Smooth Affine Manifolds which are not Complete Intersections

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1. Introduction

Let $\{y_1, \dots, y_k\}$ $(k > 0)$ be an arbitrary finite set of points on a smooth complex, irreducible projective curve X of genus $g > 1$. Then for each $i \in \{1, 2, \dots, k\}$ there is a point $x_i \in X$ nearly y_i such that the affine curve $Y = X \setminus \{x_1, \dots, x_k\}$ cannot be embedded as a closed curve into the affine plane $\mathbb{C}^2$.
Actually, the canonical divisor on such an embedding $\varphi(Y)$ would be equal to zero, i.e., there would then be a rational differential form ω on $\varphi(Y)$ without zeros or poles on $\varphi(Y)$. Therefore the support of the divisor $K = (\varphi^*(\omega))$ on X would be a subset of $\{x_1, \dots, x_k\}$. On the other hand Z. Jelonek proved that one can choose the points x_i in such a manner that $\{x_1, \dots, x_k\}$ does not contain the support of any canonical divisor of X [3].
Using the same approach, we will show that these affine curves Y cannot be embedded as closed smooth complete intersection curves into the affine space $\mathbb{C}^n$ of arbitrary dimension.

Moreover, for the higher-dimensional case Jelonek proved the following: If H denotes an irreducible hypersurface in $\mathbb{P}^n$ of degree $d > n+1$ then the rational, smooth affine variety $Y = \mathbb{P}^n \setminus H$ cannot be embedded as a closed hypersurface into $\mathbb{C}^{n+1}$. Indeed, the canonical divisor on such a hypersurface $\varphi(Y) \subset \mathbb{C}^{n+1}$ would again be equal to zero and the divisor $K = (\varphi^*(\omega))$ on $\mathbb{P}^n$ would be equal to $m \cdot H$ with $m \in \mathbb{Z}$. On the other hand $K \sim -(n+1) \cdot E \sim m \cdot H \sim m \cdot d \cdot E$ is impossible because of $d > n+1$. In this paper instead of $\mathbb{P}^n$ we take any smooth, complex, irreducible projective manifold $X \subseteq \mathbb{P}^n$ with the canonical line bundle $\omega_X = \mathcal{O}_X(n_X)$ ($n_X \in \mathbb{Z}$, $n_X \neq 0$). Moreover, an arbitrary closed affine, complete intersection will play the role of the affine hypersurface $\varphi(Y) \subset \mathbb{C}^{n+1}$.

2. Canonical divisor on affine complete intersections

Lemma 1. *The canonical divisor K_Y on any n-dimensional affine closed, smooth, irreducible, ideal-theoretically complete intersection $Y \subseteq \mathbb{C}^N$ is equal to zero, i.e., there is a rational differential form ω of degree n on Y without zeros or poles on Y. In particular, the canonical divisor K_Y on any n-dimensional affine closed, smooth, irreducible hypersurface $Y \subseteq \mathbb{C}^{n+1}$ is equal to zero.*

Proof. Let $I(Y) = (F_1, F_2, \dots , F_{N-n})$ be the ideal of Y and let $D_{k_1,k_2,\dots,k_{N-n}} = \det(\frac{\partial F_i}{\partial x_{k_j}})$ be a minor of the Jacobian matrix with $N-n$ rows. Since Y is smooth the subsets $U_{k_1,k_2,\dots,k_{N-n}} := \{D_{k_1,k_2,\dots,k_{N-n}} \neq 0\} \subseteq Y$ define an open covering of Y .

For each permutation $p = \begin{pmatrix} 1 & 2 & \dots & n & n+1 & n+2 & \dots & N \\ j_1 & j_2 & \dots & j_n & k_1 & k_2 & \dots & k_{N-n} \end{pmatrix}$ the coordinate-functions $x_{j_1}, x_{j_2}, \dots , x_{j_n}$ are local parameters on $U_{k_1,k_2,\dots,k_{N-n}}$ [4].

We consider the differential forms

$$\omega_{j_1,j_2,\dots,j_n,k_1,k_2,\dots,k_{N-n}} := \frac{1}{D_{k_1,k_2,\dots,k_{N-n}}} \cdot dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_n}$$

and in particular $\omega = \omega_{1,2,\dots,N} = \dfrac{1}{D_{n+1,n+2,\dots,N}} \cdot dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$.

Then the equations $\omega_{j_1,j_2,\dots,j_n,k_1,k_2,\dots,k_{N-n}} = \operatorname{sgn}(p) \cdot \omega_{1,2,\dots,N}$ are easy to prove. Therefore, we have the rational differential form $\omega = \omega_{1,2,\dots,N}$ without poles or zeros on Y. □

3. The case of curves

Let X be a smooth, irreducible, complex, projective curve. Since X is a real differentiable surface, a (complete) Riemannian metric d on X exists.

Theorem 1. *Let X be a smooth, irreducible, complex, projective curve of genus $g > 1$ and let $y_1, y_2, \dots, y_k$ be arbitrary points of X $(k > 0)$. Then for each $\varepsilon > 0$ the points $x_1, x_2, \dots, x_k$ exist on X with $d(x_i, y_i) < \varepsilon$ $\forall i$ such that the affine curve $Y = X \setminus \{x_1, x_2, \dots, x_k\}$ cannot be embedded as a closed, complete intersection curve into the affine space $\mathbb{C}^n$ of any dimension. In particular the curve Y cannot be embedded as a closed curve into the affine plane $\mathbb{C}^2$.*
(On the other hand it is well known that Y can be embedded as a closed curve into the affine space $\mathbb{C}^3$.)

Proof. For each canonical divisor K on X we have $\deg K = 2g - 2 > 0$ and $\mathrm{Supp}(K) \neq \emptyset$. It is sufficient to prove that for each $\varepsilon > 0$ the points $x_1, x_2, \dots, x_k$ exist on X with $d(x_i, y_i) < \varepsilon$ $\forall i$ such that there is no canonical divisor K on X with $\mathrm{Supp}(K) \subseteq \{x_1, x_2, \dots, x_k\}$. Actually, let us suppose that an embedding φ of $Y = X \setminus \{x_1, x_2, \dots, x_k\}$ as a closed, complete intersection curve into any affine space exists. Owing to lemma 1 the canonical divisor of $\varphi(Y)$ would be equal to zero, i.e., we have a rational differential form ω on $\varphi(Y)$ without zeros or poles on $\varphi(Y)$. Then the support of the canonical divisor $K = (\varphi^*(\omega))$ on X would be a subset of $\{x_1, x_2, \dots, x_k\}$.
Let us follow Jelonek's proof by induction on k [3]: In the case $k = 1$ assume that $x, x' \in X$ are points and K and K' are canonical divisors on X with $K = (2g-2) \cdot x$ and $K' = (2g-2) \cdot x'$. Then we have $(2g-2) \cdot (x - x') \sim 0$, i.e., $x - x'$ is an element of the group $Pic^0(X)$ whose order divides $2g - 2$. Since $Pic^0(X)$ containes only a finite number of elements of any fixed order (cf. [3]) there is at most a finite number of divisor classes containing those points $x \in X$. Since X is not rational the linear equivalence $y \sim y'$ of two points $y, y' \in X$ implies $y = y'$ (cf. [4]), i.e., there is at most a finite number of points $x \in X$ with the property that $\{x\}$ is the support of a canonical divisor K of X. Therefore in the ε-neighbourhood of y_1 we have a point $x_1 \in X$ such that $Y = X \setminus \{x_1\}$ cannot be embedded as a closed, complete intersection curve into the affine space of any

dimension.

Now assume $k \geq 2$. By induction condition, we have points $x_1, x_2, \dots, x_{k-1} \in X$ with $d(x_i, y_i) < \varepsilon \quad \forall i$ such that there is no canonical divisor K on X with $\mathrm{Supp}(K) \subseteq \{x_1, x_2, \dots, x_{k-1}\}$. We fix these point x_i of X and consider the set

$S := \{x \in X \ : \text{there is a canonical divisor } K \text{ with } \mathrm{Supp}(K) \subseteq \{x_1, x_2, \dots, x_{k-1}, x\}\}$.

For every point $x \in S$ one has integers $m_1, m_2, \dots, m_{k-1}, m \in \mathbb{Z}$ with

$K \sim m_1 \cdot x_1 + m_2 \cdot x_2 + \ldots + m_{k-1} \cdot x_{k-1} + m \cdot x$,

$m = 2g - 2 - m_1 - m_2 - \ldots - m_{k-1}$ and

$m \neq 0$ by induction condition.

For a second point $x' \in S$ attached to the same integers $m_1, m_2, \dots, m_{k-1}, m \in \mathbb{Z}$ we have $K \sim m_1 \cdot x_1 + m_2 \cdot x_2 + \ldots + m_{k-1} \cdot x_{k-1} + m \cdot x'$ and therefore $m \cdot (x - x') \sim 0 \quad (m \neq 0)$. This again implies that there is at most a finite number of points $x \in S$ attached to these fixed integers $m_1, m_2, \dots, m_{k-1}, m \in \mathbb{Z}$. Therefore, the set S is cauntable and the ε-neighbourhood of y_k contains a point $x_k \in X \setminus S$. That means, $Y = X \setminus \{x_1, x_2, \dots, x_k\}$ cannot be embedded as a closed, complete intersection curve into the affine space of any dimension. $\square$

Corollary 1. *For every smooth, irreducible, complex, affine curve X of genus $g > 1$ a diffeomorphic smooth, irreducible, complex, affine curve Y exists which cannot be embedded as a closed, complete intersection curve into any affine space.*

4. Higher dimensional manifolds

Theorem 2. *Let $X \subseteq \mathbb{P}^N$ be a smooth, irreducible, complex, n-dimensional, projective manifold $(n \geq 2)$ with the canonical line bundle $\omega_X = \mathcal{O}_X(n_X)$ $(n_X \in \mathbb{Z} \ , \ n_X \neq 0)$ and let $H_d \subset \mathbb{P}^N$ be an irreducible hypersurface of degree $d > 1$ such that $D = X \cap H_d \sim d \cdot E$ becomes a prime divisor on X.*
If d does not divide the integer n_X then the smooth affine manifold $Y = X \setminus H_d$ cannot be embedded as a closed complete intersection into any affine space.
(On the other hand it is well known that Y can be embedded as a closed manifold into the affine space $\mathbb{C}^{2n+1}$.)

Proof. The Veronese embedding (d-uple embedding) shows that $X \setminus H_d$ is affine closed in the affine space of dimension $\binom{N+d}{d} - 1$ [5].
The projective manifold X can be for instance a smooth projective complete intersection of multidegree $(m_1, m_2, \dots, m_{N-n})$ whose canonical line bundle ω_X is equal to $\mathcal{O}_X(n_X)$ with $n_X = \sum m_i - N - 1$.
Supposing that there is an embedding φ of Y as a closed complete intersection into any affine space. Then the canonical divisor of $\varphi(Y)$ would be equal to zero, i.e., a rational differential form ω on $\varphi(Y)$ exists without zeros or poles on $\varphi(Y)$. Therefore the support of the canonical divisor $K = (\varphi^*(\omega)) \sim n_X \cdot E$ on X would be linearly equivalent to $k \cdot D \sim k \cdot d \cdot E$ with $k \in \mathbb{Z}$ (where E denotes the hyperplane section divisor on X). But from the linear equivalence $n_X \cdot E \sim k \cdot d \cdot E$ ensues that d divides n_X . $\square$

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