category. Then $\mathscr{O} \subseteq f^{-1} \circ f(\mathscr{F})$ implies $\mathscr{O} \subseteq \mathscr{F}$ whenever $\mathscr{O} \subseteq \mathscr{X}$ is open and $\mathscr{F} \subseteq \mathscr{X}$ is closed.*

Proof. Since f is one-to-one on $\mathscr{X} \setminus \mathscr{A}_f$ we have $\mathscr{O} \setminus \mathscr{A}_f \subseteq \mathscr{F}$. Then, in view of Eq. (7), $cl(\mathscr{O} \setminus \mathscr{A}_f) \supseteq \mathscr{O}$, and the assertion follows. ❏

We get our first result for mappings satisfying an identity analogous to Eq. (15), $cl(f(M)) = f(cl(M))$, where cl denotes the closure as well in $(\mathscr{X}, \rho)$ as in $(\mathscr{X}', \rho')$. Such functions are referred to as *closed mappings*. Closed mappings are also continuous, hence the preimage $f^{-1}(\mathscr{O}')$ of an open subset $\mathscr{O}' \subseteq \mathscr{X}'$ is again open.

Theorem 20 *Let $f : \mathscr{X} \to \mathscr{X}'$ be a closed mapping and let $\mathscr{A}_f$ be of first Baire category. If $\mathscr{F}$ is nowhere dense, then $f(\mathscr{F})$ is also nowhere dense.*

Proof. Assume $f(\mathscr{F})$ to be not nowhere dense in $f(\mathscr{X}')$. Then there is an open set $\mathscr{O}' \subseteq \mathscr{X}'$ such that $cl(f(\mathscr{F})) \supseteq f(\mathscr{X}') \cap \mathscr{O}' \neq \emptyset$. Hence, $f^{-1}(cl(f(\mathscr{F}))) = f^{-1}(f(cl(\mathscr{F}))) \supseteq f^{-1}(\mathscr{O}') \neq \emptyset$. According to Lemma 19 we have $cl(\mathscr{F}) \supseteq f^{-1}(\mathscr{O}')$, and $\mathscr{F}$ contains a nonempty open subset. ❏

Similar to *cl* let *int* denote the interior operation in both spaces.

Theorem 21 *If $f : \mathscr{X} \to \mathscr{X}'$ is closed and satisfies the inequality $f(int(M)) \subseteq cl(int(f(M)))$ for arbitrary $M \subseteq \mathscr{X}$ then the preimage $f^{-1}(\mathscr{F}')$ of a nowhere dense set $\mathscr{F}' \subseteq \mathscr{X}'$ is nowhere dense in $\mathscr{X}$.*

Proof. Assume $\mathscr{F} := f^{-1}(\mathscr{F}')$ to be not nowhere dense. Then there is a nonempty open set $\mathscr{O} \subseteq cl(f^{-1}(\mathscr{F}'))$. Hence, $\emptyset \neq int(cl(f^{-1}(\mathscr{F}')))$. Applying the inequality and the fact that f is closed yields $f(int(cl(f^{-1}(\mathscr{F}')))) \subseteq cl(int(cl(f(f^{-1}(\mathscr{F}')))))$. But $cl(int(cl(f(f^{-1}(\mathscr{F}'))))) \subseteq cl(\mathscr{F}')$, and thus $cl(\mathscr{F}')$ contains the nonempty open set $int(cl(f(f^{-1}(\mathscr{F}'))))$. ❏

In the previous Section 4.1 we have seen that the function ν_r satisfies the hypotheses of Theorems 20 and 21. Thus we obtain the following.

Theorem 22 *Let $F \subseteq X_r^\omega$. Then F is nowhere dense, iff $\nu_r(F)$ is nowhere dense.*

4.3 Absolutely Disjunctive Reals

Now, we use our translation results to show that the set of absolutely disjunctive reals is residual.

As a corollary to Theorem 22 we obtain immediately the following.

Lemma 23 *The set of all absolutely disjunctive real numbers is a residual $\mathbf{G}_\delta$-set in $[0,1]$, and for every $r \in \mathbb{N}$, $r \geq 2$ the set $\nu_r^{-1}(\mathscr{D})$ is a residual $\mathbf{G}_\delta$-set in X_r^ω.*

Proof. Since $\mathscr{D} = \bigcap_{r \geq 2} \nu_r(D_r)$, it suffices to show that every $\nu_r(D_r)$ is a residual $\mathbf{G}_\delta$-set in $[0,1]$. Following Eq. (12), $X_r^\omega \setminus D_r$ is an $\mathbf{F}_\sigma$-set of first Baire category and the ambiguity set of ν_r, A_{ν_r}, satisfies $A_{\nu_r} \subseteq X_r^\omega \setminus D_r$. Thus we have $\nu_r(D_r) = [0,1] \setminus \nu_r(X_r^\omega \setminus D_r)$. In view of Theorem 22 and Property 16, the image $\nu_r(X_r^\omega \setminus D_r)$ is also an $\mathbf{F}_\sigma$-set of first Baire category.

The second assertion follows from the fact that ν_r is continuous and $\mathscr{D}$ is a residual $\mathbf{G}_\delta$-set. ❏

5 Base conversion

In the previous sections we have seen that the sets D_r of disjunctive sequences in X_r^ω as well as the preimages $\nu_r^{-1}(\mathscr{D})$ of the set of absolutely disjunctive reals are residual $\mathbf{G}_\delta$-sets. From the papers [Cassels 59, Schmidt 60, Hertling 96] it is known that the property to be disjunctive is not invariant under base conversion $\nu_b \circ \nu_r^{-1} : X_r^\omega \setminus A_{\nu_r} \to X_b^\omega$.

Thus $D_r \supset \nu_r^{-1}(\mathscr{D})$, and the constructive description of the set D_r obtained in Proposition 6 cannot be carried over directly to $\nu_r^{-1}(\mathscr{D})$. The aim of this section is to give a constructive description of the set of r-ary expansions of all absolutely disjunctive reals, $\nu_r^{-1}(\mathscr{D})$.

We conclude this section using Theorem 14 and the non-invariance of disjunctive reals under base conversion to show that the class of finite-state ω-languages is also not invariant under base conversion.

5.1 The Constructivity of $\mathscr{D}$

It is known that, in general, it is not possible to continuously (as a mapping from Cantor space (X_r^ω, ρ) to (X_b^ω, ρ)) convert base r expansions of real numbers to base b expansions. Even, if we exclude the set of ultimately periodic ω-words, *Ult*, from this conversion. More specifically, the size of the smallest ball $v \cdot X_b^\omega$ for which $\nu_b(v \cdot X_b^\omega) \supseteq \nu_r(w \cdot X_r^\omega)$ does not only depend on the length of w. For instance, if $b = 10$, $r = 2$ and $w \sqsubset \xi$ with $\nu_2(\xi) = \frac{1}{5}$ we have $|v| = 0$ independently of w.

In [Staiger 02] it is explained that admitting a small ambiguity in our conversion we can solve the problem in the following way:
For every $w \in X_r^*$ we find in a constructive way at most two words $v_-, v_+ \in X_b^*$ both of length $\lfloor |w| \cdot \log_b r \rfloor$ such that $\nu_b(v_- \cdot X_b^\omega) \cup \nu_b(v_+ \cdot X_b^\omega) \supseteq \nu_r(w \cdot X_r^\omega)$. Moreover, if two words are really necessary, then $v_-, v_+ \in X_b^*$ can be chosen in such a way that v_+ is the successor of v_- in the quasilexicographical ordering of X_b^*.

Thus we define the following computable mappings $h_+^{r\to b}, h_-^{r\to b} : X_r^* \to X_b^*$ such that $h_-^{r\to b}(w) := v_-$ and $h_+^{r\to b}(w) := v_+$ where the computation of v_- and v_+ is carried out as described above or in [Staiger 02].

The following lemma shows that the sets of infixes of successors in quasilexicographical ordering do not deviate too much from each other.

Lemma 24 *Let $w, w' \in X_r$ and let w' be the successor of w in the quasilexicographical ordering of X_r^*. Then $|(\mathbf{T}(w) \setminus \mathbf{T}(w')) \cap X_r^n| \le n+1$ and $|(\mathbf{T}(w') \setminus \mathbf{T}(w)) \cap X_r^n| \le n+1$ for all $n \le |w|$.*

Proof. In case $|w| < |w'|$ we have $w = (r-1)^{|w|}$ and $w' = 0^{|w|+1}$ whence $|(\mathbf{T}(w) \cap X_r^n| = |(\mathbf{T}(w') \cap X_r^n| = 1$, and the assertion is trivially satisfied.

Assume $|w| = |w'| = l$. Since w' is the successor of w, there is a common prefix $u \sqsubset w$, $u \sqsubset w'$ such that $w = u \cdot x \cdot (r-1)^{l-|u|-1}$ and $w' = u \cdot (x+1) \cdot 0^{l-|u|-1}$.

The infixes of w and w' can be estimated as

$$\begin{aligned}
\mathbf{T}(w) \cap X_r^n &\subseteq (\mathbf{T}(u) \cap X_r^n) \cup \{u_i x (r-1)^{n-i-1} : 0 \leq i < n\} \cup \{(r-1)^n\},\\
\mathbf{T}(w') \cap X_r^n &\subseteq (\mathbf{T}(u) \cap X_r^n) \cup \{u_i (x+1) 0^{n-i-1} : 0 \leq i < n\} \cup \{0^n\}
\end{aligned}$$

where u_i is the suffix of length i of u. ❑

Now we state our main theorem proving the constructivity of the set $\mathscr{D}$ of absolutely disjunctive reals in recursion theoretic terms analogous to the one of D_r given in Section 2.2.

Theorem 25 *For every $r \in \mathbb{N}$, $r \geq 2$ there is a recursive language W_r such that $\nu_r(W_r^\delta) = \mathscr{D}$.*

Proof. The following explicit construction of the language W_r is similar to the one in the proof of Proposition 6.

$$\begin{aligned}
W_r \ := \ & \Big\{ wx : w \in X_r^* \wedge x \in X_r \wedge \exists n \Big(n \leq |w| \wedge \\
& \forall b \left(2 \leq b \leq n \rightarrow \mathbf{T}(h_+^{r\to b}(wx)) \supseteq X_b^n \vee \mathbf{T}(h_-^{r\to b}(wx)) \supseteq X_b^n \right) \wedge \\
& \exists b \left(2 \leq b \leq n \ \wedge \ \mathbf{T}(h_+^{r\to b}(w)) \not\supseteq X_b^n \wedge \mathbf{T}(h_-^{r\to b}(w)) \not\supseteq X_b^n \right) \Big) \Big\}
\end{aligned}$$

Let $\xi \in W_r^\delta$ and let $\eta_b := \nu_b^{-1}(\nu_r(\xi))$. It suffices to show that $\eta_b \in X_b^\omega$ is disjunctive. By construction, for all sufficiently large $n \in \mathbb{N}$ there is a prefix $w_n \sqsubset \xi$ such that $h_+^{r\to b}(w_n)$ or $h_-^{r\to b}(w_n)$ is a prefix of η_b and $\mathbf{T}(h_+^{r\to b}(w_n)) = X_b^n$ or $\mathbf{T}(h_-^{r\to b}(w_n)) = X_b^n$.

In view of Lemma 24 this implies $\#\mathbf{T}(h_+^{r\to b}(w)) \cap X_b^n \geq b^n - (n+1)$ and $\#\mathbf{T}(h_-^{r\to b}(w)) \cap X_b^n \geq b^n - (n+1)$. Accordingly, $\#(\mathbf{T}(\eta_b) \cap X_b^n) \geq \frac{\log_b(b^n-n-1)}{n}$ for infinitely many $n \in \mathbb{N}$, and Eq. (11) proves $\tau(\eta_b) = 1$.

Conversely, let $\xi \in X_r^\omega$ and $\nu_r(\xi) \in \mathscr{D}$. Then every $\eta_b := \nu_b^{-1}(\nu_r(\xi))$ is disjunctive. It suffices to prove that for every $n \in \mathbb{N}$ there is a prefix $w_n \sqsubset \xi$ such that $\forall b \left(2 \leq b \leq n \rightarrow \mathbf{T}(h_+^{r\to b}(w)) \supseteq X_b^n \vee \mathbf{T}(h_-^{r\to b}(w)) \supseteq X_b^n\right)$. (Then w_n has a prefix in $v_n \in W_r$ which has $|v_n| \geq n$, thus $W_r \cap \mathbf{A}(\xi)$ is infinite.)

Choose $n \in \mathbb{N}$ and for every $\eta_b, 2 \leq b \leq n$, a prefix $v_b \sqsubset \eta_b$ such that $\mathbf{T}(v_b) \supseteq X_b^n$. If $u_{n,b} \sqsubset \xi$ has length $|u_{n,b}| \geq |v_b| \cdot \log_r b$ then $h_+^{r\to b}(u_{n,b}) \sqsupseteq v_b$ or $h_-^{r\to b}(u_{n,b}) \sqsupseteq v_b$. Now define w_r to be the longest of the words $u_{n,b}, 2 \leq b \leq n$. ❑

5.2 Non-invariance of Finite-State ω-Languages

This last part uses results of the non-invariance of disjunctivity under base conversion to show that the class of finite-state ω-languages is also not invariant under base conversion.

In order to achieve this goal we introduce the concept of box-counting dimension in $[0,1]$ (see [Falconer 90]). To this end let $\mathcal{N}_\varepsilon(M)$ be the smallest number of intervals of length ε (balls of diameter ε) which cover $m \subseteq [0,1]$. The *upper box-counting dimension* of $M \subseteq [0,1]$ is defined as

$$\overline{\mathrm{bdim}}\, M := \limsup_{\varepsilon\to 0} \frac{\log_r \mathcal{N}_\varepsilon(M)}{-\log_r \varepsilon}.$$

This formula, in some sense, resembles the definition of the entropy of ω-languages. If we define $\mathcal{N}'_{r^{-n}}(M)$ as the smallest number of intervals of the form $[\frac{i}{r^{-n}}, \frac{i+1}{r^{-n}}]$ (mesh cubes in an r^{-n}-coordinate mesh as in [Falconer 90]) which cover M, we observe that

$$\mathcal{N}'_{r^{-n}}(\nu_r(F)) \leq \mathsf{s}_F(n) \leq 3 \cdot \mathcal{N}'_{r^{-n}}(\nu_r(F)). \tag{16}$$

Proof. On the one hand, the intervals $\nu_r(wX_r^\omega)$, $|w| = n$ where $wX_r^\omega \cap F \neq \emptyset$ cover $\nu_r(F)$ and are of the required form. Thus the first inequality is evident.

On the other hand, if $w \in \mathbf{A}(F)$ and $|w| = n$ there are at most three intervals of the form $[\frac{i}{r^{-n}}, \frac{i+1}{r^{-n}}]$ not disjoint to $\nu_r(wX_r^\omega)$. Thus at least $\frac{1}{3}\cdot \mathsf{s}_F(n)$ mesh cubes are necessary to cover $\nu_r(F)$, which yields the second inequality. ❑

Eq. (16) yields

$$\mathsf{H}_F = \limsup_{n\to\infty} \frac{\log_r \mathcal{N}'_{r^{-n}}(\nu_r(F))}{n} \text{ for } F \subseteq X_r^\omega. \tag{17}$$

From the results of Section 3.1 of [Falconer 90] we have the following.

Lemma 26 $\overline{\mathrm{bdim}}\, M = \limsup_{n\to\infty} \frac{\log_r \mathcal{N}'_{r^{-n}}(M)}{n}$

As a consequence of Eq. (17) and Lemma 26 we obtain that the entropy of ω-languages is invariant under base conversion.

Lemma 27 *Let $F \subseteq X_r^\omega$, $E \subseteq X_b^\omega$ and $\nu_r(F) = \nu_b(E)$. Then $\mathsf{H}_F = \mathsf{H}_E$.*

Now Theorem 6 of [Hertling 96] and Theorem 14 yield the announced example.

Example 28 Let $F := \{0,1\}^\omega \subseteq X_4^\omega$. Then $\mathsf{H}_F = \frac{1}{2}$. Hence $\tau(\xi) \leq \frac{1}{2}$ for all $\xi \in F$.

Consider $\xi_0 \in F$ where $\nu_4(\xi_0) = \sum_{i\in\mathbb{N}} 4^{-i!-i}$. Theorem 6 of [Hertling 96] shows that $\eta_0 \in X_6^\omega$ with $\nu_4(\xi_0) = \nu_6(\eta_0)$ is disjunctive. Hence $\tau(\eta_0) = 1$.

Now assume $E := \nu_6^{-1}(\nu_4(F))$ to be finite-state. Since $\eta_0 \in E$, this implies $\mathsf{H}_E = 1$, contradicting Lemma 27. ❑

Remark 2 *Unfortunately, the example presented above does not represent a "nice" subset of the unit interval* $[0,1]$. *In contrast to the situation of the previous example, it is shown in* [Jürgensen and Staiger 01] *that the class of finite-state (in fact, regular) ω-languages $F \subseteq (\underbrace{X_r \times \cdots \times X_r}_{d \text{ times}})^\omega$ encoding geometric figures is invariant under base conversion.*

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