Global Well-Posedness and Stability of a Partial Integro-Differential Equation with Applications to Viscoelasticity

S.-O. Londen H. Petzeltová* J. Prüss

Institute of Mathematics, Helsinki University of Technology, FIN-02150 Espoo, Finland

Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic

FB Mathematik und Informatik, Martin-Luther-Universität, Halle-Wittenberg, Theodor-Lieser-Str. 5, D-60120 Halle, Germany

1 Introduction

In this paper we consider the equation

$$u_{tt}(t,x) = \int_0^t a(t-s)u_{txx}(s.x)ds + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t b(t-s)(g(u_x(s,x))_x \,\mathrm{d}s + f(t,x), \tag{1.1}$$

$t > 0,\ x \in (0,1)$, with boundary conditions

$$u(t,0) = u(t,1) = 0,\ t > 0, \tag{1.2}$$

and initial values

$$u(0,x) = u^0(x),\ u_t(0,x) = u^1(x). \tag{1.3}$$

Here $a(t)$ and $b(t)$ are creep kernels which behave like

$$a(t) \sim c_a t^{\alpha-1}, \quad b(t) \sim c_b t^{\beta-1}, \quad t \to 0,$$

where $\alpha, \beta \in (0,1]$, $c_a, c_b > 0$ are constants and $\alpha < \beta$. Thus the order of the time-derivative of the linear term is strictly higher than that of the nonlinear term. The nonlinearity g belongs to the class $C^2(\mathbb{R})$ with normalization $g(0) = 0$ and roughly speaking its primitive $G(s) = \int_0^s g(r)dr$ satisfies

$$G(r) \sim c_1 r g(r) \sim c_2 |r|^{m+1}, \quad |r| \to \infty,$$

with positive constants $c_1, c_2 > 0$ and $m \geq 1$; note that g' may change sign. The data $f(t,x)$, $u^0(x)$ and $u^1(x)$ are given. The formulation of the precise assumptions is contained in the next section.

The motivation for this problem comes from the theory of viscoelastic material behaviour. As is well-known the governing equation in this field is *balance of momentum*, which in one dimension reads

$$u_{tt} + \sigma_x = h,$$

*Work supported by the grant A1019002 of GA AV ČR

where u denotes displacement, σ means stress, and h accounts for body forces. This equation is supplemented by a constitutive law which reflects the properties of the material under consideration, the *stress-strain relation*. Here we consider the law

$$\sigma(t,x) = \int_0^\infty a(s) u_{xt}(t-s,x)ds + \frac{\partial}{\partial t}\int_0^\infty b(s) g(u_x(t-s,x))ds, \quad t \in \mathbb{R}.$$

Inserting this relation into balance of momentum we obtain (1.1), where f contains h as well as the stress history up to time $t = 0$.

Let us discuss a classical limiting case of our problem, i.e. the case $\alpha = 0$, $\beta = 1$, $f = 0$. This yields the problem

$$\begin{aligned} &u_{tt} = a_0 u_{xxt} + g(u_x)_x, \quad t > 0,\ x \in (0,1), \\ &u(t,0) = u(t,1) = 0, \quad t > 0, \\ &u(0,x) = u^0(x), \quad u_t(0,x) = u^1(x), \quad x \in (0,1). \end{aligned} \tag{1.4}$$

In case $a_0 = 0$ this is a quasilinear wave equation which in general does not admit global classical solutions because of breaking of waves, even if $g'(s) > 0$ for $s \in \mathbb{R}$. On the other hand, for $a_0 > 0$ it is an old result of GREENBERG [6] that (1.4) does admit global classical solutions, without any restriction on the growth-order m of g. In other words, the damping induced by the parabolic term u_{xxt} in (1.4) is strong enough to control the "destructive" effects of the quasilinear term $g(u_x)_x$.

For problem (1.1) the situation is similar. The linear term $a * u_{xxt}$ is trying to stabilize while the nonlinear term $(b * g(u_x)_x)_t$ wants to destroy the regularity of the solution. Which part is going to win? In general the answer is not known. Here we can add the following result. If the order of the time-derivative in the linear term (=1-α) is greater than the order of the time-derivative in the nonlinear term (=1-β), then the linear term $a * u_{txx}$ wins. Thus our key condition is $\beta > \alpha$. In addition, we require the problem to be *subcritical* in the sense that

$$\frac{m-1}{m+1} < 2\frac{\beta-\alpha}{1+\alpha}.$$

We show below that if these two assumptions hold, then the problem (1.1) is globally well-posed; not only in a weak but also in a stronger L_p-setting, and that the L_∞-norm of u_x stays globally bounded in time.

Furthermore we show global asymptotic stability of the trivial solution, provided the steady state problem

$$g(u_x)_x = 0, \quad x \in (0,1),\ u(0) = u(1) = 0,$$

admits only the trivial solution - which is equivalent to the sign-condition $sg(s) > 0$ for $s \neq 0$ - and provided the linearization at the trivial solution is asymptotically stable; the latter means

$$g'(0) > 0, \quad b_\infty := \lim_{t\to\infty} b(t) > 0.$$

Concerning the literature related to (1.1), we observe that the special case (1.4) was considered by GREENBERG [6], PEGO [10], BALL et al. [1]. Global existence of weak solutions has been obtained in GRIPENBERG, LONDEN and PRÜSS [7] under essentially the same assumptions as here, however, no uniqueness was shown and no global bounds were obtained. The special case $a(t) = t^{\alpha-1}$, $b(t) = 1$ was studied in the recent paper PETZELTOVÁ and PRÜSS [11], where the same results as in this paper are proved for this special case, assuming a stronger condition on α and m.

The plan for this paper is as follows. In Section 2 we introduce the precise assumptions and state the main results, Theorem 1 on global well-posedness, and Theorem 2 on asymptotic stability. The remaining sections are devoted to the prooofs of Theorems 1 and 2.

Acknowledgement. This work was carried out while H.P. and J.P. were visiting at Helsinki University of Technology. Thanks go to the Department of Mathematics and to the Academy of Science, Finland, for financial support. We appreciate the warm hospitality during this visit.

2 Assumptions and main results

Recall that a function $a:(0,\infty)\to\mathbb{R}$ is called **k-monotone** if a is $k-1$-times differentiable,

$$(-1)^j a^{(j)}(t)\geq 0 \quad \text{for all } t>0,\ j=0,\ldots,k-1,$$

and $(-1)^{k-1}a^{(k-1)}(t)$ is nonincreasing on $(0,\infty)$. A function $c:(0,\infty)\to\mathbb{R}$ is called **completely positive**, if there is $l_0\geq 0$ and a 1-monotone function $l:(0,\infty)\to\mathbb{R}$ such that

$$l_0c(t)+c*l(t)=l_0c(t)+\int_0^t c(t-s)l(s)ds=1, \quad t>0.$$

For properties of k-monotone resp. completely positive functions we refer to Prüss [13].

The kernels a and b in equation (1.1) are assumed to satisfy the following assumptions.

$$(\mathbf{a})\begin{cases} a\in L^1_{loc}(\mathbb{R}^+),\ a \textit{ is } 3-\textit{monotone},\\ |\arg\widehat{a}(\lambda)|\leq\theta_a<\frac{\pi}{2},\ \mathrm{Re}\ \lambda>0,\\ |\widehat{a}(\lambda)|\geq\frac{C_a}{1+|\lambda|^\alpha},\ \mathrm{Re}\ \lambda>0,\ \textit{for some } \alpha\in[0,1). \end{cases} \tag{2.1}$$

$$(\mathbf{b})\begin{cases} b(t)=b_1+b_\infty,\ b_\infty>0,\\ b_1\in L^1(\mathbb{R}^+),\ b_1 \textit{ is } 3-\textit{monotone and completely positive},\\ |\widehat{b}_1(\lambda)|\leq\frac{C_b}{1+|\lambda|^\beta},\ \mathrm{Re}\ \lambda>0,\ \textit{for some } \beta\in(\alpha,1]. \end{cases} \tag{2.2}$$

$$(\mathbf{c})\begin{cases} \textit{There exists } c\in L^1_{loc}(\mathbb{R}^+),\ \textit{1-monotone}\\ \textit{and completely positive, such that} \quad b=a*c. \end{cases} \tag{2.3}$$

Note that, due to $\beta>\alpha$, (2.1) and (2.2) already imply the existence of an absolutely continuous function c such that $a*c=b$; in (2.3) we assume that a and b are connected through a kernel having the indicated structural properties. Typical examples of kernels satisfying these conditions are

$$a(t)=t^{\alpha-1}e^{-\omega_a t}, \quad b_1(t)=t^{\beta-1}e^{-\omega_b t},\ t>0,$$

where $0\leq\alpha<\beta\leq 1$, $\omega_a\geq 0$, $\omega_b>0$.

The basic assumptions on the nonlinearity that allow for global existence are

$$(\mathbf{g})\begin{cases} g\in C^2(\mathbb{R}),\ g(0)=0,\ G(s):=\int_0^s g(r)dr.\\ \textit{There are constants } \eta>0,\ C_g>0,\ p>1 \textit{ such that}\\ g'(s)\geq-\eta,\ s\in\mathbb{R},\quad \frac{1}{p}<2\frac{\beta-\alpha}{1+\alpha},\\ |g(s)|+|g(s)|^p/|s|^p\leq C_g(1+G(s)). \end{cases} \tag{2.4}$$

To obtain bounded solutions we need in addition

$$\textit{there are } \varepsilon > 0,\ R \geq 0, \textit{such that } g(s)s \geq \varepsilon |s|^2 \textit{ for } |s| \geq R. \tag{2.5}$$

$$\lim_{|s|\to\infty} \frac{|g(s)|^p}{|s|^p G(s)} = 0. \tag{2.6}$$

Condition (2.5) with $R = 0$ allows us to prove the convergence result.

Note that in case $G(s) \sim c_1 s g(s) \sim c_2 |s|^{m+1}$ as $|s| \to \infty$ the growth conditions on g are satisfied if $m \geq 1$ with $p < \frac{m+1}{m-1}$. This is the subcriticality assumption mentioned in Section 1.

For the function f we assume

$$(\mathbf{f}) \quad f \in L^1_{loc}(\mathbb{R}^+; L^q(0,1)); \quad q \geq \max\Big\{2, \frac{1+\alpha}{1-\alpha}\Big\}. \tag{2.7}$$

We will require $f \in L^1(\mathbb{R}^+; L^q(0,1))$ for boundedness and convergence of solutions.

As to the initial data, we suppose

$$(\mathbf{IC}) \begin{cases} u^0 \in X = W_0^{1,\infty}(0,1) \cap W^{s,q}(0,1); \textit{ with } s > 2 - \frac{2\beta}{1+\alpha} \textit{ if } \beta \leq \frac{1+\alpha}{2}, \\ u^1 \in L^q(0,1), \end{cases} \tag{2.8}$$

The assumptions on the kernels have the following consequences that will be used in the sequel.

- *There exist $k_0 \geq 0$, and $k_1 \in L^1(\mathbb{R}^+)$, 1-monotone and completely positive, such that*

$$k_0 b(t) + b * k_1(t) = 1, \quad t > 0. \tag{2.9}$$

- *Let $l = k_0 a + a * k_1$. Then l is 1-monotone, completely positive, and*

$$l \in L^1_{loc}(\mathbb{R}^+),\ \mathrm{Re}\ \widehat{l}(\lambda) > 0 \textit{ for all } \lambda \textit{ with } \mathrm{Re}\ \lambda \geq 0. \tag{2.10}$$

- *Given $\theta \in (0,1)$, for any $\eta > 0$ there exists $C_\eta > 0$ such that*

$$\eta \mathrm{Re}\ \widehat{b}_1(\lambda) \leq \theta \mathrm{Re}\ \widehat{a}(\lambda) + C_\eta \mathrm{Re}\ \widehat{l}(\lambda),\ \mathrm{Re}\ \lambda > 0. \tag{2.11}$$

- *Assumption (2.1) implies that a is 2-regular, i.e., there is $C > 0$ such that*

$$|\lambda^n \hat{a}^{(n)}(\lambda)| \leq c|\widehat{a}(\lambda)| \textit{ for } \mathrm{Re}\ \lambda > 0,\ n = 1, 2, \tag{2.12}$$

- *and of strictly positive type, i.e., there exist $a_1 \in L^1(\mathbb{R}^+)$, $a_1 \not\equiv 0$, $C > 0$ such that*

$$\int_0^t (a * \phi)(s)\phi(s)\ \mathrm{d}s \geq C \int_0^t |a_1 * \phi(s)|^2\ \mathrm{d}s \textit{ for all } t > 0,\ \phi \in C(\mathbb{R}^+). \tag{2.13}$$

Characterization and some useful properties of kernels satisfying (2.1)-(2.2) are given in Appendix.

We define operators A_p, $1 \leq p \leq \infty$, by means of

$$\mathcal{D}(A_p) = W^{2,p}(0,1) \cap W_0^{1,p}(0,1), \quad A_p \phi = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi = -D^2\phi. \tag{2.14}$$

Note that $\mathcal{D}(A_p^\sigma) = H_0^{2\sigma,p}(0,1)$ for $1 + 1/p > 2\sigma > 1/p$, in particular, $\mathcal{D}(A_p^{1/2}) = H_0^{1,p}(0,1)$. Here $H_0^{2\sigma,p}(0,1)$ denote the Bessel-potential spaces on $(0,1)$ with zero trace.

The following theorems contain the main results of this paper.

Theorem 2.1 (Global existence).

Let the assumptions (2.1)-(2.4), (2.7), (2.8) be satisfied. Then there exists a unique weak solution u to the problem (1.1)-(1.3) such that

$$u \in C(\mathbb{R}^+; W_0^{1,\infty}(0,1)) \cap C^1(\mathbb{R}^+; L^q(0,1)), \tag{2.15}$$

$$u_x \in C^{\frac{1-\alpha}{2}}(\mathbb{R}^+, L^q(0,1)). \tag{2.16}$$

If, in addition, $u^0 \in \mathcal{D}(A_q)$, $u^1 \in \mathcal{D}(A_q^{\frac{1}{2}})$, $f \in L^1_{loc}(\mathbb{R}^+; \mathcal{D}(A_q^{\frac{1}{2}}))$ *then*

$$u \in C(\mathbb{R}^+; \mathcal{D}(A_q)) \cap C^1(\mathbb{R}^+; \mathcal{D}(A_q^{\frac{1}{2}})).$$

We want to mention that for global existence we may allow for $b_\infty = 0$ and $b_1 \notin L^1(\mathbb{R}^+)$, however, then we will not be able to get global bounds on the solution.

Theorem 2.2 (Asymptotic stability).

Let the assumptions of Theorem 2.1 be satisfied. Moreover, let (2.5) and (2.6) hold and $f \in L^1(\mathbb{R}^+; L^q(0,1))$. *Then the unique global solution of problem (1.1)-(1.3) is bounded and satisfies*

$$u \in BUC(\mathbb{R}^+; W_0^{1,\infty}(0,1)) \cap BUC^1(\mathbb{R}^+; L^2(0,1)).$$

If (2.5) holds with $R = 0$, *then* $u(t) \to 0$ *in* $W^{1,\infty}(0,1)$, *and* $u_t(t) \to 0$ *in* $L^2(0,1)$, *as* $t \to \infty$.

If the initial values satisfy in addition, for some $r \in [1, \infty)$, $u^0 \in W^{2,r}(0,1)$, $u^1 \in W_0^{1,r}(0,1)$, $f \in L^1(\mathbb{R}^+; \mathcal{D}(A_r^{\frac{1}{2}}))$, *then*

$$u \in BUC(\mathbb{R}^+; W^{2,r}(0,1)) \cap BUC^1(\mathbb{R}^+; W_0^{1,r}(0,1)),$$

and $u(t) \to 0$ *in* $W^{2,r}(0,1)$, $u_t(t) \to 0$ *in* $W^{1,r}(0,1)$ *as* $t \to \infty$.

The remaining part of the paper is devoted to the proofs of these theorems.

3 Preliminaries

Let us briefly outline the proof of local existence. First, the equation is rewritten in the form (3.2) and solved by the variation of parameters formula. This gives the equation (3.4). To apply the contraction method to this equation, we require some rather technical estimates on the solution operators. In particular, observe that in order to obtain the key estimates (3.16), (3.18) we need to work with the fractional powers A^δ (Lemma 3.2).

Let $\|\cdot\|_p$ denote the norm in $L^p(0,1)$, $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(0,1)$, $D = d/dx$, $\dot{v} = v_t$, $k * v = \int_0^t k(t-s)v(s)\,\mathrm{d}s$; C stands for a generic positive constant. The Laplace transform of a function f we denote by $\widehat{f}$ or $\mathcal{L}(f)$.

We introduce equation (1.1) in a functional analytic setting.

It is well known that A_p defined by (2.14) is an invertible, sectorial operator on $L^p(0,1)$, A_2 is selfadjoint, positive definite in $L^2(0,1)$, $\sigma(A_p) \subset [\pi^2, +\infty)$ consists only of simple eigenvalues and there is $M(\theta) > 0$ such that

$$\|A_p^\delta(z+A_p)^{-1}\|_{\mathcal{B}(L^p)} \le \frac{M}{|\pi^2+z|^{1-\delta}},\ 0\le\delta\le 1,\ |\arg(z+\pi^2)|\le \pi-\theta,\ \theta>0. \tag{3.1}$$

For simplicity we drop the subscript p where there is no danger of confusion. Also, we will frequently write L^p instead of $L^p(0,1)$.

We reformulate (1.1) as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}u + a * \frac{\mathrm{d}}{\mathrm{d}t}Au = \frac{\mathrm{d}}{\mathrm{d}t}b * Dg(Du) + f \tag{3.2}$$

and proceed as in [11].

The Laplace transform of the equation (3.2) yields

$$\lambda(\lambda+\widehat{a}(\lambda)A)\widehat{u}(\lambda) = (\lambda+\widehat{a}(\lambda)A)u^0 + u^1 + \lambda\widehat{b}(\lambda)\mathcal{L}(Dg(u_x)) + \widehat{f}(\lambda).$$

Let S be the solution of the operator equation

$$\dot{S}\ + a * AS = 0,\ S(0) = I.$$

By (**a**), $\widehat{a}(\lambda)$ can be extended to Re $\lambda \ge 0$, $\lambda \ne 0$, and for λ in this set we have $\lambda[\widehat{a}(\lambda)]^{-1} \notin (-\infty, 0]$. Therefore,

$$\widehat{S}(\lambda) = (\lambda+\widehat{a}(\lambda)A)^{-1},\ \mathrm{Re}\ \lambda\ge 0,\ \lambda\ne 0$$

is well defined. Denote

$$S_1 = 1 * S, \quad S_b = b * S. \tag{3.3}$$

Then the solution u can be expressed by means of the variation of parameters formula

$$u(t) = u^0 + S_1(t)u^1 + S_b * Dg(Du)(t) + S_1 * f(t). \tag{3.4}$$

To prove existence of weak solutions we need to commute the operators S_b and D. To do so, we realize that

$$A = DA^NQ,$$

where $Q : L^p(0,1) \to W^{1,p}(0,1)$ is defined by

$$Q\phi(x) = \int_0^x y\phi(y)\ \mathrm{d}y - \int_x^1 (1-y)\phi(y),$$

and A^N denotes the Laplace operator with Neumann boundary conditions defined on the space of zero mean functions, i.e.

$$\mathcal{D}(A_p^N) = \{\phi\in W^{2,p}|\ \int_0^1 \phi(x)\ \mathrm{d}x = 0, \phi_x(0)=\phi_x(1)=0\},\ A_p^N\phi = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi = -D^2\phi.$$

We have

$$DQ\phi = \phi,\ QD\phi = \phi - \int_0^1 \phi(x)\ \mathrm{d}x = P\phi,$$

where P denotes the projector of L^p to the subspace of zero mean functions. This implies

$$S(t) = DS^N(t)Q,\ S_b(t) = DS_b^N(t)Q = Db * S^N(t)Q,$$

S^N being the solution operator corresponding to A^N, $\widehat{S}^N(\lambda) = (\lambda + \widehat{b}(\lambda)A^N)^{-1}$. So we can express u as follows:

$$u = u^0 + S_1 u^1 + DS_b^N * Pg(Du) + S_1 * f. \tag{3.5}$$

As all formulae, estimates and proofs for A, S and A^N, S^N are exactly the same, we omit the script ^{N} throughout the further text.

Now, we derive estimates for the solution operators S, S_1, S_b that we will need in the existence proofs.

The condition $(2.1)_2$ implies existence of a positive number $\theta < \frac{\pi}{2}$ such that

$$|\arg(\lambda/\widehat{a}(\lambda))| \leq \pi - \theta \ \text{ when } |\arg(\lambda)| \leq \frac{\pi}{2}.$$

Moreover, according to (3.1) and $(2.1)_3$, we have

$$\Big\|\frac{1}{\widehat{a}(\lambda)}\Big(\frac{\lambda}{\widehat{a}(\lambda)} + A\Big)^{-1}\Big\|_{\mathcal{B}(L^p)} \leq \frac{C}{1+|\lambda|}.$$

We use the L^2- theory of the Laplace transform and express $S(t)$ by integrating over the imaginary axis.

$$S(t) = \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} e^{\lambda t}(\lambda + \widehat{a}(\lambda)A)^{-1}\, \mathrm{d}\lambda. \tag{3.6}$$

Set

$$H_\delta(\lambda) = A^\delta \widehat{S}(\lambda) = \frac{1}{\widehat{a}(\lambda)}A^\delta\Big(\frac{\lambda}{\widehat{a}(\lambda)} + A\Big)^{-1}, \ H = H_0 = \widehat{S}(\cdot),$$

and estimate $\|A^\delta S(t)\|_{\mathcal{B}(L^p)}$. To this end, we shall take advantage of the following result on vector-valued analytic functions in $\mathbb{C}^+ = \{\lambda \in \mathbb{C};\ \mathrm{Re}\ \lambda > 0\}$. (See [12, Theorem 1]).

Lemma 3.1 *Suppose $h : \mathbb{C}^+ \to X$ is holomorphic and satisfies*

$$\|h(\lambda)\| + \|\lambda h'(\lambda)\| \leq c|\lambda|^{-\gamma}, \ \mathrm{Re}\ \lambda > 0,$$

for some $\gamma \in (0,1)$. Then there is a continuous function $v : (0,\infty) \to X$ such that $\widehat{v}(\lambda) = h(\lambda)$ for $\mathrm{Re}\lambda > 0$. In addition, the following estimates hold.

$$\|v(t)\| \leq Mt^{\gamma-1}, \ t > 0,$$

$$\|tv(t) - sv(s)\| \leq M|t-s|^\gamma, \ 0 < s < t < \infty.$$

Here M denotes a constant depending only on γ and c.

We have for $\delta < 1/(1+\alpha)$

$$\begin{aligned}\|H_\delta(\lambda)\phi\|_p = \|\frac{1}{\widehat{a}(\lambda)}A^\delta(\frac{\lambda}{\widehat{a}(\lambda)} + A)^{-1}\phi\|_p &\leq C\frac{1}{|\widehat{a}(\lambda)|}\frac{1}{|1+\lambda/\widehat{a}(\lambda)|^{1-\delta}}\|\phi\|_p \\ &\leq C|\lambda|^{-1+\delta(1+\alpha)}\|\phi\|_p \ \text{ for Re } \lambda > 0. \end{aligned}\tag{3.7}$$

We get the same estimate for $\|A^\delta\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\widehat{S}(\lambda)\phi\|_p$, because of (2.12). Applying Lemma 3.1 gives

$$\|A^\delta S(t)\phi\|_p \leq Ct^{-\delta(1+\alpha)}\|\phi\|_p \ \text{ if } \ \delta < \frac{1}{1+\alpha}. \tag{3.8}$$

Next, we shall improve the estimate at infinity, specifically, we show that

$$A^\delta S \in L^1(\mathbb{R}^+;\mathcal{B}(L^p)) \ \text{ for } \ \delta < \frac{1}{1+\alpha}.$$

To do so, we decompose $S(t)$ into two parts

$$S(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\rho)H(\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho + \frac{1}{2\pi}\int_{-\infty}^{\infty}[1-\psi(\rho)]H(\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho,$$

where $\psi \in C^\infty(\mathbb{R})$, $\psi(\rho) = 1$ for $|\rho| \le N$, $\psi(\rho) = 0$ for $|\rho| \ge N+1$, $0 \le \psi(\rho) \le 1$ elsewhere, $N > 0$ arbitrary fixed. After applying A^δ and performing two integrations by parts in the second integral we get (note that by (3.7) the substitution terms vanish)

$$\begin{aligned} A^\delta S(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\rho)H_\delta(\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho - \frac{1}{2\pi t^2}\int_{-\infty}^{\infty}[(1-\psi(\rho))H_\delta(\mathrm{i}\rho)]''\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho \\ &= \mathcal{S}_1(t) + \mathcal{S}_2(t), \quad t \in \mathbb{R}^+. \end{aligned}$$

Since $A^\delta S(t)$ satisfies (3.8) and $\mathcal{S}_1$ is obviously bounded on $\mathbb{R}^+$, $\mathcal{S}_2$ satisfies (3.8) for $t \in (0,1)$ too. Also, (2.12), $(2.1)_3$ imply

$$\|H_\delta'(\lambda)\|_{\mathcal{B}(L^p)} \le |\lambda|^{-2+\delta(1+\alpha)}, \ \|H_\delta''(\lambda)\|_{\mathcal{B}(L^p)} \le |\lambda|^{-3+\delta(1+\alpha)}.$$

which gives $\mathcal{S}_2(t) \le C/t^2$. These estimates on $\mathcal{S}_2$ imply $\mathcal{S}_2 \in L^1(\mathbb{R}^+;\mathcal{B}(L^p))$.

To show that $\mathcal{S}_1 \in L^1(\mathbb{R}^+;\mathcal{B}(L^p))$ we use local analyticity of $1/\widehat{a}(\lambda)$ on $\mathrm{i}\mathbb{R}$, the fact that the product of a function that is locally analytic on $\mathrm{i}\mathbb{R}$ with a compactly supported C^∞ function is the Fourier transform of a function in $L^1(\mathbb{R})$, and the Paley-Wiener theorem. We can proceed literally as in the proof of [13, Theorem 10.1].

To estimate Du we will need bounds for AS_b. We decompose AS_b by means of its Laplace transform as follows.

$$\widehat{AS_b}(\lambda) = \frac{\widehat{b}(\lambda)}{\widehat{a}(\lambda)}\Big(I - \frac{\lambda}{\widehat{a}(\lambda)}(\frac{\lambda}{\widehat{a}(\lambda)} + A)^{-1}\Big) = \widehat{c}(\lambda) - \widehat{T}(\lambda)$$

so

$$AS_b(t) = c(t) - T(t) \ \text{ where } \ T(t) = \frac{\mathrm{d}}{\mathrm{d}t} c * S(t), \ c \text{ given in (2.3)}, \tag{3.9}$$

and

$$T = T_1 + T_\infty, \quad \widehat{T}_1(\lambda) = \frac{\lambda\widehat{b}_1(\lambda)}{\widehat{a}(\lambda)}\widehat{S}(\lambda), \quad \widehat{T}_\infty(\lambda) = \frac{b_\infty}{\widehat{a}(\lambda)}\widehat{S}(\lambda). \tag{3.10}$$

In the following Lemma, we collect some estimates that will be used in the sequel.

Lemma 3.2 *Let S, S_1, and $T = T_1 + T_\infty$ be given by the formulae (3.6), (3.3), (3.9), (3.10), $S_{b_1} = b_1 * S$, $1 < p < +\infty$, $0 \le \delta < \frac{1}{1+\alpha}$. Then there is a constant C such that*

$$\|A^\delta S(t)\phi\|_p \le Ct^{-\delta(1+\alpha)}\|\phi\|_p, \tag{3.11}$$

$$A^\delta S(t) \in L^1(\mathbb{R}^+;\mathcal{B}(L^p)), \tag{3.12}$$

$$\|A^\delta S_1(t)\phi\|_p \le C\min\{t^{1-\delta(1+\alpha)}, 1\}\|\phi\|_p, \tag{3.13}$$

$$\|A^\delta S_{b_1}(t)\phi\|_p \le Ct^{\beta-\delta(1+\alpha)}\|\phi\|_p, \tag{3.14}$$

$$A^\delta S_{b_1} \in L^1(\mathbb{R}^+;\mathcal{B}(L^p)), \tag{3.15}$$

$$\|AS_b(t)\phi\|_p \le \left\{ \begin{array}{l} C(1+t^{-1+\beta-\alpha})\|\phi\|_p, \ \textit{for } \phi \in L^p, \ \alpha<\beta<1, \\ C(1+t^{-\frac{1+\alpha}{2}})\|A^\gamma\phi\|_p, \ \textit{for } \phi\in\mathcal{D}(A^\gamma), \ \alpha<\beta\le\frac{\alpha+1}{2}, \ \gamma\in(\frac{1}{2}-\frac{\beta}{1+\alpha},\frac{1}{2}), \end{array}\right. \tag{3.16}$$

$$\|D\dot{S}_{b_1}(t)\phi\|_p \le C\min\{t^{-1+\beta-\frac{1+\alpha}{2}}, t^{-2+\beta-\alpha}\}\|\phi\|_p, \tag{3.17}$$

$$D\dot{S}_b \in L^1(\mathbb{R}^+,\mathcal{B}(L^p)) \ \ \textit{if} \ \ \beta > \frac{1+\alpha}{2}, \tag{3.18}$$

$$T\in L^1(\mathbb{R}^+,\mathcal{B}(L^p,L^\infty)) \ \textit{when} \ \frac{1}{2p} < \frac{\beta-\alpha}{1+\alpha}. \tag{3.19}$$

Proof: We have already proved (3.11) and (3.12). To obtain (3.13), we use (3.11), (3.12) and the definition $S_1 = 1 * S$. The estimates (3.14), (3.15) follow by $S_{b_1} = b_1 * S$, (3.11), (3.12), and by the size estimates that we can prove for b_1 (see Lemma 6.3).

The Laplace transform of $AS_b(t)\phi = AS_{b_1}(t)\phi + b_\infty AS_1\phi$ can be written as

$$\widehat{AS_b}\phi(\lambda) = \frac{\widehat{b}_1(\lambda)}{\widehat{a}(\lambda)}A^{1-\gamma}\Big(\frac{\lambda}{\widehat{a}(\lambda)}+A\Big)^{-1}A^\gamma\phi + \frac{b_\infty}{\lambda\widehat{a}(\lambda)}A\Big(\frac{\lambda}{\widehat{a}(\lambda)}+A\Big)^{-1}\phi.$$

Taking into account $(2.1)_3$, $(2.2)_3$ and (3.1) we conclude that the first part is bounded by $C|\lambda|^{\alpha-\beta-\gamma(\alpha+1)}$. Now take first $\gamma=0$, then, in case $\beta\le\frac{1+\alpha}{2}$, $\gamma=\frac{1}{2}-\frac{\beta}{1+\alpha}$. In both cases. apply Lemma 3.1 (in which the γ-exponents are, respectively, $\beta-\alpha>0$ and $\frac{1-\alpha}{2}>0$). This yields

$$\|AS_{b_1}(t)\phi\|_p \le \left\{ \begin{array}{l} Ct^{-1+\beta-\alpha}\|\phi\|_p, \\ Ct^{-\frac{1+\alpha}{2}}\|A^\gamma\phi\|_p \ \text{ if } \ \beta\le\frac{1+\alpha}{2}, \ \gamma=\frac{1}{2}-\frac{\beta}{1+\alpha}. \end{array}\right.$$

The inverse Laplace transform of $\frac{1}{\lambda\widehat{a}(\lambda)}$ can be estimated by $C(1+t^{-\alpha})$ combining proofs of Theorem 1 in [12] and Proposition 0.2 in [13]. Hence the inverse of the second term is bounded by $C(1+t^{-\frac{1+\alpha}{2}})\|\phi\|_p$ and (3.16) follows.

To prove (3.17) distinguish two cases. If $\beta>\frac{1+\alpha}{2}$, the Laplace transform of $D\dot{S}_{b_1}$ satisfies

$$|\mathcal{L}(D\dot{S}_{b_1})(\lambda)| \le C|\lambda|^{-\beta+\frac{1+\alpha}{2}}$$

and we can apply Lemma 3.1 directly to get the first part of (3.17). In the case $\beta\le\frac{1+\alpha}{2}$ we take advantage of the fact that

$$\mathcal{L}(tf(t))(\lambda) = -\frac{\mathrm{d}}{\mathrm{d}\lambda}\widehat{f}(\lambda). \tag{3.20}$$

Hence

$$\mathcal{L}(tD\dot{S}_{b_1}(t))(\lambda) = -\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\widehat{DS_{b_1}}(\lambda) - \widehat{DS_{b_1}}(\lambda)$$

and the estimates $\|\widehat{DS_{b_1}}(\lambda)\|_{\mathcal{B}(L^p)} \le C|\lambda|^{\alpha/2-\beta-1/2}$, $\|\widehat{DS_{b_1}}(\lambda)\|_{\mathcal{B}(L^p)} \le |\lambda|^{\alpha-\beta}$ respectively, together with the corresponding expressions for their derivatives and Lemma 3.1 yield the remaining assertion in (3.17) and, consequently also (3.18).

It remains to prove (3.19). It is sufficient to show that

$$A^\delta T\in L^1(\mathbb{R}^+,\mathcal{B}(L^p(0,1)) \text{ when } \frac{1}{2p}<\delta<\frac{\beta-\alpha}{1+\alpha}.$$

Then

$$T \in L^1(\mathbb{R}^+, \mathcal{B}(L^p, C^\mu)) \text{ for } \mu < 2\delta - \frac{1}{p}$$

due to the embedding $\mathcal{D}(A_p^\delta) \hookrightarrow C^\mu(0,1)$. We decompose T as in (3.10) and employ Lemma 3.1 again to show

$$\|A^\delta T_1(t)\phi\|_p \le Ct^{-1+\beta-\alpha-\delta(1+\alpha)}\|\phi\|_p, \text{ for } \delta < \frac{\beta-\alpha}{1+\alpha}, \tag{3.21}$$

$$\|A^\delta T_\infty(t)\phi\|_p \le Ct^{-\alpha(1+\delta)-\delta}\|\phi\|_p \text{ for } \delta < \frac{1-\alpha}{1+\alpha}.$$

In the same way as we proved (3.12) we prove

$$A^\delta T_\infty \in L^1(\mathbb{R}^+, \mathcal{B}(L^p(0,1)) \text{ when } \delta < \frac{1-\alpha}{1+\alpha}.$$

For T_1 we use (3.21), (3.20) and the estimate

$$\left\| \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[A^\delta \widehat{T}_1(\lambda)\phi\right]\right\|_p \le C|\lambda|^{\alpha-\beta}\|\phi\|_p.$$

The same holds also for $\lambda\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left[A^\delta \widehat{T}_1(\lambda)\right]$ and Lemma 3.1 applies to give

$$\|A^\delta T_1(t)\phi\|_p \le Ct^{-2+\beta-\alpha}\|\phi\|_p.$$

The proof is complete. □

We employ the last Lemma to prove the local existence result.

Proposition 3.1 *Let (2.1)-(2.2) hold, $g \in C^1(\mathbb{R})$, $g(0) = 0$, $q \ge \max\{2, \frac{1+\alpha}{1-\alpha}\}$, $\gamma = \max\{0, \frac{1}{2} - \frac{\beta}{1+\alpha}\}$. Given any data $u^0 \in W_0^{1,\infty} \cap H_0^{1+\gamma,q}$, $u^1 \in L^q$, $f \in L^1_{loc}(\mathbb{R}^+; L^q)$ there exists a unique weak solution of (1.1)*

$$u \in C([0,t_0); W_0^{1,\infty}(0,1)) \cap C^1([0,t_0); L^q(0,1)), \tag{3.22}$$

defined on a maximal time interval $[0,t_0)$. If $t_0 < \infty$ then $\limsup_{t\to t_0}\|Du\|_\infty = \infty$. In addition,

$$Du \in C^{\frac{1-\alpha}{2}}([0,t_0); L^q(0,1)) \tag{3.23}$$

and

$$\|D\dot{u}(t)\|_q \le C(T)t^{-\frac{1+\alpha}{2}} \text{ for } 0 < t \le T < t_0. \tag{3.24}$$

Moreover, if $u^0 \in \mathcal{D}(A_q) = W^{2,q} \cap W_0^{1,q}$, $u^1 \in \mathcal{D}(A_q^{\frac{1}{2}}) = W_0^{1,q}$, and $f \in L^1_{loc}(\mathbb{R}^+; \mathcal{D}(A_q^{1/2})$, then

$$u \in C([0,t_0); \mathcal{D}(A_q)) \cap C^1([0,t_0); \mathcal{D}(A_q^{\frac{1}{2}})).$$

Proof: According to (3.5) we have

$$u = u^0 + S_1u^1 + DS_b * Pg(Du) + S_1 * f.$$

We apply the standard contraction method to the operator given by the right hand side of this equation in the space $C([0,t_1]; W_0^{1,\infty}(0,1))$ with suitable $t_1 > 0$.

We use estimates from Lemma 3.2, the relations

$$Du = Du^0 + DS_1 u^1 - AS_b * Pg(Du) + DS_1 * f, \tag{3.25}$$

$$\dot{u} = Su^1 + D\dot{S}_b * Pg(Du) + S * f, \tag{3.26}$$

obtained from (3.5) by differentiating, and the fact that

$$A^{\frac{1}{2}}Q,\ DA^{-\frac{1}{2}} \text{ are bounded operators in } L^q.$$

The inclusion

$$\mathcal{D}(A_q^\delta) \subset C^\mu(0,1) \ \text{ if } 0 < \mu < 2\delta - \frac{1}{q} \tag{3.27}$$

with $\delta > \frac{1}{2q}$, allows for the estimate

$$\|DS_1(t)u^1\|_{L^\infty} \le C\|A^\delta A^{\frac{1}{2}} S_1(t)u^1\|_q \le Ct^{\frac{1-\alpha}{2}-\delta(1+\alpha)}\|u^1\|_q. \tag{3.28}$$

Observe that the exponent on the right hand side of this equation can be chosen positive provided $\frac{1}{q} < \frac{1-\alpha}{1+\alpha}$. In the same way we estimate $\|DS_1 * f\|_{C([0,t_1],L^\infty)}$. From (3.9), (3.19) we get $AS_b \in L^1_{loc}(\mathbb{R}^+; \mathcal{B}(L^\infty(0,1)))$, so $AS_b * Pg(Du) \in BUC([0,t_1]; L^\infty(0,1))$, $AS_b * Pg(Du)(0) = 0$.

To prove (3.24) we have to solve the following equation for

$$v = D\dot{u} \text{ in } L^r\Big((0,t_0); L^q(0,1)\Big),$$

$$v(t) = DS(t)u^1 - \int_0^t AS_b(t-s)P\Big(g'(Du(s))v(s)\Big)\,\mathrm{d}s \ - AS_b(t)Pg(Du^0) + DS * f(t). \tag{3.29}$$

We use (3.11) with $\delta = \frac{1}{2}$ and (3.16) to get

$$\|v(t)\|_q \le Ct^{-\frac{1+\alpha}{2}} + C\int_0^t (t-s)^{-1+\beta-\alpha}\|v(s)\|_q\,\mathrm{d}s\ .$$

The generalized Gronwall inequality then yields (3.24) and, consequently, (3.23).

To obtain the estimates for $\dot{u}$, use (3.26), (3.10) and (3.17) provided that $\beta > \frac{1+\alpha}{2}$. To prove that $u \in C^1([0,t_0); L^q)$ also for $\beta \le \frac{1+\alpha}{2}$, we make use of (3.23). We write $\dot{u}$ in the form

$$\dot{u}(t) = S(t)u^1 + S * f(t) + DS_b(t)Pg(Du^0) + \int_0^t DS_b(s)P\Big(g'(Du(t-s))D\dot{u}(t-s)\Big)\,\mathrm{d}s$$

and use (3.13), (3.14), (3.24) and the fact that $g(Du^0) \in \mathcal{D}(A^\gamma)$.

We can continue the solution as long as $\|u(t)\|_{W^{1,\infty}}$ remains bounded, this implies boundedness of $\|\dot{u}(t)\|_q$ on the same interval.

The contraction principle applied in the space $C([0,t_1]; \mathcal{D}(A)) \cap C^1([0,t_1]; \mathcal{D}(A^{\frac{1}{2}}))$ yields the existence of a strong solution to (1.1). □

To derive energy inequalities we will need the following lemma, which contains some well known facts on Volterra equations.

Lemma 3.3 *Assume $g \in C^2(\mathbb{R})$, $g(0) = 0$, $g'(s) \geq -\eta$ for all $s \in \mathbb{R}$, let a, b be 2-monotone, k 1-monotone and $v \in W^{1,2}_{loc}(\mathbb{R}^+, L^2(0,1))$. Then*

$$\int_0^T \langle \frac{\mathrm{d}}{\mathrm{d}t} k * v(t), v(t)\rangle \ \mathrm{d}t \geq \frac{1}{2}\int_0^T k(t)(\|v(T-t)\|_2^2 + \|v(t)\|_2^2) \ \mathrm{d}t \tag{3.30}$$

$$\int_0^T \langle \frac{\mathrm{d}}{\mathrm{d}t} b * g(v)(t), \dot{v}(t)\rangle \ \mathrm{d}t \geq -\eta \int_0^T \langle b * \dot{v}(t), \dot{v}(t)\rangle \ \mathrm{d}t + \int_0^T b(t)\langle g(v(0)), \dot{v}(t)\rangle \ \mathrm{d}t. \tag{3.31}$$

$$\begin{aligned} 2\int_0^T \langle a * \dot{v}(t), \dot{v}(t)\rangle \ \mathrm{d}t &= \int_0^T \int_0^t \|v(t-s) - v(t)\|_2^2 \ \mathrm{d}\dot{a}(s) \ \mathrm{d}t \\ &+ \int_0^T -\dot{a}(s)[\|v(s) - v(0)\|_2^2 + \|v(T-s) - v(T)\|_2^2] \ \mathrm{d}s + a(T)\|v(T) - v(0)\|_2^2. \end{aligned} \tag{3.32}$$

Proof: First, we approximate the kernels by an increasing sequence of regular functions k_ϵ, b_ϵ of the same type and apply results from [8], Chapter 18. The Hilbert space version of Lemma 4.1, [8] gives

$$\int_0^T \langle \frac{\mathrm{d}}{\mathrm{d}t} k_\epsilon * v(t), v(t)\rangle \ \mathrm{d}t \geq \frac{1}{2}\int_0^T k_\epsilon(t)(\|v(T-t)\|_2^2 + \|v(t)\|_2^2) \ \mathrm{d}t.$$

To estimate $\int_0^T \langle \frac{\mathrm{d}}{\mathrm{d}t} b * g(v)(t), \dot{v}(t)\rangle \ \mathrm{d}t$, we denote $h(s) = g(s) + \eta s$ and make use of Lemma 5.4, [8]. The function h is increasing and we get

$$\int_0^T \langle \frac{\mathrm{d}}{\mathrm{d}t} b_\epsilon * h(v)(t), \dot{v}(t)\rangle \ \mathrm{d}t \geq \int_0^T \langle b_\epsilon(t) h(v(0)), \dot{v}(t)\rangle \ \mathrm{d}t.$$

Going back to the function $g(s) = h(s) - \eta s$, and passing to the limit with b_ϵ, k_ϵ we obtain (3.30), (3.31).

The third formula is proved, replacing $a(t)$ by $a(t+\varepsilon)$, by a direct computation and then by passing to the limit $\varepsilon \to 0$. □

We close this section by results that enable us to prove global existence.

Theorem 3.1 *Let $g \in C^1(\mathbb{R})$, $g(s)s \geq \epsilon|s|^2$ for $|s| \geq R$ and c be completely positive. Assume that h_1, h_2 are bounded continuous functions on $J = [0, T]$ and v is a solution of the equation*

$$v + c * g(v) = h_1 + c * h_2. \tag{3.33}$$

Then

$$\|v\|_{L^\infty(0,T)} \leq R + 2\|h_1\|_{L^\infty(0,T)} + \frac{1}{\epsilon}\|h_2\|_{L^\infty(0,T)}. \tag{3.34}$$

Proof: First, consider the equation

$$v_\sigma + c * g(v_\sigma) + \sigma g(v_\sigma) = h_1 + c * h_2 + \sigma h_2 \tag*{$(3.33)_\sigma$}$$

with some $\sigma > 0$ fixed. This equation is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} r_\sigma * (v_\sigma - h_1) = -g(v_\sigma) + h_2,$$

with r_σ satisfying

$$\sigma r_\sigma + c * r_\sigma = 1, \quad \hat{r}_\sigma(\lambda) = \frac{1}{\lambda(\hat{c}(\lambda)+\sigma)}.$$

By Prüss [13, Proposition 4.5], r_σ is nonnegative and nonincreasing,

$$r_\sigma(0+) = 1/\sigma < \infty. \tag{3.35}$$

Set $\phi = v_\sigma - h_1$ and $\phi_\infty = R + \|h_1\|_{L^\infty(J)} + \frac{1}{\epsilon}\|h_2\|_{L^\infty(J)}$ and show $|\phi(t)| \le \phi_\infty$ on J.

By the assumptions, ϕ is continuous and $|\phi(0)| < \phi_\infty$ (since, without loss of generality, we can take R large enough). Let $t_0 > 0$ be the first t such that $|\phi(t_0)| = \phi_\infty$ and suppose $\phi(t_0) = \phi_\infty$, the case $\phi(t_0) = -\phi_\infty$ being completely analogous. Then $\phi(t) < \phi_\infty$ for $t < t_0$.

Further

$$v_\sigma(t_0) = \phi(t_0) + h_1(t_0) \ge \phi_\infty - \|h_1\|_{L^\infty(J)} = R + \frac{\|h_2\|_{L^\infty(J)}}{\epsilon} \ge R,$$

so $g(v_\sigma(t_0)) \ge \epsilon v_\sigma(t_0) \ge \epsilon R + \|h_2\|_{L^\infty(J)}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} r_\sigma * \phi(t_0) = -g(v_\sigma(t_0)) + h_2(t_0) \le -\epsilon R - \|h_2\|_{L^\infty(J)} + h_2(t_0) \le -\epsilon R < 0. \tag{3.36}$$

On the other hand,

$$\begin{aligned} &\frac{1}{h}\Big[r_\sigma * \phi(t_0) - r_\sigma * \phi(t_0 - h)\Big] \\ &= \frac{1}{h}\Big[r_\sigma * (\phi - \phi_\infty)(t_0) - r_\sigma * (\phi - \phi_\infty)(t_0 - h)\Big] + \frac{1}{h}\phi_\infty \int_{t_0-h}^{t_0} r_\sigma(s)\,\mathrm{d}s \\ &= \frac{1}{h}\int_0^{t_0-h} [r_\sigma(t_0 - s) - r_\sigma(t_0 - h - s)][\phi(s) - \phi_\infty]\,\mathrm{d}s + \frac{1}{h}\int_{t_0-h}^{t_0} r_\sigma(t_0 - s)(\phi(s) - \phi_\infty)\,\mathrm{d}s \\ &+ \frac{1}{h}\phi_\infty \int_{t_0-h}^{t_0} r_\sigma(s)\,\mathrm{d}s \ \ge \frac{1}{h}\int_{t_0-h}^{t_0} r_\sigma(t_0 - s)(\phi(s) - \phi_\infty)\,\mathrm{d}s \ \ge r_\sigma(h)\frac{1}{h}\int_0^h (\phi(t_0 - s) - \phi_\infty)\,\mathrm{d}s. \end{aligned}$$

The last term tends to zero as $h \to 0$ due to the continuity of ϕ and (3.35). This gives $\frac{\mathrm{d}}{\mathrm{d}t} r_\sigma * \phi(t_0) \ge 0$, a contradiction to (3.36).

Now, the assertion of the theorem follows by approximation of the solution v of (3.33) by solutions v_σ of $(3.33)_\sigma$. The function g is locally Lipschitz continuous which implies that solutions of (3.33), $(3.33)_\sigma$ are uniquely determined and $v_\sigma \to v$ as $\sigma \to 0$ uniformly on compact sets. Hence (3.34) holds also for v. □

We will also use a modified version of the preceding theorem, specifically, we replace the assumption $g(s)s \ge \epsilon|s|^2$ by $g'(s) \ge -\eta$.

Proposition 3.2 *Let $g \in C^1(\mathbb{R})$, $g(0) = 0$, $g'(s) \ge -\eta$, $\eta > 0$, $s \in \mathbb{R}$ and c be completely positive. Assume that h_1, h_2 are bounded continuous functions on $J = [0,T]$ and v is a solution of the equation*

$$v + c * g(v) = h_1 + c * h_2.$$

Then

$$\|v\|_{L^\infty(0,t)} \le 2\|h_1\|_{L^\infty(0,t)} + \frac{1}{\epsilon}\|h_2\|_{L^\infty(0,T)} + 2(\eta + \epsilon)\int_0^t c(t - s)\|v\|_{L^\infty(0,s)}\,\mathrm{d}s \tag{3.37}$$

holds for any $\epsilon > 0$, $t \in [0,T]$.

Proof: We write (3.33) in the form

$$v + c * g_1(v) = h_1 + c * h_2 + c * g_2(v) \tag{3.38}$$

with

$$g_1(s) = \frac{1}{s}\max\{sg(s), \epsilon s^2\}, \quad g_2(s) = g_1(s) - g(s).$$

The function g_1 satisfies the assumptions of the preceding Theorem with $R = 0$ and

$$|g_2(s)| \le (\eta + \epsilon)|s|, \; g_2(s)s \ge 0. \tag{3.39}$$

The formula (3.34) applied to (3.38 now gives that

$$\|v\|_{L^\infty(0,\tau)} \le 2\|h_1\|_{L^\infty(0,\tau)} + \frac{1}{\epsilon}\|h_2\|_{L^\infty(0,\tau)} + 2\sup_{0\le t\le\tau}\int_0^t c(t-s)|g_2(v(s))|\;\mathrm{d}s$$

holds for any $\tau \le T$ which together with (3.39) leads to (3.37).

□

4 A priori estimates and global existence

Now, we are going to deduce *a priori* estimates that allow us to prove global existence and boundedness of solutions. Observe that an important feature of (1.1) is the fact that energy inequalities do hold.

One common way to obtain energy estimates is to multiply the equation by $\dot{u}$ and integrate over the time-space interval. Here, because of the convolution of the nonlinearity with the kernel b, we would not get an estimate of u_x directly. To overcome this difficulty, we derive two types of energy inequalities.

First, we apply the inverse operator of $\frac{\mathrm{d}}{\mathrm{d}t}b*$ to (1.1), i.e., we convolve the equation with $k_0\delta + k_1$ such that $k_0 b + k_1 * b = 1$, equivalently $k_0 + \hat{k}_1(\lambda) = \frac{1}{\lambda\hat{b}(\lambda)}$. The existence of such k_0, k_1 is stated in (2.9).

Denoting $l = k_0 a + k_1 * a$ we have

$$k_0\ddot{u} + k_1 * \ddot{u} = l * D^2\dot{u} + Dg(Du) + k_0 f + k_1 * f.$$

Then we multiply formally the resulting equation by $\dot{u}$, integrate over $[0,\tau]\times[0,1]$, and integrate by parts to get

$$\int_0^\tau\int_0^1 [k_0\ddot{u} + k_1 * \ddot{u}]\cdot\dot{u}\;\mathrm{d}x\;\mathrm{d}t + \int_0^\tau\int_0^1 l * D\dot{u}\cdot D\dot{u}\;\mathrm{d}x\;\mathrm{d}t + \int_0^\tau \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 G(Du)\;\mathrm{d}x\;\mathrm{d}t =$$

$$= \int_0^\tau\int_0^1 [k_0 f + k_1 * f]\cdot\dot{u}\;\mathrm{d}x\;\mathrm{d}t,$$

with $G(s) = \int_0^s g(r)\;\mathrm{d}r$. We employ Lemma 3.3 to obtain

$$\int_0^\tau \langle k_1 * \ddot{u}(t), \dot{u}(t)\rangle\;\mathrm{d}t = \int_0^\tau \langle \frac{\mathrm{d}}{\mathrm{d}t}k_1 * \dot{u}(t), \dot{u}(t)\rangle\;\mathrm{d}t - \int_0^\tau \langle k_1(t)\dot{u}(0), \dot{u}(t)\rangle\;\mathrm{d}t$$

$$\ge \frac{1}{2}\int_0^\tau k_1(t)(\|\dot{u}(\tau - t)\|_2^2 + \|\dot{u}(t)\|_2^2)\;\mathrm{d}t - \int_0^\tau k_1(t)\langle\dot{u}(0), \dot{u}(t)\rangle\;\mathrm{d}t$$

$$\geq \frac{1}{4}\int_0^\tau k_1(t)\|\dot u(t)\|_2^2 \, \mathrm{d}t - \|k_1\|_{L^1(R^+)}\|u^1\|_2^2,$$

where we made use of the estimate

$$\int_0^\tau k_1(t)\langle \dot u(0), \dot u(t)\rangle \, \mathrm{d}t \leq \int_0^\tau k_1(t)\|u^1\|_2\|\dot u(t)\|_2 \, \mathrm{d}t \leq \int_0^\tau k_1(t)\|u^1\|_2^2 \, \mathrm{d}t + \frac{1}{4}\int_0^\tau k_1(t)\|\dot u(t)\|_2^2 \, \mathrm{d}t.$$

Thus we have

$$\frac{k_0}{2}\|\dot u(\tau)\|_2^2 + \frac{1}{4}\int_0^\tau k_1(t)\|\dot u(t)\|_2^2 \, \mathrm{d}t + \int_0^\tau \langle l * D\dot u(t), D\dot u(t)\rangle \, \mathrm{d}t + \int_0^1 G(Du(\tau,x)) \, \mathrm{d}x$$

$$\leq \int_0^\tau \langle [k_0 f(t) + k_1 * f(t)], \dot u(t)\rangle \, \mathrm{d}t + \int_0^1 G(Du(0)) \, \mathrm{d}x + [\frac{k_0}{2} + \|k\|_{L^1(R^+)}]\|u^1\|_2^2. \tag{4.1}$$

The second energy inequality is obtained as follows. Multiplying (1.1) by $\dot u$, integrating over $[0,\tau]\times[0,1]$ and again integrating by parts we obtain

$$\frac{1}{2}\int_0^\tau \frac{\mathrm{d}}{\mathrm{d}t}\|\dot u(t)\|_2^2 \, \mathrm{d}t + \int_0^\tau \langle a * D\dot u(t), D\dot u(t)\rangle \, \mathrm{d}t + \int_0^\tau \langle \frac{\mathrm{d}}{\mathrm{d}t} b * g(Du)(t), D\dot u(t)\rangle \, \mathrm{d}t$$

$$= \int_0^\tau \langle f(t), \dot u(t)\rangle \, \mathrm{d}t.$$

With $b(t) = b_1(t) + b_\infty$ and using (3.31) we get the inequality

$$\|\dot u(\tau)\|_2^2 + 2\int_0^\tau \langle a * D\dot u(t), D\dot u(t)\rangle \, \mathrm{d}t + 2b_\infty \int_0^1 G(Du(\tau,x)) \, \mathrm{d}x \tag{4.2}$$

$$\leq 2\int_0^\tau \langle f(t), \dot u(t)\rangle \, \mathrm{d}t + \|u_1\|_2^2 + 2b_\infty \int_0^1 G(Du^0(x)) \, \mathrm{d}x$$

$$+2\eta \int_0^\tau \langle b_1 * D\dot u(t), D\dot u(t)\rangle \, \mathrm{d}t - 2\int_0^\tau b_1(t)\langle g(Du^0), D\dot u(t)\rangle \, \mathrm{d}t.$$

Now, we multiply (4.1) by $2C_\eta$ where C_η is given by (2.11) and add to (4.2).

$$(1 + k_0 C_\eta)\|\dot u(\tau)\|_2^2 + 2\int_0^\tau \langle (C_\eta l + \theta a - \eta b_1) * D\dot u(t), D\dot u(t)\rangle \, \mathrm{d}t +$$

$$2\int_0^\tau (1-\theta)\langle a * D\dot u(t), D\dot u(t)\rangle \, \mathrm{d}t + \frac{C_\eta}{2}\int_0^\tau k_1(t)\|\dot u(t)\|_2^2 \, \mathrm{d}t + 2(C_\eta + b_\infty)\int_0^1 G(Du(\tau,x)) \, \mathrm{d}x$$

$$\leq 2C_\eta \int_0^\tau \langle k_1 * f(t), \dot u(t)\rangle \, \mathrm{d}t + 2(1 + k_0 C_\eta)\int_0^\tau \langle f(t), \dot u(t)\rangle \, \mathrm{d}t$$

$$-2\int_0^\tau b_1(t)\langle g(Du^0), D\dot u(t)\rangle \, \mathrm{d}t + (1 + k_0 C_\eta + 2C_\eta\|k\|_1)\|u^1\|_2^2 + 2(b_\infty + C_\eta)\int_0^1 G(Du^0(x)) \, \mathrm{d}x.$$

The second term on the left hand side of the last inequality is nonnegative due to (2.11).

For $f \in L^1((0,T); L^2(0,1))$, we can estimate further

$$2C_\eta \int_0^\tau \langle k_1 * f(t), \dot u(t)\rangle \, \mathrm{d}t + 2(1 + k_0 C_\eta)\int_0^\tau \langle f(t), \dot u(t)\rangle \, \mathrm{d}t$$

$$\leq 2(1 + C_\eta k_0 + C_\eta\|k\|_{L^1(R^+)})\|f\|_{L^1(0,T;L^2)} \sup_{t\in(0,\tau)} \|\dot u(t)\|_2.$$

The remaining term on the right hand side can be controlled in a simple way:

$$\int_0^\tau b_1(t)\langle g(Du^0), D\dot u(t)\rangle \ \mathrm{d}t = -\int_0^\tau b_1(t)\langle Dg(Du^0), \dot u(t)\rangle \ \mathrm{d}t,$$

however, this requires the additional condition $u^0 \in W^{2,2}$.

If we do not want to make this assumption we have to work harder, but we may proceed as follows. Integration by parts yields

$$\int_0^\tau b_1\langle g(Du^0), D\dot u\rangle \ \mathrm{d}t = b_1(\tau)\langle g(Du^0), Du(\tau) - Du^0\rangle + \int_0^\tau (-\dot b_1(t)\langle g(Du^0), Du(t) - Du^0\rangle \ \mathrm{d}t, \tag{4.3}$$

hence

$$|\int_0^\tau b_1\langle g(Du^0), D\dot u\rangle \ \mathrm{d}t| \le [b_1(\tau)\tau^\delta + \int_0^\tau |\dot b_1(t)|t^\delta \ \mathrm{d}t] \ \|g(Du^0)\|_2\|Du\|_{\dot C^\delta([0,\tau];L^2)},$$

where $\delta = (1-\alpha)/2$ and $\|\cdot\|_{\dot C^\delta([0,\tau]}$ denotes the homogeneous Hölder norm on $[0,\tau]$. For $\beta > (1+\alpha)/2$ this estimate is good enough to give control on this term for finite τ, say $\tau \le 1$. In fact, we have by Lemma 6.3, after some simple estimates,

$$b_1(\tau)\tau^{(1-\alpha)/2} + \int_0^\tau |\dot b_1|(t)t^{(1-\alpha)/2} \ \mathrm{d}t \le C\int_0^1 b_1(t)t^{-(1+\alpha)/2} \ \mathrm{d}t.$$

Condition **(a)** gives

$$\mathrm{Re}\ \widehat a(\lambda) \ge C\mathrm{Re}\ \frac{1}{(1+\lambda)^\alpha}, \quad \mathrm{Re}\ \lambda \ge 0,$$

hence by (3.32)

$$\begin{aligned} t^{\alpha-1}e^{-t}||Du(t) - Du^0||_2^2 &\le 2\Gamma(\alpha)\int_0^t \langle s^{\alpha-1}e^{-s} * D\dot u(s), D\dot u(s)\rangle \ \mathrm{d}s \\ &\le C\int_0^t \langle a * D\dot u(s), D\dot u(s)\rangle \ \mathrm{d}s, \end{aligned} \tag{4.4}$$

which implies

$$||Du||_{\dot C^{(1-\alpha)/2}([0,\tau];L^2)} \le C \sup_{0\le t\le 1}\int_0^t \langle a * D\dot u(s), D\dot u(s)\rangle \ \mathrm{d}s.$$

In case $\beta \le (1+\alpha)/2$, $\tau \le 1$, choose $1/2 > \gamma > \frac{1}{2} - \frac{\beta}{1+\alpha}$, $\delta = \frac{1-\alpha}{2} + \gamma(1+\alpha)$, and apply the same reasoning as above to get the result

$$|\int_0^\tau b_1\langle g(Du^0), D\dot u\rangle \ \mathrm{d}t| \le [b_1(\tau)\tau^\delta + \int_0^\tau |\dot b_1(t)|t^\delta \ \mathrm{d}t] \ \|A^\gamma g(Du^0)\|_2\|A^{-\gamma}Du\|_{\dot C^{1-\theta(1+\alpha)/2}([0,1];L^2)},$$

where $\theta = 1-2\gamma < 2\beta/(1+\alpha)$, $\delta = 1-\theta(1+\alpha)/2 > 1-\beta$. The norm $\|A^{-\gamma}Du\|_{\dot C^{1-\theta(1+\alpha)/2}([0,1];L^2)}$ can be controlled as follows. By the moment inequality we obtain

$$\begin{aligned} \|A^{-\gamma}(Du(t) - Du^0)\|_2 &\le C\|A^{1/2-\gamma}(u(t) - u^0)\|_2 \\ &\le C\|A^{1/2}(u(t) - u^0)\|_2^{1-2\gamma}\|u(t) - u^0\|_2^{2\gamma} \\ &\le C\|D(u(t) - u^0)\|_2^{1-2\gamma} \sup_{0\le t\le\tau}\|\dot u\|_2^{2\gamma}t^{2\gamma} \\ &\le C[\int_0^\tau \langle a * D\dot u, D\dot u\rangle \ \mathrm{d}t]^{1/2-\gamma} \sup_{0\le t\le\tau}\|\dot u(t)\|_2^{2\gamma}. \end{aligned}$$

where the last inequality is a consequence of (4.4) which holds for any $\alpha \in (0,1)$. Young's inequality shows that we have control also in this case.

For $\tau \geq 1$ we proceed differently and do not need to distinguish the cases $\beta > \frac{1+\alpha}{2}$ and $\beta \leq \frac{1+\alpha}{2}$. We have

$$\text{Re } \widehat{b}_1(\lambda) \leq \frac{C_b}{1+|\lambda|^\beta} \leq \frac{C}{1+|\lambda|^\alpha} \leq C\text{Re } \widehat{a}(\lambda), \quad \text{Re } \lambda > 0.$$

This implies by (3.32) that the first on the right hand side of (4.3) can be estimated as follows:

$$\begin{aligned}
&|b_1(t)\langle g(Du^0), D(u(t)-u^0)\rangle| \leq \frac{b_1(t)}{2\varepsilon}\|g(Du^0)\|_2^2 + \frac{\varepsilon}{2} b_1(t)\|D(u(t)-u^0)\|_2^2 \\
&\leq \frac{b_1(t)}{2\varepsilon}\|g(Du^0)\|_2^2 + \varepsilon \int_0^t \langle b_1 * D\dot{u}, D\dot{u}\rangle \text{ d}s \\
&\leq \frac{b_1(t)}{2\varepsilon}\|g(Du^0)\|_2^2 + \varepsilon C \int_0^t \langle a * D\dot{u}, D\dot{u}\rangle \text{ d}s.
\end{aligned}$$

In a similar way we get for the second term,

$$\int_1^t |\dot{b}_1(s)| \langle g(Du^0), D(u(s)-u^0)\rangle \text{ d}s \leq \frac{b_1(1)}{2\varepsilon}\|g(Du^0)\|_2^2 + \varepsilon C \int_0^t \langle a * D\dot{u}, D\dot{u}\rangle \text{ d}s.$$

These arguments show how the bad term $\int_0^\tau b_1(t)\langle g(Du^0), D\dot{u}(t)\rangle \text{ d}t$ can be dominated.

Summing up, we have

Proposition 4.1 *Let* **(a)**, **(b)** *be satisfied, and assume* $g \in C^2(\mathbb{R})$, $g(0)=0$, *is such that* $g'(s)$ *and* $G(s)$ *are bounded from below. Let* $u^1 \in L^2$, $u^0 \in X = W^{s,2} \cap W_0^{1,\infty}$, $f \in L^1(0,T;L^2)$, *where* $s > 2 - 2\beta/(1+\alpha)$. *Then there is a constant* $C = C(\|u^1\|_2, \|u^0\|_X, \|f\|_{L^1(0,T;L^2)})$ *such that any weak solution of (1.1) satisfies the inequality*

$$\left.\begin{aligned}
&\|\dot{u}(t)\|_2 + \textstyle\int_0^t \langle a * D\dot{u}(s), D\dot{u}(s)\rangle \text{ d}s + \int_0^1 G(Du(t,x)) \text{ d}x \leq C, \quad 0 \leq t \leq T, \\
&\textstyle\int_0^t \|a_1 * D\dot{u}(s)\|_2^2 \text{ d}s \leq C, \quad 0 \leq t \leq T.
\end{aligned}\right\} \tag{4.5}$$

Here a_1 *stems from strong positivity of* a.

Observe that once the first part of (4.5) holds, then the second part follows by (2.13).

This is the starting point of our discussion of global existence and long time behaviour of the solutions of (1.1).

The growth assumption $(2.4)_3$ together with (4.5) yield

$$\|G(Du(t))\|_1 + \|g(Du(t))\|_1 \leq C, \ 0 \leq t < t_0, \tag{4.6}$$

and $\|g(Du(t))\|_p < \infty$ for each $t < t_0$. Another important consequence of (4.5) is

$$a_1 * D\dot{u} \in L^2(\mathbb{R}^+; L^2). \tag{4.7}$$

for every a_1 satisfying (2.13).

Now, we use (4.6) and Proposition 3.2 to obtain global existence.

Proof of Theorem 1.1. Let t_0 be the maximal existence time from the Proposition 3.1. We show that $\|Du(t)\|_\infty$ remains bounded on any interval $(0,T)$ with $T \leq t_0$. The decomposition (3.9) allows us to rewrite the formula (3.25) in the form

$$v + c * g(v) = h_1 + c * h_2 \tag{4.8}$$

with $v = Du$, $h_1 = Du^0 + DS_1u^1 + DS_1 * f + T * Pg(Du)$, $h_2 = (I-P)g(Du)$. Observe that $\|h_2\|_\infty$ is bounded on $(0,t_0)$. Moreover, $\|Du^0 + DS_1u^1 + DS_1 * f\|_\infty$ is bounded on $(0,t_0)$ due to (3.13), (3.27), (3.28). The convolution $T * Pg(Du)$ is estimated by

$$\|T * Pg(Du)(t)\|_\infty \leq C\int_0^t \|T(t-s)\|_{B(L^p;L^\infty)}\Big\|\frac{g(Du)}{|Du|}(s)\Big\|_p \|Du(s)\|_\infty \, \mathrm{d}s.$$

Then we infer from $(2.4)_3$ and (4.6), that $\frac{g(Du)}{|Du|} \in L^\infty((0,T);L^p(0,1))$. Theorem 3.1 and Proposition 3.2 apply and we obtain the estimate

$$\|Du\|_{L^\infty(0,t;L^\infty)} \leq C + C\int_0^t d(t-s)\|Du\|_{L^\infty(0,s;L^\infty)} \, \mathrm{d}s, \tag{4.9}$$

where $d(t) = c(t) + \|T(t)\|_{B(L^p;L^\infty)}$. Since $d \in L^1_{loc}(R^+)$ due to (2.3) and (3.19), (4.9) gives boundedness of $\|Du(t)\|_\infty$ on $[0,T]$, and, consequently, global existence. As to strong solutions, having

$$u^0 \in \mathcal{D}(A_q), \quad u^1 \in \mathcal{D}(A_q^{\frac{1}{2}}), \;\; f \in L^1_{loc}(\mathbb{R}^+, \mathcal{D}(A_q^{\frac{1}{2}})),$$

we get boundedness of A_qu, $A_q^{\frac{1}{2}}\dot{u}$, using expressions

$$Au(t) = Au^0 + AS_1(t)u^1 + AS_b * g'(Du)Au(t) + AS_1 * f(t),$$

(3.29), and Lemma 3.2. For both $\|Au(t)\|_{L^q}$, $\|A^{\frac{1}{2}}\dot{u}(t)\|_q$, we have estimates of the type

$$\|v(t)\|_q \leq C_1 + C_2\int_0^t (t-s)^{-1+\beta-\alpha}\|v(t)\|_q \, \mathrm{d}s$$

and may use the generalized Gronwall lemma. □

Now, we are going to show that the global solution is bounded on $\mathbb{R}^+$, i.e. the first assertion of Theorem 2.2.

Proposition 4.2 *Let the assumptions of Theorem 2.1 be satisfied. Moreover, let $f \in L^1(\mathbb{R}^+, L^q)$ and let g satisfy (2.5), (2.6). Then there exists a constant $K = K(\|u^1\|_2, \|u^0\|_{W^{s,2}}, \|f\|_{L^1(R^+,L^q)})$ such that*

$$\|u(t)\|_{W^{1,\infty}} + \|\dot{u}(t)\|_2 \leq K, \quad t \geq 0. \tag{4.10}$$

Proof: . We proceed in the same way as above, using the formula

$$Du + c * g(Du) = Du^0 + DS_1u^1 + T * Pg(Du) + DS_1 * f + c * (I-P)g(Du),$$

rewritten in the form (3.33), Theorem 3.1 and the Gronwall inequality, this time with a kernel of L^1-norm less than 1. In fact, we denote $m(t) = \|T(t)\|_{B(L^p;L^\infty)}$ and use (2.5) and the energy inequality (4.5), to estimate

$$\Big\|\int_0^t T(t-s)Pg(Du(s)) \, \mathrm{d}s\Big\|_\infty.$$

For this purpose note that (2.6) implies that for each $\rho > 0$ there is constant $C_\rho \geq 0$ such that

$$|g(s)|^p \leq C_\rho + \rho |s|^p G(s), \quad s \in \mathbb{R}.$$

Fix any $\rho > 0$ small enough and estimate in the following way.

$$\Big\| \int_0^t T(t-s) P g(Du(s)) \,\mathrm{d}s \Big\|_\infty \leq$$

$$\int_0^t m(t-s) \|g(Du(s))\|_p \,\mathrm{d}s \leq \int_0^t m(t-s)(C_\rho + \rho \|Du(s)\|_\infty^p \|G(Du(s)\|_1)^{\frac{1}{p}} \,\mathrm{d}s$$

$$\leq C_\rho^{1/p} |m|_1 + \rho^{1/p} \sup_s \|G(Du(s)\|_1^{1/p} \int_0^t m(t-s) \|Du(s)\|_\infty \,\mathrm{d}s$$

$$\leq C_1 + \rho^{1/p} C_2 \int_0^t m(t-s) \|Du(s)\|_\infty \,\mathrm{d}s.$$

Now choose ρ so small that $\rho^{1/p} C_2 \|m\|_{L^1(R^+)} < 1$.

Applying Theorem 3.1 to Du (recall (4.8)), we obtain for $z(t) = \|Du\|_{L^\infty((0,t);L^\infty)}$ the inequality

$$z(t) \leq C + \rho^{1/p} C_2 m * z(t)$$

which yields boundedness of z and hence of Du in the L^∞-norm on $\mathbb{R}^+$.

The time derivative of the solution is estimated in the following Lemma.

We set

$$a_1(t) = c_1 t^{\alpha/2-1} e^{-t} \tag{4.11}$$

where the constant c_1 is chosen such that

$$|\widehat{a}_1(\mathrm{i}\rho)|^2 \leq \mathrm{Re}\, \widehat{a}(\mathrm{i}\rho), \quad \rho \in \mathbb{R} \tag{4.12}$$

Then a is a_1-positive and we have

Lemma 4.1 *Let a_1 be given by (4.11), (4.12), $f \in L^1(\mathbb{R}^+; L^2)$ and u be a solution of (1.1) satisfying the energy inequality (4.5). Then*

$$Du \in BUC(\mathbb{R}^+; L^2), \tag{4.13}$$

$$a_1 * \frac{\mathrm{d}}{\mathrm{d}t} g(Du) \in L^2(\mathbb{R}^+; L^2), \tag{4.14}$$

$$\dot{u}(t) \to 0 \textit{ in } L^2(0,1) \textit{ as } t \to \infty. \tag{4.15}$$

Proof: The first assertion follows from the formula

$$Du = Du^0 + DS_1 u^1 + DS_1 * f + T * Pg(Du) - c * Pg(Du).$$

In fact, it follows from assumptions **(IC)**, **(f)** and by uniform boundedness of Du that the first four terms are in $BUC(\mathbb{R}^+; L^2)$, by Lemma 3.2. It is easily checked that the last term belongs to this class since $c(t)$ is 1-monotone.

For the second statement we use the abbreviation

$$\langle\langle v\rangle\rangle(t)=\int_0^t\langle a*v(s),v(s)\rangle\,\mathrm{d}s.$$

With this notation we have by (3.32) and (4.5)

$$\begin{aligned}\int_0^\tau\|a_1*\frac{\mathrm{d}}{\mathrm{d}t}g(Du)\|_2^2\,\mathrm{d}t &\leq C\langle\langle\frac{\mathrm{d}}{\mathrm{d}t}g(Du)\rangle\rangle(\tau)\\ &\leq C\sup_{|s|\leq L}|g'(s)|^2\langle\langle D\dot{u}\rangle\rangle(\tau)<\infty,\end{aligned}$$

where $L=\sup_{t>0}\|Du(t)\|_\infty$.

Finally, the last statement follows as in [11] from $\dot{u}\in BUC(\mathbb{R}^+;L^2)$. To prove this we write

$$\dot{u}=Su^1+S*f+b_\infty DS*Pg(Du)+T_2*\frac{\mathrm{d}}{\mathrm{d}t}a_1*Pg(Du),$$

where T_2 is defined by $\widehat{T}_2(\lambda)=D\widehat{S}_{b_1}(\lambda)/\widehat{a}_1(\lambda)$. Since $u_1\in L^q$, $f\in L^1(\mathbb{R}^+;L^q)$, $q\geq 2$, by assumption, and $Pg(Du)\in L^\infty(\mathbb{R}^+;L^\infty)$ by Proposition 4.2, Lemma 3.2 implies that the first three terms belong even to $BUC(\mathbb{R}^+;L^q)$. By the second statement of this lemma we know that $\frac{\mathrm{d}}{\mathrm{d}t}a_1*Pg(Du)\in L^2(\mathbb{R}^+;L^2)$, hence it remains to show that $T_2\in L^2(\mathbb{R}^+;\mathcal{B}(L^2)$. To see this, observe that the estimate

$$|\widehat{T}_2(\lambda)|_{\mathcal{B}(L^2)}+|\lambda\widehat{T}_2'(\lambda)|_{\mathcal{B}(L_2)}\leq C/|\lambda|^{\beta-\alpha+1/2},\quad \mathrm{Re}\ \lambda>0,$$

holds. Hence Lemma 3.1 yields

$$|T_2(t)|_{\mathcal{B}(L_2)}\leq Ct^{\beta-\alpha-1/2},\quad t>0,$$

which implies $T_2\in L^2((0,1);\mathcal{B}(L^2))$, as well as $T_2\in L^\infty((1,\infty);\mathcal{B}(L^2))$. A local analyticity argument as in the proof of Lemma 3.2 also yields $T_2\in L^1((0,\infty);\mathcal{B}(L^2))$, hence $T_2\in L^2((0,\infty);\mathcal{B}(L^2))$ follows. □

5 Global Asymptotic Stability

In this section, we prove the remaining parts of Theorem 2.2. First, we show $Du\in L^r(1,\infty;L^2)$, for some $r\in[2,\infty)$, which together with (4.13) implies $Du\to 0$ in $L^2(0,1)$, i.e. $u(t)\to 0$ as $t\to\infty$ in $W^{1,2}(0,1)$. To this end, we require that (2.5) holds with $R=0$. This assumption assures that the problem has a unique stationary state, the zero solution, and that $g'(0)>0$.

We write Du as a sum corresponding to the decomposition $AS_b=c-T$,

$$Du(t)=Du^0+DS_1(t)u^1-AS_b*Pg(Du)(t)+DS_1*f(t)=w(t)-c*Pg(Du)(t),\tag{5.1}$$

where

$$w(t)=Du^0+DS_1(t)u^1+T*Pg(Du)(t)+DS_1*f(t).\tag{5.2}$$

Let B denote the inverse operator to $c*$, i.e.,

$$\widehat{Bv}(\lambda)=\frac{1}{\widehat{c}(\lambda)}\widehat{v}(\lambda).$$

From (5.1) we have

$$B(Du - w) + Pg(Du) = 0.$$

The scalar product with $Du\|Du\|_2^{r-2}$ gives

$$\langle B(Du-w)(t), Du(t)\rangle \|Du(t)\|_2^{r-2} + \langle g(Du(t)), Du(t)\rangle \|Du(t)\|_2^{r-2} = 0.$$

We assume (2.5) with $R = 0$, so we have

$$\langle g(Du)(t), Du(t)\rangle \|Du(t)\|_2^{r-2} \geq \epsilon \|Du(t)\|_2^r.$$

It is well-known that the operator B is accretive in $L^p([0,T]; L^2(0,1))$ for each $p \in [1,\infty]$, since $c(t)$ is completely positive; see CLÉMENT and PRÜSS [3]. This implies

$$\int_0^T \langle BDu(t), Du(t)\rangle \|Du(t)\|_2^{r-2}\, \mathrm{d}t \geq 0.$$

Therefore we obtain

$$\epsilon \int_0^T \|Du(t)\|_2^r\, \mathrm{d}t \leq \int_0^T \langle Bw(t), Du(t)\rangle \|Du(t)\|_2^{r-2}\, \mathrm{d}t, \qquad \text{for all } T > 0. \tag{5.3}$$

We now show $Bw = w_1 + w_2$ with $w_1 \in L^1(\mathbb{R}^+; L^2)$ and $w_2 \in L^r(\mathbb{R}^+; L^2)$, which by the uniform bound on $\|Du(t)\|_\infty$ yields $Du \in L^r(\mathbb{R}^+; L^2)$.

Lemma 5.1 *Let u be a weak solution of (1.1) and w be given by (5.2). Then*

$$Bw = w_1 + w_2,\ \ w_1 \in L^1(\mathbb{R}^+; L^2),\ \ w_2 \in L^r(\mathbb{R}^+; L^2), \tag{5.4}$$

for each $r > \frac{1}{\mu}$ with μ such that (6.6) holds.

Proof: By the definitions of $c(t)$, $l(t)$, and $T(t)$ we have

$$\begin{aligned} Bw &= \frac{\mathrm{d}}{\mathrm{d}t} l * w = lDu^0 + l * DSu^1 + l * DS * f + \frac{\mathrm{d}}{\mathrm{d}t} l * \frac{\mathrm{d}}{\mathrm{d}t} c * S * Pg(Du) \\ &= a * (k_1 Du^0 + k_0 DSu^1 + k_1 * DSu^1 + k_0 DS * f + k_1 * DS * f) \\ &+ k_0 a Du^0 + SPg(Du^0) + S * \frac{\mathrm{d}}{\mathrm{d}t} Pg(Du). \end{aligned}$$

The kernel a belongs to $L^r(1,\infty)$ with $r > \frac{1}{\mu}$, μ such that (6.6) holds, and it belongs to $L^1(0,1)$. Hence a decomposes as $a = a_2 + a_3$ where $a_2 \in L^1(\mathbb{R}^+)$ and $a_3 \in L^r(\mathbb{R}^+)$. This gives the decomposition $Bw = w_1 + w_2$ where

$$\begin{aligned} w_1 &= a_2 * (k_1 Du^0 + k_0 DS * f + k_1 * DSu^1 + k_1 * DS * f) + a_2 k_0 Du^0 + SPg(Du^0) \\ w_2 &= a_3 * (k_1 Du^0 + k_0 DS * f + k_1 * DSu^1 + k_1 * DS * f) + a_3 k_0 Du^0 + S * \tfrac{\mathrm{d}}{\mathrm{d}t} Pg(Du). \end{aligned}$$

In view of $S \in L^1(\mathbb{R}^+; \mathcal{B}(L^2))$, $DS \in L^1(\mathbb{R}^+; \mathcal{B}(L^2))$, $f \in L^1(\mathbb{R}^+, L^2)$, $k_0 \geq 0$, and $k_1 \in L^1(\mathbb{R}^+)$, we have $w_1 \in L^1(\mathbb{R}^+; L^2)$ and, as soon as we show that $S * \frac{\mathrm{d}}{\mathrm{d}t} Pg(Du) \in L^r(\mathbb{R}^+; L^2)$, we get $w_2 \in L^r(\mathbb{R}^+; L^2)$.

To prove the latter, as in [11] we use the generalization of the Paley-Wiener theorem to the Laplace transform of Hilbert space valued L^2 functions. We have

$$\mathcal{L}(S * \frac{\mathrm{d}}{\mathrm{d}t} g(u_x)) = \mathcal{L}(T_2 * a_1 * \frac{\mathrm{d}}{\mathrm{d}t} g(u_x)),$$

where

$$\widehat{T}_2(\lambda) = \frac{1}{\widehat{a}_1(\lambda)} \widehat{S}(\lambda), \quad a_1 \text{ given by (4.11).}$$

The function $(1 + \lambda^{1-\frac{\alpha}{2}})\widehat{T}_2(\lambda)$ is bounded in $\mathrm{Re}\lambda > 0$ and $a_1 * \frac{\mathrm{d}}{\mathrm{d}t} g(u_x) \in L^2(\mathbb{R}^+, L^2)$ according to (4.14). Hence, by the vector-valued Paley-Wiener theorem,

$$S * \frac{\mathrm{d}}{\mathrm{d}t} g(u_x) \in L^2(\mathbb{R}^+; L^2).$$

We also know that $T_2 \in L^2(\mathbb{R}^+; \mathcal{B}(L^2))$ and so

$$S * \frac{\mathrm{d}}{\mathrm{d}t} g(u_x) \in L^\infty(\mathbb{R}^+; L^2).$$

Combining these results we get $w_2 \in L^r(\mathbb{R}^+; L^2)$. $\square$

From the relation (5.3) we have

$$\epsilon \int_0^T \|Du(t)\|_2^r \, \mathrm{d}t \le \int_0^T \langle Bw(t), Du(t)\rangle \|Du(t)\|_2^{r-2} \, \mathrm{d}t \le$$

$$\le \|w_1(t)\|_{L^1(R^+;L^2)} \sup_{t\in R^+} \|u_x(t)\|_{L^\infty}^{r-1} + \|w_2\|_{L^r(R^+;L^2)} \|Du\|_{L^r((0,T),L^2)}^{r-1} \le$$

$$\le C_1 + C_2 \|Du\|_{L^r((0,T),L^2)}^{r-1}$$

for all $T > 0$. This implies $Du \in L^r(\mathbb{R}^+, L^2)$. By $Du \in BUC(\mathbb{R}^+; L^2)$ we obtain $Du(t) \to 0$ in L^2 as $t \to \infty$, and then also $Du(t) \to 0$ in every $L^q(0,1)$ because of uniform boundedness of Du in $L^\infty(0,1)$.

To prove the remaining assertions of Theorem 2.2 we write the problem (1.1)-(1.3) in the following form

$$\ddot{u} + a * A\dot{u} + \frac{\mathrm{d}}{\mathrm{d}t} b * \nu A u = f + \frac{\mathrm{d}}{\mathrm{d}t} b * (\nu A u + Dg(Du)); \; t > 0 \quad u(0) = u^0, \; \dot{u}(0) = u^1. \quad (5.5)$$

This is rewritten as

$$\ddot{u} + (a + \nu b_1) * A\dot{u} + \nu b_\infty A u = f + \frac{\mathrm{d}}{\mathrm{d}t} b * (\nu A u + Dg(Du)) - \nu b_1 A u^0. \quad (5.6)$$

The solution of the linear problem

$$\ddot{u} + d * A\dot{u} + \nu b_\infty A u = f, \quad u(0) = u^0, \; \dot{u}(0) = u^1, \quad d = a + \nu b_1,$$

is given by the variation of parameters formula

$$u(t) = C(t)u^0 + R(t)u^1 + \int_0^t R(t-s) f(s) \, \mathrm{d}s, \; t \ge 0, \quad (5.7)$$

where the operator families $C(t)$ and $R(t)$ correspond to the operator-cosine and -sine families in the case $d \equiv 0$. Their Laplace transforms are given by

$$\widehat{C}(\lambda) = (\lambda + \widehat{d}(\lambda)A)\widehat{R}(\lambda),$$

and

$$\widehat{R}(\lambda) = (\lambda^2 + \lambda\widehat{d}(\lambda)A + b_\infty A)^{-1},$$

for Re $\lambda > 0$. Some properties of operator families of this type are collected in [11], Lemma 4.4. They are based on results from PRÜSS [13], FAŠANGOVÁ and PRÜSS [4], [5]). Making use of these results and Lemma 3.1, we get the estimates

$$\|C(t)\|_{B(L^\infty)} \to 0, \quad \|AR(t)\|_{B(L^q)} \leq C\min\{t^{-\alpha}, t^{\alpha-2}\}, \quad AR \in L^1(\mathbb{R}^+, \mathcal{B}(L^\infty)) \tag{5.8}$$

and, analogously to the decomposition of the operator AS_b, we write

$$AR(t) = k_\nu(t) - T_\nu(t)$$

where now

$$\widehat{k}_\nu(\lambda) = \frac{1}{\lambda\widehat{d}(\lambda) + \nu b_\infty}, \text{ and } \widehat{T}_\nu(\lambda) = \lambda^2\widehat{k}_\nu(\lambda)\widehat{R}(\lambda).$$

We may now proceed as we did above, to obtain the following equation for $Du(t)$:

$$Du(t) = C(t)Du^0 + DR(t)u^1 - AR * P\Big[\frac{\mathrm{d}}{\mathrm{d}t}b * (g(Du) - \nu Du) - \nu b_1 Du^0\Big](t) + DR * f(t).$$

Hence, denoting $k_\nu^b = \frac{\mathrm{d}}{\mathrm{d}t}b * k_\nu$, $T_\nu^b = \frac{\mathrm{d}}{\mathrm{d}t}b * T_\nu$, we have

$$Du + k_\nu^b * [g(Du) - \nu Du] = w,$$

where

$$\begin{aligned} w(t) = &C(t)Du^0 + DR(t)u^1 + T_\nu^b * P(g(Du) - \nu Du)(t) \\ &+ k_\nu^b(I-P)g(Du)(t) + AR * b_1(t)Du^0 + DR * f(t). \end{aligned}$$

By the asssumptions on g there is a constant $\epsilon > 0$ such that $sg(s) - \epsilon s^2 \geq 0$. The conditions satisfied by the kernels a, b, k_ν with $\nu = \epsilon$ imply that $k_\epsilon^b \in L^1(\mathbb{R}^+)$. Therefore we may apply the second part of Theorem 3.1 of Chapter 20 in [8] to obtain $\|Du(t)\|_\infty \to 0$ as $t \to \infty$, once we have $\|w(t)\|_\infty \to 0$. This will follow from the fact that $\|Du(t)\|_r \to 0$ as $t \to \infty$, for any $r \in [1, \infty)$, and properties of the operator families $C(t)$, $R(t)$. In fact, we have $C(t)Du^0 \to 0$ by (5.8), $\|(I-P)g(Du)(t)\|_\infty \to 0$ implies $k_\epsilon^b * (I-P)g(Du)(t) \to 0$, and the estimates on $R(t)$ show

$$\|DR(t)u^1\|_\infty \leq \|DA^{-1}\|_{\mathcal{B}(L^q, L^\infty)}\|AR(t)u^1\|_q \leq Ct^{\alpha-2}\|u^1\|_q \to 0.$$

In the same way we estimate $DR * f$. To conclude, observe that in the same way as for T we obtain

$$T_\epsilon^b \in L^1(\mathbb{R}^+; \mathcal{B}(L^r(0,1), C^\mu[0,1])), \quad \text{for } 0 < \mu < 2\frac{1-\alpha}{1+\alpha} - \frac{1}{r}.$$

This then implies convergence to 0 in $L^\infty(0,1)$ of the third term in the definition of $w(t)$, hence $\|Du(t)\|_\infty \to 0$ as $t \to \infty$.

As to the convergence of strong solutions, with $\nu = g'(0) > 0$ we may rewrite problem (1.1)~(1.3) as

$$u(t) = C(t)u^0 + R(t)u^1 + R * f(t) - \nu R * b_1(t)Au^0$$

$$+\int_0^t \frac{\mathrm{d}}{\mathrm{d}t} R * b(t-s)[g'(0) - g'(Du(s))]Au(s)\ \mathrm{d}s,\ t \geq 0,$$

which yields the following inequality for $\phi(t) = \|Au(t)\|_X$:

$$\phi(t) \leq \phi_0(t) + \int_0^t \gamma(t-s)\|g'(0) - g'(Du(s))\|_\infty \phi(s)\ \mathrm{d}s,\ t \geq 0,$$

where

$$\phi_0(t) = \|C(t)Au^0 + A^{1/2}R(t)A^{1/2}u^1 + A^{1/2}R(t) * A^{1/2}f(t) - \nu AR * b_1 Au^0\|_X$$

$$\text{and } \gamma(t) = \|\frac{\mathrm{d}}{\mathrm{d}t} b * AR(t)\|_{\mathcal{B}(X)} \in L^1(\mathbb{R}^+).$$

Because of $\|Du(t)\|_\infty \to 0$ as $t \to \infty$, and $\phi_0(t) \to 0$ this implies $\phi(t) = \|Au(t)\|_X \to 0$. Here X may be any of the spaces $X = L^p(0,1)$, $1 \leq p < \infty$ or $X = C_0[0,1]$. In a similar way one also gets $\|A^{1/2}\dot{u}(t)\|_X \to 0$ as $t \to \infty$. This proves the last statement of Theorem 2.2.

□

6 Appendix

In this section, we collect some properties of the kernels satisfying hypotheses (2.1)-(2.3), in particular we give some necessary and sufficient conditions in the time domain. We begin with the argument condition for $a(t)$.

Lemma 6.1 *Suppose a is 3-monotone, $a \not\equiv$ constant. Then the following assertions are equivalent:*

(i) $|\arg \widehat{a}(\lambda)| \leq \theta_a < \frac{\pi}{2}$ *for* $\mathrm{Re}\ \lambda > 0$*;*

(ii) *There is a constant $c_a < 1$ such that* $ta(t) \leq c_a \int_0^t a(s)\ \mathrm{d}s$

(iii) $\limsup_{t \to 0,\infty} \frac{ta(t)}{\int_0^t a(s)\ \mathrm{d}s} < 1$.

Proof: By [13, Proposition 3.10] we have

$$\frac{3}{5}\int_0^{\frac{1}{\rho}} -t\dot{a}(t)\ \mathrm{d}t \leq \mathrm{Re}\ \widehat{a}(\mathrm{i}\rho) \leq 4\int_0^{\frac{1}{\rho}} -t\dot{a}(t)\ \mathrm{d}t \tag{6.1}$$

as well as

$$\frac{1}{2\sqrt{2}}\int_0^{\frac{1}{\rho}} a(t)\ \mathrm{d}t \leq |\widehat{a}(\mathrm{i}\rho)| \leq 2\int_0^{\frac{1}{\rho}} a(t)\ \mathrm{d}t. \tag{6.2}$$

The equivalence between (i) and (ii) now follows easily:

$$|\arg \widehat{a}(\lambda)| \leq \theta_a \iff \frac{|\mathrm{Im}\ \widehat{a}(\lambda)|}{\mathrm{Re}\ \widehat{a}(\lambda)} \leq C \iff |\widehat{a}(\lambda)| \leq C\mathrm{Re}\ \widehat{a}(\lambda)$$

$$\iff \int_0^t a(s)\ \mathrm{d}s \leq C\int_0^t -s\dot{a}(s)\ \mathrm{d}s = C(-ta(t) + \int_0^t a(s)\ \mathrm{d}s)$$

$$\iff (C-1)\int_0^t a(s)\ \mathrm{d}s \geq Cta(t) \iff ta(t) \leq (1 - \frac{1}{C})\int_0^t a(s)\ \mathrm{d}s.$$

The equivalence between (ii) and (iii) follows since $ta(t) \le \int_0^t a(s)\ \mathrm{d}s$ for all $t > 0$ and the equality holds if and only if $a(s) \equiv a(t)$ on $(0, t]$. Since a is convex, this yields a constant. □

In the following corollary some sufficient conditions for Lemma 6.1 (iii) are given.

Corollary 6.1 *Suppose a is 3-monotone. Then*

(i) *If* $\lim_{t\to 0} \frac{a(t)}{t^{\alpha_0 - 1}} = a_{\alpha_0} \in (0, \infty)$ *exists for some* $\alpha_0 \in (0,1)$
then $\limsup_{t\to 0} \frac{ta(t)}{\int_0^t a(s)\ \mathrm{d}s} = \alpha_0 < 1.$

(ii) *If* $\lim_{t\to \infty} \frac{a(t)}{t^{\alpha_1 - 1}} = a_{\alpha_1} \in (0, \infty)$ *exists for some* $\alpha_1 \in (0,1)$
then $\limsup_{t\to \infty} \frac{ta(t)}{\int_0^t a(s)\ \mathrm{d}s} = \alpha_1 < 1.$

(iii) *If* $a \in L^1(\mathbb{R}^+)$ *then* $\limsup_{t\to \infty} \frac{ta(t)}{\int_0^t a(s)\ \mathrm{d}s} = 0.$

The lower bound on $\widehat{a}$ can be characterized as follows.

Lemma 6.2 *Suppose a is 2-monotone, $\alpha \in (0, 1]$, $a \not\equiv 0$. Then*

$$|\widehat{a}(\lambda)| \ge \frac{C}{1 + |\lambda|^\alpha}, \quad \mathrm{Re}\ \lambda > 0$$

iff

$$\frac{1}{t}\int_0^t a(s)\ \mathrm{d}s \ge ct^{\alpha - 1}, \quad t \in (0,1),$$

for some constant $c > 0$.

Proof: By Prüss, [13, Proposition 3.3] a is 1-regular and [13, Lemma 8.1] shows that

$$C|\widehat{a}(|\lambda|)| \le |\widehat{a}(\lambda)| \le C^{-1}|\widehat{a}(|\lambda|)|, \quad \mathrm{Re}\ \lambda > 0. \tag{6.3}$$

Since a is nonnegative, $a \not\equiv 0$, we have $\widehat{a}(\lambda) > 0$ for $\lambda > 0$. Hence it is sufficient to show the equivalence

$$\widehat{a}(\lambda) \ge \frac{C}{\lambda^\alpha},\ \lambda \ge 1 \Longleftrightarrow \int_0^t a(s)\ \mathrm{d}s \ge ct^\alpha, \quad t \in (0,1). \tag{6.4}$$

For $\lambda > 0$ we have

$$\frac{1}{\mathrm{e}}\int_0^{\frac{1}{\lambda}} a(s)\ \mathrm{d}s \le \int_0^{\frac{1}{\lambda}} e^{-\lambda s} a(s)\ \mathrm{d}s \le \widehat{a}(\lambda),$$

$$\widehat{a}(\lambda) \le \int_0^{\frac{1}{\lambda}} a(s)\ \mathrm{d}s + a(\frac{1}{\lambda})\int_{\frac{1}{\lambda}}^\infty e^{-\lambda s}\ \mathrm{d}s \le \int_0^{\frac{1}{\lambda}} a(s)\ \mathrm{d}s + a(\frac{1}{\lambda})\frac{1}{\lambda}\frac{1}{\mathrm{e}} \le (1 + \frac{1}{\mathrm{e}})\int_0^{\frac{1}{\lambda}} a(s)\ \mathrm{d}s.$$

This yields (6.4). □

In a similar way the upper bound on $\widehat{b}$ can be characterized in the time domain.

Lemma 6.3 *Suppose b is 2-monotone, $\beta \in (0,1]$. Then*

$$|\widehat{b}(\lambda)| \leq \frac{C}{1+|\lambda|^\beta}, \quad \mathrm{Re}\ \lambda > 0$$

iff

$$b \in L^1(\mathbb{R}^+) \quad \textit{and} \quad \frac{1}{t}\int_0^t b(s)\ \mathrm{d}s \leq Ct^{\beta-1}, \quad t \in (0,1).$$

Proof: As in the proof of the preceding Lemma we have

$$\widehat{b}(\lambda) \leq \frac{C}{\lambda^\beta},\ \lambda \geq 1 \Longleftrightarrow \frac{1}{t}\int_0^t b(s)\ \mathrm{d}s \leq Ct^{\beta-1}, \quad t \in (0,1). \tag{6.5}$$

On the other hand, since b is nonnegative, we have

$$\widehat{b}(\lambda) \leq \widehat{b}(0) = \int_0^\infty b(s)\ \mathrm{d}s,\ \lambda > 0.$$

Hence $b \in L^1(\mathbb{R}^+) \Longleftrightarrow \widehat{b}(\lambda) \leq C$ for $\lambda \in (0,1]$. □

Remark. In Lemma 6.3 we may replace $\frac{1}{t}\int_0^t b(s)\ \mathrm{d}s$ by $b(t)$ while in Lemma 6.2 it is not clear.

The following upper bound on $\widehat{a}$ is used in the proof of Theorem 2.

Lemma 6.4 *Suppose $a \in L^1_{loc}(\mathbb{R}^+)$ is 3-monotone and such that $|arg\ \widehat{a}(\lambda)| \leq \theta_a < \frac{\pi}{2}$ for $\mathrm{Re}\ \lambda > 0$. Then there is $\mu > 0$ such that*

$$\widehat{a}(\lambda) \leq \frac{C}{\lambda^{1-\mu}} \quad \textit{for} \quad 0 < \lambda \leq 1 \quad \textit{and} \quad a(t) \leq Ct^{-\mu} \quad \textit{for} \quad t \geq 1. \tag{6.6}$$

Proof: The argument condition implies

$$\mathrm{Re}\ \widehat{a}(\mathrm{i}\rho) \leq |\widehat{a}(\mathrm{i}\rho)| \leq \mathrm{Re}\ \widehat{a}(\mathrm{i}\rho)/\cos\theta_a,\ \rho \in \mathbb{R}\backslash\{0\}.$$

Moreover, recall (6.1) and (6.2). Now for $0 < \lambda < 1$

$$\log\widehat{a}(\lambda) = \log\widehat{a}(1) + \int_\lambda^1 -\xi\frac{\widehat{a}'(\xi)}{\widehat{a}(\xi)}\frac{\mathrm{d}\xi}{\xi},$$

and

$$-\xi\frac{\widehat{a}'(\xi)}{\widehat{a}(\xi)} = \xi\frac{\widehat{ta}(\xi)}{\widehat{a}(\xi)} = \frac{\widehat{\frac{\mathrm{d}}{\mathrm{d}t}(ta)}(\xi)}{\widehat{a}(\xi)} = 1 - \frac{-\widehat{t\dot{a}}(\xi)}{\widehat{a}(\xi)}.$$

Further

$$\frac{-\widehat{t\dot{a}}(\xi)}{\widehat{a}(\xi)} \geq \frac{\frac{1}{\mathrm{e}}\int_0^{\frac{1}{\xi}} -t\dot{a}(t)\ \mathrm{d}t}{2\int_0^{\frac{1}{\xi}} a(t)\ \mathrm{d}t} \geq \mu \text{ for all } \xi > 0,$$

with some constant $\mu > 0$; in the last step we used the argument condition once more. Therefore

$$\log\widehat{a}(\lambda) \leq \log\widehat{a}(1) + \int_\lambda^1 (1-\mu)\frac{\mathrm{d}\xi}{\xi} = \log\widehat{a}(1) + (\mu-1)\log\lambda,$$

which yields

$$\widehat{a}(\lambda) \le \widehat{a}(1)\lambda^{\mu-1}, \quad 0 < \lambda < 1.$$

Finally,

$$\frac{1}{\lambda}a(1/\lambda) \le \int_0^{\frac{1}{\lambda}} a(t)\, \mathrm{d}t \le C\widehat{a}(\lambda) \le C\lambda^{\mu-1}$$

gives

$$a(t) \le Ct^{-\mu} \text{ for } t \ge 1,$$

taking $t = 1/\lambda$. □

References

[1] J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego, P.J. Stewart. On the dynamics of fine structure. *J. Nonlinear Science*, **1**:17:70, 1991.

[2] R. Chill. Tauberian theorems for vector-valued Fourier and Laplace transforms. *Studia Math.***128**:55–69, 1998.

[3] Ph. Clément and J. Prüss. Completely positive measures and Feller semigroups. *Math. Ann.*, **287**:73–105, 1990.

[4] E. Fašangová and J. Prüss. Asymptotic behaviour of a semilinear viscoelastic beam model. *Archiv Math.*, to appear 2001.

[5] E. Fašangová and J. Prüss. Evolution equations with dissipation of memory type. *Topics in Nonlinear Analysis*, 213–250, Birkhäuser 1998.

[6] J.M. Greenberg. On the existence, uniqueness and stability of the equation $\rho_0 \mathrm{x}_{tt} = e(\mathrm{x}_x)\mathrm{x}_{xx} + \lambda \mathrm{x}_{xxt}$. *J. Math. Anal. Appl.*, **25**:575–591, 1969.

[7] G. Gripenberg, S.-O. Londen, and J. Prüss. On a fractional partial differential equation with dominating linear part. *Math. Meth. Appl. Sci.*, **20**:1427–1448, 1997.

[8] G. Gripenberg, S.-O. Londen, and O. Staffans. *Volterra integral and functional equations.* Cambridge University Press, Cambridge, 1990.

[9] D. Henry. *Geometric theory of semilinear parabolic equations.* Lect. Notes in Math. 840, Springer-Verlag, New York, 1981.

[10] R.L. Pego. Phase transitions in one-dimensional viscoelasticity: admissibility and stability. *Arch. Rat. Mech. Anal.*, **97**:353–394, 1987.

[11] H. Petzeltová and J. Prüss. Global stability of a fractional partial differential equations. *J. Integral Equations Appl.*, **12**:323–347, 2000

[12] J. Prüss. Laplace transforms and regularity of solutions of evolutionary integral equations. Preprint 1996.

[13] J. Prüss. *Evolutionary integral equations and applications.* Birkhäuser, Basel, Boston, Berlin, 1993.

[14] H. Triebel. *Theory of function spaces.* Akademische Verlagsgesellschaft, Leipzig, 1983.