# Maximal Regularity for Abstract Parabolic Problems with Inhomogeneous Boundary Data in $L_p$-Spaces

Jan Prüss

*Fachbereich Mathematik und Informatik, Martin-Luther-Universiät Halle-Wittenberg, Theodor-Lieser Str. 5,* **D-60120 Halle**, *Germany*

**Abstract**

Several abstract model problems of elliptic and parabolic type with inhomogeneous initial and boundary data are discussed. By means of a variant of the Dore-Venni theorem, real and complex interpolation, and trace theorems, optimal $L_p$-regularity is shown. By means of this purely operator theoretic approach classical results on $L_p$-regularity of the diffusion equation with inhomogeneous Dirichlet or Neumann or Robin condition are recovered. An application to a dynamic boundary value problem with surface diffusion for the diffusion equation is included.

**AMS Classification** 35K20, 35K22, 45K05, 47D05

**Keywords and Phrases**

Maximal regularity, sectorial operators, interpolation, trace theorems, elliptic and parabolic initial-boundary value problems, dynamic boundary conditions.

## 1. Introduction

The operator-sum method as developed by Da Prato and Grisvard [1], Dore and Venni [2], and recently by Kalton and Weis [7] has been employed successfully to solve abstract Cauchy problems of the form

$$\dot{u}(t) + Au(t) = f(t),\ t > 0, \quad u(0) = 0,$$

where $A$ denotes the generator of a bounded analytic $C_0$-semigroup in the Banach space $X$. It is the goal of this paper to show how this method can be used to obtain maximal $L_p$-regularity for a variety of abstract elliptic and parabolic problems with *inhomogeneous* initial and boundary values. Such problems arise in the real pde world as model problems. Results on such model problems can be transferred to elliptic and parabolic boundary value problems on domains with smooth boundary by well-known techniques like localization, perturbation and coordinate transformation.

As an example of this strategy let us consider the diffusion equation with dynamic boundary condition involving *surface diffusion*. So let $\Omega \subset \mathbb{R}^n$ be an open bounded

domain with boundary $\Gamma = \partial\Omega$ of class $C^2$, such that $\Gamma$ decomposes into three disjoint parts $\Gamma = \Gamma_0 \cup \Gamma_n \cup \Gamma_d$, where $\Gamma_j$ is open and closed in $\Gamma$. Consider the problem

$$\begin{aligned}
&\partial_t u(t,x) - \Delta u(t,x) = f(t,x), \quad t > 0,\ x \in \Omega,\\
&u(t,x) = \phi(t,x),\ x \in \Gamma_0, \quad \partial_n u(t,x) = \psi(t,x),\ x \in \Gamma_n,\ t > 0,\\
&\partial_t u(t,x) + \partial_n u(t,x) - \Delta_\Gamma u(t,x) = h(t,x),\ x \in \Gamma_d,\ t > 0,\\
&u(0,x) = u_0(x),\ x \in \Omega,
\end{aligned} \tag{1.1}$$

where $\partial_n$ means the normal derivative at the boundary of $\Omega$, and $\Delta_\Gamma$ denotes the Laplace-Beltrami operator on the manifold $\Gamma$. By means of localization, transformation and perturbation this problem can be reduced to four model problems, namely the diffusion equation on $\mathbb{R}^n$, and the diffusion equation on the half-space $\mathbb{R}^n_+$ with Dirichlet or Neumann or dynamic surface diffusion condition on the boundary of $\mathbb{R}^n_+$.

This paper deals with the abstract version of these model problems. To show the strength of our results we have included surface diffusion, which plays the role of an unbounded Robin condition.

As an application, let us present the $L_p$-maximal regularity result for (1.1) which follows from Theorem 7 below.

**Theorem.** *Let $1 < p < \infty$, $p \neq 3/2, 3$, and let $J = [0, a]$. Then problem (1.1) admits a unique solution*

$$u \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega))$$

*such that*

$$u|_{\Gamma_d} \in W^{3/2-1/2p}_p(J; L_p(\Gamma_d)) \cap L_p(J; W^{3-1/p}_p(\Gamma_d)),$$

*if and only if $f \in L_p(J \times \Omega)$, $u_0 \in W^{2-2/p}_p(\Omega)$,*

$$\phi \in W^{1-1/2p}_p(J; L_p(\Gamma_0)) \cap L_p(J; W^{2-1/p}_p(\Gamma_0)),$$

$$\psi \in W^{1/2-1/2p}_p(J; L_p(\Gamma_n)) \cap L_p(J; W^{1-1/p}_p(\Gamma_n)),$$

$$h \in W^{1/2-1/2p}_p(J; L_p(\Gamma_d)) \cap L_p(J; W^{1-1/p}_p(\Gamma_d)),$$

*and the compatibility conditions*

$$u_0|_{\Gamma_0} = \phi|_{t=0} \ \text{ for } p > 3/2,\ \partial_n u_0|_{\Gamma_n} = \psi|_{t=0} \ \text{ for } p > 3,\ u_0|_{\Gamma_d} \in W^{3-3/p}_p(\Gamma_d), \ \text{ for } p > 1,$$

*are satisfied.*

Specializing the results proved below, we are able to recover classical results on parabolic initial-boundary value problems. So e.g. in case $Y = L_p(\mathbb{R}^n)$ and $\mathcal{B} = -\Delta$, $D = 0$, Theorems 5 and 6 reduce to the famous results of Ladyzhenskaya, Solonnikov and Uralt́seva [9] on maximal $L_p$-regularity of second order parabolic initial-boundary value problems. Many other problems fit into the framework of our approach. So for example, in (1.1) we can allow for a Robin condition like $\partial_n u - \Delta_\Gamma u = \psi$ instead of the Neumann condition on $\Gamma_n$, and surface diffusion can be dropped in the dynamic boundary condition on $\Gamma_d$.

In this paper we only employ a theorem of Dore-Venni type, [2], [10], real and complex interpolation, and trace theorems. We would like to point out that our techniques have been applied successfully also to the study of free boundary value problems like the *Stefan problem with surface tension*, see Escher, Prüss and Simonett [3] or the one and two phase free boundary value problems for the Navier-Stokes equation, see Escher, Prüss, and Simonett [4].

Some of our results may be generalized using a more recent theorem of Kalton and Weis [7]. For example, in Theorems 2 and 3, the involved operators $A$ and $F$, need only to be $\mathcal{R}$-sectorial instead of class $\mathcal{BIP}$, and their power angles can be replaced by their $\mathcal{R}$-angles. In Theorem 4, say $D$ needs only to be $\mathcal{R}$-sectorial if in addition $F$ is assumed to admit an $\mathcal{H}^\infty$-calculus and $\phi_F^\infty + \phi_D^R < \pi$. However, extensions in this direction are not clear for Theorems 5 to 7.

## 2. Preliminaries

To introduce some notation used below recall that a Banach space $X$ belongs to the class $\mathcal{HT}$ if the Hilbert transform is bounded on $L_2(\mathbb{R}; X)$. Recall also that a closed linear operator $A$ in $X$ is called *nonnegative* or *pseudo-sectorial* if $(-\infty, 0)$ is contained in the resolvent set of $A$ and the resolvent estimate

$$t|(t + A)^{-1}|_{\mathcal{B}(X)} \leq M_0, \ t > 0,$$

holds, for some constant $M_0 > 0$. If in addition the domain $\mathcal{D}(A)$ and the range $\mathcal{R}(A)$ of $A$ are dense in $X$ then $A$ is called *sectorial*. We emphasize that for (pseudo)-sectorial operators the Dunford functional calculus is available and in particular the fractional powers $A^z$, $z \in \mathbb{C}$ are well-defined closed linear operators in $X$; see e.g. Komatsu [8]. We also recall that in case $X$ is reflexive and $A$ is pseudo-sectorial then the space $X$ decomposes according to $X = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$, where $\mathcal{N}(A)$ designates the kernel of $A$. Thus in such a situation $A$ is sectorial on $\overline{\mathcal{R}(A)}$.

We say that an operator $A$ in $X$ belongs to $\mathcal{BIP}(X)$ if $A$ is sectorial and the imaginary powers $A^{is}$ of $A$ form a bounded $C_0$-group on $X$. The type $\theta_A$ of this group will be called the power angle of $A$. We begin by repeating a variant of the Dore-Venni theorem; see Dore and Venni [2] and Prüss and Sohr [10].

**Theorem 1.** *Suppose $X$ belongs to the class $\mathcal{HT}$, and assume $A, B \in \mathcal{BIP}(X)$ commute and satisfy the* **strong parabolicity condition** *$\theta_A + \theta_B < \pi$, and let $t > 0$. Then*
*(i) $A + tB$ is closed and sectorial;*
*(ii) $A + tB \in \mathcal{BIP}(X)$ with $\theta_{A+tB} \leq \max\{\theta_A, \theta_B\}$;*
*(iii) there is a constant $C > 0$, independent of $t > 0$, such that*

$$|Ax| + t|Bx| \leq C|Ax + tBx|, \quad x \in \mathcal{D}(A) \cap \mathcal{D}(B). \tag{2.1}$$

*In particular, if $A$ or $B$ is invertible, then $A + tB$ is invertible as well.*

Some consequences of this result concerning complex interpolation are contained in the next corollary. By $[X, Z]_\theta$ we denote the complex interpolation spaces between $X$ and $Z$.

**Corollary 1.** *Suppose $X$ is a Banach space of class $\mathcal{HT}$, $A, B \in \mathcal{BIP}(X)$ are commuting in the resolvent sense, and their power angles satisfy the parabolicity condition $\theta_A + \theta_B < \pi$. Let $A$ or $B$ be invertible and $\alpha \in (0, 1)$. Then*
*(a) $A^\alpha (A + B)^{-\alpha}$ and $B^\alpha (A + B)^{-\alpha}$ are bounded in $X$;*
*(b) $\mathcal{D}((A + B)^\alpha) = [X, \mathcal{D}(A + B)]_\alpha = [X, \mathcal{D}(A)]_\alpha \cap [X, \mathcal{D}(B)]_\alpha = \mathcal{D}(A^\alpha) \cap \mathcal{D}(B^\alpha)$.*

The proof of this result is not difficult, but due to limations of space we refer to the forthcoming monograph Hieber and Prüss [6].

The following result is due to Grisvard [5], even in a more general context. We denote the real interpolation spaces between $X$ and $Z$ by $(X, Z)_{\alpha,p}$.

**Proposition 1.** *Suppose $A, B$ are sectorial linear operators in a Banach space $X$, commuting in the resolvent sense and let $\alpha \in (0,1)$, $p \in [1,\infty]$. Then*

$$(X, \mathcal{D}(A) \cap \mathcal{D}(B))_{\alpha,p} = (X, \mathcal{D}(A))_{\alpha,p} \cap (X, \mathcal{D}(B))_{\alpha,p}.$$

The next result is known as the *mixed derivative theorem* and is due to Sobolevskii [11].

**Proposition 2.** *Suppose $A$ and $B$ are sectorial linear operators in a Banach space $X$ with spectral angles $\phi_A + \phi_B < \pi$, which commute and are* coercively positive, *i.e. $A + tB$ with natural domain $\mathcal{D}(A + tB) = \mathcal{D}(A) \cap \mathcal{D}(B)$ is closed for each $t > 0$ and there is a constant $M > 0$ such that*

$$|Ax|_X + t|Bx|_X \le M|Ax + tBx|_X, \quad \text{for all } x \in \mathcal{D}(A) \cap \mathcal{D}(B),\ t > 0.$$

*Then there is constant $C > 0$ such that*

$$|A^\alpha B^{1-\alpha} x|_X \le C|Ax + Bx|_X, \quad \text{for all } x \in \mathcal{D}(A) \cap \mathcal{D}(B),\ \alpha \in [0,1].$$

*In particular, $A^\alpha B^{1-\alpha}(A+B)^{-1}$ is bounded in $X$, for each $\alpha \in [0,1]$.*

Observe that Proposition 2 applies in particular to the situation of the Dore-Venni theorem, Theorem 1. The next result deals with traces of vector-valued functions, where we employ the standard notation $D_A(\alpha, p) = (X, X_A)_{\alpha,p}$, for the real interpolation spaces, where $X_A$ means the Banach space $\mathcal{D}(A)$ equipped with the graph norm of $A$.

**Proposition 3.** *Let $Y$ be a Banach space of class $\mathcal{HT}$, $A$ a sectorial operator $Y$ which belongs to $\mathcal{BIP}(Y)$ with power angle $\theta_A < \pi/s$, $1 < p < \infty$, $s > 1/p$, and let $J$ be an interval. Then*

$$H^s_p(J;Y) \cap L_p(J;\mathcal{D}(A^s)) \hookrightarrow BUC^n(J; D_A(s - n - 1/p, p)),$$

*and also*

$$W^s_p(J;Y) \cap L_p(J;D_A(s,p)) \hookrightarrow BUC^n(J; D_A(s - n - 1/p, p)),$$

*where $n$ means any integer smaller than $s - 1/p$.*

Here as usual, $H^s_p(J;Y)$ resp. $W^s_p(J;Y)$ mean the vector-valued Bessel potential spaces resp. Sobolev–Slobodecky spaces on an open interval $J \subset \mathbb{R}$, where $1 < p < \infty$ and $s > 0$. For the case $\theta_A = 0$ a proof is given in Escher, Prüss, and Simonett [3], the general case is proved in Hieber and Prüss [6].

The next result is a direct consequence of the definition of the real interpolation spaces $D_A(\alpha, p)$, which nevertheless is very useful.

**Proposition 4.** *Let $1 < p < \infty$, $1/p < \alpha < 1$, suppose $A$ is an invertible pseudo-sectorial operator in $X$ with $\phi_A < \pi/2$, and set $u(t) = e^{-At}x$, $x \in X$.*

*Then the following statements are equivalent.*

*(i)* $x \in D_A(\alpha - 1/p, p)$;
*(ii)* $u \in L_p(\mathbb{R}_+; D_A(\alpha, p))$;
*(iii)* $u \in W^\alpha_p(\mathbb{R}_+; X)$.

Observe that $u \in W^\alpha_p(\mathbb{R}_+;X) \cap L_p(\mathbb{R}_+; D_A(\alpha,p))$ holds for all $x \in X$, in case $0 \le \alpha < 1/p$.

## 3. Abstract Equations on the Halfline

We consider now the following abstract theorem on evolution equations

$$\dot{u} + Au = f, \quad t > 0, \quad u(0) = u_0, \tag{3.1}$$

in a Banach space $Y$. The main result on maximal $L_p$-regularity for (3.1) is a well-known consequence of Theorem 1.

**Theorem 2.** *Suppose $Y$ is a Banach space of class $\mathcal{HT}$, $1 < p < \infty$, let $A \in \mathcal{BIP}(Y)$ be invertible with power angle $\theta_A < \pi/2$, and let $D_A$ denote the domain $\mathcal{D}(A)$ of $A$ equipped with the graph norm of $A$.*

*Then (3.1) has precisely one solution in $Z := H^1_p(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_A)$ if and only if*

$$f \in X := L_p(\mathbb{R}_+; Y) \quad \textit{and} \quad u_0 \in D_A(1 - 1/p, p).$$

A result much easier to obtain -but nevertheless useful- is the following.

**Proposition 5.** *Let $1 < p < \infty$, $1/p < \alpha < 1$, suppose $A$ is an invertible pseudo-sectorial operator in $X$ with $\phi_A < \pi/2$, let $f \in L_p(\mathbb{R}_+; X)$ and $x \in X$.*

*Then the following statements for the solution $u$ of (3.1) are equivalent.*

*(i)* $u \in W^{1+\alpha}_p(\mathbb{R}_+; X) \cap W^{\alpha}_p(\mathbb{R}_+; \mathcal{D}(A))$;

*(ii)* $f \in W^{\alpha}_p(\mathbb{R}_+; X)$, $x \in \mathcal{D}(A)$, $Ax - f(0) \in D_A(\alpha - 1/p, p)$.

Our next theorem concerns the abstract second order problem with Dirichlet condition

$$\begin{aligned} -u''(y) + F^2 u(y) &= g(y), \quad y > 0, \\ u(0) &= \phi, \end{aligned} \tag{3.2}$$

in $L_p(\mathbb{R}_+; Y)$.

**Theorem 3.** *Suppose $Y$ is a Banach space of class $\mathcal{HT}$, $1 < p < \infty$, let $F \in \mathcal{BIP}(Y)$ be invertible with power angle $\theta_F < \pi/2$, and let $D^j_F$ denote the domain $\mathcal{D}(F^j)$ of $F^j$ equipped with its graph norm, $j = 1, 2$.*

*Then (3.2) has precisely one solution $u$ in $Z := H^2_p(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D^2_F)$ if and only if*

$$g \in X := L_p(\mathbb{R}_+; Y) \quad \textit{and} \quad \phi \in D_F(2 - 1/p, p).$$

*If this is the case we have in addition $u \in H^1_p(\mathbb{R}_+; D^1_F)$.*

*Proof.* Apply Theorem 1 in $X = L_p(\mathbb{R}_+; Y)$ to $A = F^2$ and $B = -d^2/dy^2$ with domain $\mathcal{D}(B) = H^2_p(\mathbb{R}_+; Y) \cap {}_0H^1_p(\mathbb{R}_+; Y)$ to see that (3.2) admits a unique solution $u \in Z$, for each $g \in X$, $\phi = 0$. It is given explicitly by the formula

$$u(y) = \frac{1}{2} F^{-1} \int_0^\infty [e^{-F|y-s|} - e^{-F(y+s)}] g(s) ds, \quad t > 0.$$

On the other hand, if $-u'' + F^2 u = 0$, and $u$ is bounded then $u' + Fu = 0$. Hence the unique solution of (3.2) in $X$ with $g = 0$ is given by

$$u(y) = e^{-Fy}\phi,$$

which belongs to $Z$ if and only if $\phi \in D_F(2 - 1/p, p)$. In fact, by Theorem 2, $F\phi \in D_F(1 - 1/p, p)$ is equivalent to $v(y) := e^{-Fy} F\phi \in H^1_p(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D^1_F)$, which by

invertibility of $F$ in turn is equivalent to $u \in Z$. The last assertion follows from the mixed derivative theorem with $\alpha = 1/2$. □

There is a companion result for the abstract second order problem with abstract Robin condition which for $D = 0$ becomes the Neumann condition.

$$\begin{aligned} -u''(y) + F^2 u(y) &= g(y), \quad y > 0, \\ -u'(0) + Du(0) &= \psi, \end{aligned} \tag{3.3}$$

For this problem the maximal regularity result in $L_p(\mathbb{R}_+; Y)$ reads as follows.

**Theorem 4.** *Suppose $Y$ is a Banach space of class $\mathcal{HT}$, $1 < p < \infty$, let $F \in \mathcal{BIP}(Y)$ be invertible with power angle $\theta_F < \pi/2$, and let $D_F^j$ denote the domain $\mathcal{D}(F^j)$ of $F^j$ equipped with its graph norm, $j = 1, 2$. Suppose that $D$ is pseudo-sectorial, belongs to $\mathcal{BIP}(\overline{\mathcal{R}(D)})$, commutes with $F$, and is such that $\theta_F + \theta_D < \pi$.*

*Then (3.3) has precisely one solution in $Z := H_p^2(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_F^2)$ with $u(0) \in \mathcal{D}(D)$ and $Du(0) \in D_F(1 - 1/p, p)$ if and only if*

$$g \in X := L_p(\mathbb{R}_+; Y) \quad \text{and} \quad \psi \in D_F(1 - 1/p, p).$$

*If this is the case we have in addition $u \in H_p^1(\mathbb{R}_+; D_F^1)$.*

*Proof.* Since $\mathcal{N}(D) \oplus \overline{\mathcal{R}(D)} = Y$ by reflexivity of $Y$, and $D$ and $F$ commute, Theorem 1 implies that $F + D$ with domain $\mathcal{D}(F + D) = \mathcal{D}(F) \cap \mathcal{D}(D)$ is invertible. The solution $u$ of (3.3) can be written explicitly as will be shown below.

$$u(y) = e^{-Fy}(F + D)^{-1}\psi + \frac{1}{2}F^{-1}\int_0^\infty [e^{-F|y-s|} + (F - D)(F + D)^{-1}e^{-F(y+s)}]g(s)ds.$$

This time we proceed as follows. Apply Theorem 1 to $A = F^2$ and $B = -d^2/dy^2$ in $X = L_p(\mathbb{R}; Y)$, i.e. for the problem on the entire line. This way, for each $g \in X$, we obtain a unique solution $v \in H_p^2(\mathbb{R}; Y) \cap L_p(\mathbb{R}; D_F^2)$. It is given by the formula

$$v(y) = \frac{1}{2}F^{-1}\int_{-\infty}^\infty e^{-F|y-s|}g(s)ds, \quad y \in \mathbb{R}.$$

We write $v = (A + B)^{-1}g$. Now suppose first $\psi = 0$ and let $g \in L_p(\mathbb{R}_+; Y)$. Let $E_0 : L_p(\mathbb{R}_+; Y) \to L_p(\mathbb{R}; Y)$ denote the operator of extension by 0, i.e.

$$(E_0 f)(y) = f(y), \ \ y > 0, \quad (E_0 f)(y) = 0, \ \ y < 0,$$

$P_+ : L_p(\mathbb{R}; Y) \to L_p(\mathbb{R}_+; Y)$ the restriction to $\mathbb{R}_+$, and $R : L_p(\mathbb{R}; Y) \to L_p(\mathbb{R}; Y)$ reflection at 0, i.e.

$$(Rf)(y) = f(-y), \quad y \in \mathbb{R}.$$

these operators are all bounded with norm 1. The solution formula for $u$ may then be rewritten as

$$u = P_+(A + B)^{-1}E_0 g + (F - D)(F + D)^{-1}P_+ R(A + B)^{-1}E_0 g,$$

which shows that the solution $u$ belongs to the maximal regularity space $Z = H_p^2(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_F^2)$. The trace of $u$ at $y = 0$ exists and equals

$$u(0) = (F + D)^{-1}\int_0^\infty e^{-Fs}g(s)ds.$$

This shows $u(0) \in \mathcal{D}(D)$ and

$$Du(0) = D(F+D)^{-1} \int_0^\infty e^{-Fs} g(s) ds \in D_F(1-1/p,p),$$

since $F$ and $D$ commute by assumption, $D(F+D)^{-1}$ is bounded and is leaving invariant $D_F(1-1/p,p)$. In case $g=0$ the solution $u$ is given by

$$u(y) = e^{-Fy}(F+D)^{-1}\psi,$$

hence $u(0) = (F+D)^{-1}\psi \in \mathcal{D}(D)$, as well as

$$Du(0) = D(F+D)^{-1}\psi \in D_F(1-1/p,p),$$

but also $u(0) \in D_F(2-1/p,p)$, hence $u$ belongs to $Z$.

Conversely, if a function $u \in Z$ with $u(0) \in \mathcal{D}(D)$, $Du(0) \in D_F(1-1/p,p)$ is given then $g := -u'' + F^2 u \in L_p(\mathbb{R}_+;Y)$, hence the condition on $g$ is necessary. Now solve the problem with right hand side $g$ and homogeneous boundary condition to obtain $v \in Z$. Then $w := u - v$ satisfies $-w'' + F^2 w = 0$ on $\mathbb{R}_+$, hence is given by $w(y) = e^{-Fy}w(0)$. This implies $w(0) \in D_F(2-1/p,p)$, $w'(0) \in D_F(1-1/p,p)$ as well as $w(0) \in \mathcal{D}(D)$, hence

$$\psi = -w'(0) + Dw(0) = -u'(0) + Du(0) \in D_F(1-1/p,p).$$

Therefore the condition on $\psi$ is also necessary, and in particular the solutions are unique. The last statement follows again from the mixed derivative theorem. □

It is worthwhile to imagine which kind of operators $F$ and $D$ are covered by Theorem 4. To examine this, we let $Y = L_p(\mathbb{R}^n)$ and $F = (1+D_n)^{1/2}$, where $D_n$ means the negative Laplacian on $\mathbb{R}^n$. For $D$ we may choose an oblique derivative operator of the form $Du = a \cdot \nabla u + bu$, where $a \in \mathbb{R}^n$ and $b \geq 0$. This covers the case of a second order elliptic equation with oblique boundary condition. We even may more generally put $Du = D_n u + a \cdot \nabla u + bu$.

On the other hand, we may consider in $Y = L_p(\mathbb{R}; L_p(\mathbb{R}_n))$ an elliptic equation, e.g. $F = (1+D_n)^{1/2}$ canonically extended to $Y$ and a dynamic boundary condition like $Du = \partial_t u + bu$ where $b \geq 0$, or more generally $Du = \partial_t u + D_n u + a \cdot \nabla u + bu$, where $b > 0$.

Third, let again $Y = L_p(\mathbb{R}; L_p(\mathbb{R}^n))$ but $F = (\partial_t + D_n)^{1/2}$ this time, i.e. the underlying equation is second order parabolic. In virtue of $\theta_F = \pi/4$ we may also choose oblique boundary conditions $Du = a \cdot \nabla u + bu$, as well as $Du = D_n u + a \cdot \nabla u + bu$, and dynamic boundary conditions like $Du = \partial_t u + bu$ and $Du = \partial_t u + D_n u + bu$ are still allowed.

If we consider $Y = L_p(\mathbb{R}_+; L_p(\mathbb{R}^n))$ then the above choices are also possible, but we have to add homogeneous initial conditions at $t = 0$ for whenever the operators $F$ or $D$ contain $\partial_t$.

## 4. Parabolic Problems on a Half-Space

We now consider the vector-valued problem

$$\begin{aligned} &\partial_t u - \partial_y^2 u + \mathcal{B}u = f, \quad t,y > 0 \\ &u(t,0) = \phi(t), \quad t > 0, \\ &u(0,y) = \psi(y), \quad y > 0. \end{aligned} \tag{4.1}$$

Here $\mathcal{B}$ denotes an invertible sectorial operator in a Banach space $Y$ which belongs to the class $\mathcal{HT}$, and the data $f$, $\phi$, and $\psi$ are given. We are interested in solutions $u$ which belong to the maximal regularity class of type $L_p$, i.e.

$$u \in H^1_p(\mathbb{R}_+; L_p(\mathbb{R}_+;Y)) \cap L_p(\mathbb{R}_+; H^2_p(\mathbb{R}_+;Y)) \cap L_p(\mathbb{R}_+; L_p(\mathbb{R}_+; D_B)),$$

where $D_B$ denotes the space $\mathcal{D}(\mathcal{B})$ equipped with the graph norm of $\mathcal{B}$. The main result reads as follows.

**Theorem 5.** *Suppose $Y$ is a Banach space of class $\mathcal{HT}$, let $\mathcal{B} \in \mathcal{BIP}(Y)$ be invertible with power-angle $\theta_{\mathcal{B}} < \pi/2$, and let $p \in (1,\infty)$, $p \neq 3/2$. Let $D_B$ denote the Banach space $\mathcal{D}(\mathcal{B})$ equipped with the graph norm of $\mathcal{B}$. Then the problem (4.1) has exactly one solution*

$$u \in Z := H^1_p(\mathbb{R}_+; L_p(\mathbb{R}_+; Y)) \cap L_p(\mathbb{R}_+; H^2_p(\mathbb{R}_+; Y)) \cap L_p(\mathbb{R}_+; L_p(\mathbb{R}_+; D_B))$$

*if and only if the data $f$, $\varphi$, $\psi$ satisfy the following conditions.*

1. $f \in X := L_p(\mathbb{R}_+; L_p(\mathbb{R}_+; Y))$;
2. $\phi \in W^{1-1/2p}_p(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1-1/2p, p))$;
3. $\psi \in W^{2-2/p}_p(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1-1/p, p))$;
4. $\phi|_{t=0} = \psi|_{y=0}$ *in case* $p > 3/2$.

*If this is the case then $\phi|_{t=0} = \psi|_{y=0} \in D_B(1-3/2p, p)$ for $p > 3/2$.*

*Proof.* Suppose $u$ is a solution of (4.1). Then evidently we have $f = \partial_t u - \partial_y^2 u + \mathcal{B}u \in X$, i.e. the first condition is necessary. Let $B$ denote the natural extension of $\mathcal{B}$ to $E := L_p(\mathbb{R}_+; Y)$, with domain $\mathcal{D}(B) = L_p(\mathbb{R}_+; D_B)$. Then $B$ is again invertible, sectorial, and $B \in \mathcal{BIP}(E)$ with power angle $\theta_B < \pi/2$. Let $G = -\partial_y^2$ with domain $\mathcal{D}(G) = H^2_p(\mathbb{R}_+; Y) \cap {}_0H^1_p(\mathbb{R}_+; Y)$; then $G$ is sectorial, belongs to $\mathcal{BIP}(E)$ with power angle $\theta_G = 0$. Since both operators commute, Theorem 1 yields that $A := G + B$ with domain $\mathcal{D}(A) = \mathcal{D}(G) \cap \mathcal{D}(B)$ is sectorial, belongs to $\mathcal{BIP}(E)$ with power angle $\theta_A < \pi/2$. Then by Theorem 2 we obtain the function $u_0 := e^{-At} * f \in Z$, hence $v := u - u_0$ satisfies (4.1) with $f = 0$ and the traces of $u$ and $v$ for $y = 0$ and also for $t = 0$ coincide. Thus we may assume $f = 0$ in the sequel.

Next we extend $u$ and $\phi$ for $t < 0$ by symmetry, i.e. $u(t,y) = u(-t,y)$ and $\phi(t) = \phi(-t)$ for $t < 0$. Then $u$ satisfies a problem of the form

$$\begin{aligned} &\partial_t u - \partial_y^2 u + \mathcal{B}u = g, \quad y > 0,\ t \in \mathbb{R} \\ &u(t,0) = \phi(t), \quad t \in \mathbb{R}, \end{aligned} \tag{4.2}$$

with $g \in L_p(\mathbb{R}; L_p(\mathbb{R}_+; Y))$. Define $B$ in $L_p(\mathbb{R}; Y)$ again by pointwise extension, and let $D = \partial_t$ with domain $\mathcal{D}(D) = H^1_p(\mathbb{R}; Y)$. Both operators are sectorial, they commute and belong to $\mathcal{BIP}(L_p(\mathbb{R}; Y))$, their power angles satisfy $\theta_D + \theta_B \leq \pi/2 + \theta_B < \pi$. Therefore, by Theorem 1, $D + B$ with domain $\mathcal{D}(D+B) = \mathcal{D}(B) \cap \mathcal{D}(D)$ is invertible, sectorial, belongs to $\mathcal{BIP}(L_p(\mathbb{R}; Y))$, with power angle smaller than $\pi/2$. Now we are in position to apply Theorem 3 to $F := \sqrt{D+B}$, to the result that $\phi \in D_F(2-1/p, p)$. By means of Corollary 1 we have

$$\mathcal{D}(F) = \mathcal{D}((D+B)^{1/2}) = \mathcal{D}(D^{1/2}) \cap \mathcal{D}(B^{1/2}),$$

and Proposition 1 gives

$$D_F(\alpha, p) = D_{D^{1/2}}(\alpha, p) \cap D_{B^{1/2}}(\alpha, p).$$

The reiteration theorem yields $D_{B^{1/2}}(\alpha, p) = D_B(\alpha/2, p)$, similarly for $D$, hence we get

$$D_F(2-1/p, p) = W^{1-1/2p}_p(\mathbb{R}; Y) \cap L_p(\mathbb{R}; D_B(1-1/2p, p)).$$

Therefore by restriction to $\mathbb{R}_+$ we see that the second condition in Theorem 5 is necessary.

In a similar way we prove necessity of the third condition. This time we extend $u$ w.r.t. $y$ to all of $\mathbb{R}$, say e.g. by $u(t,y) = 3u(t,-y) - 2u(t,-2y)$. The resulting function belongs to

$$H^1_p(\mathbb{R}_+; L_p(\mathbb{R}; Y)) \cap L_p(\mathbb{R}_+; H^2_p(\mathbb{R}; Y)) \cap L_p(\mathbb{R}_+; L_p(\mathbb{R}; D_B)).$$

This time we let $A = B + G$ in $L_p(\mathbb{R}; Y)$, where $\mathcal{D}(G) = H^2_p(\mathbb{R}; Y)$, and apply Theorem 1 and Theorem 2 to the result that $u|_{t=0} \in D_A(1-1/p, p)$. Proposition 1 yields

$$D_A(1-1/p,p) = D_G(1-1/p,p) \cap D_B(1-1/p,p) = W^{2-2/p}_p(\mathbb{R}; Y) \cap L_p(\mathbb{R}; D_B(1-1/p,p)),$$

and so after restriction to $y \in \mathbb{R}_+$ we obtain necessity of the third condition.

Last but not least, by the mixed derivative theorem, Proposition 2, we have

$$u \in H^s_p(\mathbb{R}_+; H^{2-2s}_p(\mathbb{R}_+; Y)),$$

for each $s \in [0,1]$. This space embeds into $BUC(\mathbb{R}^2_+; Y)$ if $1/p < s$ and $1/p < 2 - 2s$, i.e. if $1/p < s < 1 - 1/2p$. This shows that the compatibility condition is necessary for $p > 3/2$. Taking the trace odf $\phi$ at $t = 0$, Proposition 3 yields $\phi(0) \in D_B(1 - 3/2p, p)$, the last assertion of Theorem 5.

Obviously the solution is unique, since $-A$ generates an analytic $C_0$-semigroup in $L_p(\mathbb{R}_+; Y)$, as was observed before.

Conversely, let the data $f$, $\phi$, $\psi$ be given such that the compatibilty condition holds in case $p > 3/2$. Then with $\phi_0 = \phi(0)$ we write the solution in the following way.

$$u = e^{-At} * f + e^{-At}[\psi - e^{-\mathcal{B}^{1/2}y/\sqrt{2}}\phi_0] + e^{-Fy}[\phi - e^{-\mathcal{B}t/2}\phi_0] + e^{-\mathcal{B}t/2}e^{-\mathcal{B}^{1/2}y/\sqrt{2}}\phi_0. \quad (4.3)$$

Here $A$ is defined as above in $L_p(\mathbb{R}_+; Y)$ and $F$ as above in $L_p(\mathbb{R}_+; Y)$ where now $\mathcal{D}(G) = {}_0H^1_p(\mathbb{R}_+; Y) \cap H^2_p(\mathbb{R}_+; Y)$. This formula is written for the case $p > 3/2$; for $p < 3/2$ simply set $\phi(0) = 0$. Observe that the last term $v$ satisfies $\partial_t v - \partial_y^2 v + \mathcal{B}v = 0$ and has traces $e^{-\mathcal{B}^{1/2}y/\sqrt{2}}\phi(0)$ at $t = 0$ and $e^{-\mathcal{B}t/2}\phi(0)$ at $y = 0$. According to Theorems 2 and 3 each term in (4.3) belongs to the space $Z$, which completes the proof. □

There is an analogous result for the problem

$$\begin{aligned} &\partial_t u - \partial_y^2 u + \mathcal{B}u = f, \quad t, y > 0 \\ &-\partial_y u(t,0) + \mathcal{D}u(t,0) = \phi(t), \quad t > 0, \\ &u(0,y) = \psi(y), \quad y > 0. \end{aligned} \quad (4.4)$$

It reads as follows

**Theorem 6.** *Suppose $Y$ is a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $p \neq 3$, let $\mathcal{B} \in \mathcal{BIP}(Y)$ be invertible with power-angle $\theta_{\mathcal{B}} < \pi/2$, $\mathcal{D}$ pseudo-sectorial, $\mathcal{D} \in \mathcal{BIP}(\overline{\mathcal{R}(\mathcal{D})})$ with $\theta_{\mathcal{D}} < 3\pi/4$, and suppose $\mathcal{B}$ and $\mathcal{D}$ commute in the resolvent sense. Let $D_B$ denote the Banach space $\mathcal{D}(\mathcal{B})$ equipped with the graph norm of $\mathcal{B}$. Then the problem (4.4) has exactly one solution*

$$u \in Z := H^1_p(\mathbb{R}_+; L_p(\mathbb{R}_+; Y)) \cap L_p(\mathbb{R}_+; H^2_p(\mathbb{R}_+; Y)) \cap L_p(\mathbb{R}_+; L_p(\mathbb{R}_+; D_B))$$

*with*

$$\mathcal{D}u(\cdot, 0) \in W^{1/2-1/2p}_p(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1/2 - 1/2p, p)),$$

*if and only if the data $f$, $\phi$, $\psi$ satisfy the following conditions.*

1. $f \in X := L_p(\mathbb{R}_+; L_p(\mathbb{R}_+; Y))$;

2. $\phi \in W_p^{1/2-1/2p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1/2 - 1/2p, p))$;

3. $\psi \in W_p^{2-2/p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1 - 1/p, p))$;

4. $-\partial_y \psi(0) + \mathcal{D}\psi(0) = \phi(0) \in Y$ *in case* $p > 3$.

*If this is the case then* $\phi(0) \in D_B(1/2 - 3/2p, p)$ *and* $\psi(0) \in \mathcal{D}(\mathcal{D}) \cap D_B(1 - 3/2p, p)$ *for* $p > 3/2$, $\mathcal{D}\psi(0) \in D_B(1/2 - 3/2p, p)$ *for* $p > 3$.

*Proof.* Splitting again the space $Y$ if necessary we may assume $\mathcal{N}(\mathcal{D}) = 0$. Suppose that $u$ is a solution with the stated properties. Extend $u$ in $t$ to $\mathbb{R}$ by symmetry, i.e. $u(t) = u(-t)$ for $t \leq 0$. Then

$$u \in H_p^1(\mathbb{R}; L_p(\mathbb{R}_+; Y)) \cap L_p(\mathbb{R}; H_p^2(\mathbb{R}_+; Y)) \cap L_p(\mathbb{R}; L_p(\mathbb{R}_+; D_B)).$$

Define $A$ in $X := L_p(\mathbb{R}; Y)$ by means of

$$Av = \partial_t v + \mathcal{B}v, \quad \mathcal{D}(A) = H_p^1(\mathbb{R}; Y) \cap L_p(\mathbb{R}; D_B),$$

and let $F = A^{1/2}$. Then (4.4) is equivalent to (3.2). Hence employing Theorem 3 we obtain $\partial_t u + \mathcal{B}u \in L_p(\mathbb{R}; L_p(\mathbb{R}; Y))$ as well as $\partial_y u|_{y=0} \in D_F(1 - 1/p, p)$. Restricting to $t > 0$ this yields 1. and 2., since by the results of Section 3 we have

$$D_F(1 - 1/p, p) = W_p^{1/2-1/2p}(\mathbb{R}; Y) \cap L_p(\mathbb{R}; D_B(1/2 - 1/2p, p)).$$

Similarly extend $u$ w.r.t. $y$ in the same regularity class, and consider $A = -\partial_y^2 + \mathcal{B}$ in $X = L_p(\mathbb{R}; Y)$. By Theorem 1, $A$ is sectorial, admits bounded imaginary powers with power angle $\theta_A < \pi/2$, in particular $A$ generates a bounded analytic $C_0$-semigroup in $X$. Then appply Theorem 2 to obtain $\psi \in D_A(1 - 1/p, p)$. Corollary 1 and Proposition 1 yield

$$D_A(1 - 1/p, p) = W_p^{2-2/p}(\mathbb{R}; Y) \cap L_p(\mathbb{R}; D_B(1 - 1/p, p)),$$

hence we obtain 3. by restriction of $y$ to $\mathbb{R}_+$.

The compatibility condition follows from the embedding

$$W_p^{1/2-1/2p}(J; Y) \cap L_p(J; D_B(1/2 - 1/2p, p)) \hookrightarrow C(J; D_B(1/2 - 3/2p, p),$$

which is valid for $p > 3$, by Proposition 3.

Conversely, if the data $f$, $\phi$, $\psi$ are given, then as in the proof of Theorem 4, the solution can be written in the form

$$\begin{aligned} u &= e^{-At}\psi_1 + u_1 + e^{-Fy}(F + D)^{-1}\phi_1 + \\ &+ \frac{1}{2}F^{-1}\int_0^\infty [e^{-F|y-s|} + (F - D)(F + D)^{-1}e^{-F(y+s)}]f(s)ds. \end{aligned} \tag{4.5}$$

Here $F = (\partial_t + B)^{1/2}$ in $L_p(\mathbb{R}_+; Y)$, $\mathcal{D}(F) = {}_0H_p^{1/2}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; \mathcal{D}(\mathcal{B}^{1/2}))$, by Theorem 1 and Corollary 1 is subject to the assumptions of Theorem 4. The function $u_1$ is defined as

$$u_1(t, y) = e^{-\mathcal{B}t/2}e^{-\mathcal{B}^{1/2}y/\sqrt{2}}\psi_0, \ \psi_0 := \psi|_{y=0},$$

and $\psi_1$ by

$$\psi_1 = \psi - e^{-\mathcal{B}^{1/2}y/\sqrt{2}}\psi_0.$$

Further, $A = -\partial_y^2 + \mathcal{B}$ in $L_p(\mathbb{R}_+;Y)$ with domain

$$\mathcal{D}(A) = H_p^2(\mathbb{R}_+;Y) \cap {}_0H_p^1(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;\mathcal{D}(\mathcal{B})).$$

Finally, the function $\phi_1$ is given by

$$\phi_1 = \phi - (D + \mathcal{B}^{1/2}/\sqrt{2})e^{-\mathcal{B}t/2}\psi_0 + \partial_y e^{-At}\psi_1|_{y=0}.$$

Note that the traces of $\psi_1$ and of $\phi_1$ at $t = 0$ are zero, by construction. It is now easy to check by means of Theorems 2 and 4 that each term in this decomposition of $u$ belongs to $Z$; cp. Theorem 5. □

The last result we want to discuss here is for the problem

$$\begin{aligned} &\partial_t u - \partial_y^2 u + \mathcal{B}u = f, \quad t, y > 0 \\ &\partial_t u|_{y=0} - \partial_y u|_{y=0} + \mathcal{D}u|_{y=0} = \phi, \quad t > 0, \\ &u(0,y) = \psi(y), \quad y \geq 0. \end{aligned} \tag{4.6}$$

It reads as follows

**Theorem 7.** *Suppose $Y$ is a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $p \neq 3$, let $\mathcal{B} \in \mathcal{BIP}(Y)$ be invertible with power-angle $\theta_{\mathcal{B}} < \pi/2$, $\mathcal{D}$ pseudo-sectorial, $\mathcal{D} \in \mathcal{BIP}(\overline{\mathcal{R}(\mathcal{D})})$ with $\theta_{\mathcal{D}} < \pi/2$, and suppose $\mathcal{B}$ and $\mathcal{D}$ commute in the resolvent sense. Let $D_B$ denote the Banach space $\mathcal{D}(\mathcal{B})$ equipped with the graph norm of $\mathcal{B}$, and let $Y_\alpha = D_B(\alpha, p)$, $\alpha = 1/2 - 1/2p$.*

*Then the problem (4.6) has exactly one solution*

$$u \in Z := H_p^1(\mathbb{R}_+;L_p(\mathbb{R}_+;Y)) \cap L_p(\mathbb{R}_+;H_p^2(\mathbb{R}_+;Y)) \cap L_p(\mathbb{R}_+;L_p(\mathbb{R}_+;D_B))$$

*with*

$$\mathcal{D}u|_{y=0} \in W_p^{1/2-1/2p}(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;D_B(1/2 - 1/2p, p)),$$

*and*

$$u|_{y=0} \in W_p^{3/2-1/2p}(\mathbb{R}_+;Y) \cap H_p^1(\mathbb{R}_+;D_B(1/2 - 1/2p, p)),$$

*if and only if the data $f$, $\phi$, $\psi$ satisfy the following conditions.*

1. $f \in X := L_p(\mathbb{R}_+;L_p(\mathbb{R}_+;Y))$;
2. $\phi \in W_p^{1/2-1/2p}(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;D_B(1/2 - 1/2p, p))$;
3. $\psi \in W_p^{2-2/p}(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;D_B(1 - 1/p, p))$,
4. $\psi_0 := \psi(0) \in D_B(1 - 1/p, p)$ *and* $\psi_0 \in (Y_\alpha, D(\mathcal{D}|_{Y_\alpha})_{1-1/p,p}$;
5. $\mathcal{D}\psi_0 \in D_B(1/2 - 3/2p, p)$ *and* $\psi_1 := \phi(0) + \partial_y\psi(0) - \mathcal{D}\psi_0 \in D_{\mathcal{D}}(172 - 3/2p, p)$ *in case* $p > 3$.

*Proof.* Suppose $u \in Z$ is a solution of (4.6). Then by Theorem 6 we have 1. and 3. as well as

$$\partial_y u|_{y=0} \in V := W_p^{1/2-1/2p}(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;D_B(1/2 - 1/2p, p)).$$

The additional properties $\mathcal{D}u|_{y=0} \in V$ and $\partial_t u|_{y=0} \in V$ yield 2. Since the trace of $u$ at $y = 0$ satisfies

$$u|_{y=0} \in H_p^1(\mathbb{R}_+;D_B(1/2 - 1/2p, p)) \cap L_p(\mathbb{R}_+;D_B(1 - 1/2p, p)),$$

we obtain 4. and 5. by taking the traces of $u|_{y=0}$, $\partial_t u$ and $\mathcal{D}u$ at $t = 0$, according to Proposition 3.

Conversely, let the data $f$, $\phi$, and $\psi$ with properties 1., 2., 3., 4., and 5. be given. As before we let $F$ denote the operator $F = (\partial_t + \mathcal{B})^{1/2}$ in $L_p(\mathbb{R}_+; Y)$. Then

$$\mathcal{D}(F) = {}_0H_p^{1/2}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; \mathcal{D}(\mathcal{B}^{1/2})),$$

and

$$D_F(1 - 1/p, p) = {}_0W_p^{1/2-1/2p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1/2 - 1/2p, p)).$$

Suppose

$$w := u|_{y=0} \in W_p^{1-1/2p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1 - 1/2p, p))$$

is already known. Then by Theorem 5 we obtain a unique solution $u$ of (4.1), with $\phi$ replaced by $w$, which belongs to $Z$. Theorem 6 then yields $\partial_y u|_{y=0} \in V$. We write $w = v + z$ where $v$ has traces zero and $z$ solves the problem

$$\dot{z} + (D + B^{1/2})z = e^{-Bt}(\psi_1 + (D + B^{1/2})\psi_0), \quad z(0) = \psi_0.$$

Then $u$ is of the form $u = u_1 + e^{-Fy}v$, where $u_1$ is determined by the data. This then yields

$$\phi + \partial_y u|_{y=0} = \phi + \partial_y u_1|_{y=0} - Fv = \phi_1 - Fv.$$

Inserting this identity into the dynamic boundary condition we obtain

$$\begin{aligned} \partial_t v + Fv + Dv &= \phi_1 - [\partial_t + D]z \\ &= \phi_1 + B^{1/2}z - e^{-Bt}(\psi_1 + (D + B^{1/2})\psi_0 =: \phi_2, \end{aligned}$$

which means that we have reduced the problem to the integro-differential equation

$$\partial_t v + Fv + Dv = \phi_2, \quad t > 0, \ v(0) = 0. \tag{4.7}$$

Here the function $\phi_2$ is determined by the data of the problem and has zero trace, by construction.

Defining $G = \partial_t$ in $L_p(\mathbb{R}_+; Y)$ with domain $\mathcal{D}(G) = {}_0H_p^1(\mathbb{R}_+; Y)$, and applying Theorem 1 twice we see that the operator $G + F + D$ is invertible in $D_F(\alpha, p)$, for each $\alpha$. Therefore (4.7) admits a unique solution

$$v \in {}_0W_p^{3/2-1/2p}(\mathbb{R}_+; Y) \cap {}_0H_p^1(\mathbb{R}_+; D_B(1/2 - 1/2p, p)) \cap L_p(\mathbb{R}_+; D_B(1 - 1/2p, p)),$$

and $\mathcal{D}v \in {}_0W_p^{1/2-1/2p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1/2 - 1/2p, p))$, once we know $\phi_2 \in D_F(1 - 1/p, p)$.

Now, $\phi \in V$ and, according to Proposition 3, by 2. and 3.

$$\psi_2 := \psi_1 + (D + B^{1/2})\psi_0 = \phi|_{t=0} + \partial_y \psi|_{y=0} + \mathcal{B}^{1/2}\psi_0 \in D_B(1/2 - 3/2p, p),$$

hence, by Proposition 4, $e^{-Bt}\psi_2 \in V$. Thus we obtain $\phi_2 \in D_F(1 - 1/p, p)$, provided $z \in W_p^{1-1/2p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1 - 1/2p, p))$. Therefore we have reduced to showing that $z$ enjoys the regularity claimed for $w = u|_{y=0}$.

To prove this regularity of $z$, we employ once more Propositions 4, but also Proposition 5. Using Proposition 4, by 4. and 5. we have $\psi_0 \in D_{\mathcal{D}+\mathcal{B}^{1/2}}(1 - 1/p, p)$, and $\mathcal{D}e^{-Bt}\psi_2 \in V$. Therefore Proposition 5 yields

$$z \in {}_0W_p^{3/2-1/2p}(\mathbb{R}_+; Y) \cap {}_0H_p^1(\mathbb{R}_+; D_B(1/2 - 1/2p, p)) \cap L_p(\mathbb{R}_+; D_B(1 - 1/2p, p)),$$

as well as $\mathcal{D}z \in W_p^{1/2-1/2p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1/2 - 1/2p, p))$.

Since $w$ is unique, the solution $u \in Z$ of problem (4.6) is also unique, by Theorem 5. □

# References


[1] G. Da Prato and P. Grisvard. Sommes d'opérateurs linéaires et équations différentielles opérationelles. *J. Math. Pures Appl.*, 54:305–387, 1975.

[2] G. Dore and A. Venni. On the closedness of the sum of two closed operators. *Math. Z.*, 196:189–201, 1987.

[3] J. Escher, J. Prüss, and G. Simonett. Analytic solutions of the Stefan problem with Gibbs-Thomson correction. to appear, 1999.

[4] J. Escher, J. Prüss, and G. Simonett. Analytic solutions of the free boundary value problem for the Navier- Stokes equation. to appear, 2001.

[5] P. Grisvard. Spaci di trace e applicazioni. *Rend. Math.*, 5:657–729, 1972.

[6] M. Hieber and J. Prüss. Maximal regularity of parabolic problems. Monograph in Preparation, 2001.

[7] N. Kalton and L. Weis. The $H^\infty$-calculus and sums of closed operators. Preprint, 2000.

[8] H. Komatsu. Fractional powers of operators. *Pacific J. Math.*, 1:285–346, 1966.

[9] O.A. Ladyženskaya, V.A. Solonnikov, and N.N. Ural'ceva. *Linear and Quasilinear Equations of Parabolic Type*, volume 23 of *Transl. Math. Monographs*. Amer. Math. Soc., 1968.

[10] J. Prüss and H. Sohr. On operators with bounded imaginary powers in Banach spaces. *Math. Z.*, 203:429–452, 1990.

[11] P.E. Sobolevskii. Fractional powers of coercively positive sums of operators. *Soviet Math. Dokl.*, 16:1638–1641, 1975.