Some Remarks on Maximal Regularity of Parabolic Problems

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Abstract

It is shown that a pseudo-sectorial operator A is $\mathcal{R}$-sectorial in its associated real interpolation spaces $D_A(\alpha, p)$ and admits there an operator-valued $\mathcal{H}^\infty$-calculus, for $\alpha \in (0,1)$, $p \in [1, \infty]$. This result is applied to obtain maximal regularity for the principal symbol of the Stefan problem with surface tension. An operator-valued extension of the Stein multiplier theorem for Symmetric Diffusion Semigroups is also included.

1. Introduction.

It has become apparent that the concept of families of *R-bounded* operators plays an important role in operator theory, in the theory of operator-valued Fourier multipliers, and also in maximal regularity of elliptic and parabolic problems. Although this theory is only of recent origin, already a number of nontrivial applications have been appeared and we expect that in the near future there will be many more.

A family $\mathcal{T} \subset \mathcal{B}(X,Y))$ of bounded linear operators on a Banach space X is called **R-bounded** if there is a constant $C > 0$ such that for all $N \in \mathbb{N}$, elements $x_j \in X$, selections $T_j \in \mathcal{T}$, and N independent symmetric $\{-1,1\}$-valued random variables ε_j on a probability space (Ω, μ) the inequality

$$||\sum_{j=1}^{N} \varepsilon_j T_j x_j||_{L_p(\Omega;Y)} \leq C||\sum_{j=1}^{N} \varepsilon_j x_j||_{L_p(\Omega;X)} \tag{1.1}$$

is valid. This can be seen as a strengthening of the concept of uniform boundedness of the family $\mathcal{T}$.

To draw the connection to maximal regularity, recall than an operator A in a Banach space X is called **sectorial** if its domain and range are dense, $(-\infty, 0)$ is contained in the resolvent set of A, and the resolvent estimate

$$|t(t+A)^{-1}|_{\mathcal{B}(X)} \leq M_0, \quad t > 0,$$

is valid. A sectorial operator A is called **$\mathcal{R}$-sectorial** if in addition the set $\{t(t+A)^{-1} : t > 0\}$ is $\mathcal{R}$-bounded. This concept was introduced in the same volume of the proceedings of this series of conferences in Clément and Prüss [5] and in Weis [16].

In that paper Weis proved, that in case X is a UMD-Banach space, $-A$ generates an exponentially stable analytic C_0-semigroup in X, then maximal L_p-regularity of the abstract Cauchy problem

$$\dot{u} + Au = f, \; t > 0, \quad u(0) = 0,$$

is equivalent to $\mathcal{R}$-boundedness of $\{i\rho(i\rho + A)^{-1} : \rho \in \mathbb{R}\}$. This can be reformulated as: A is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle less than $\pi/2$; see Section 2 for more explanations.

For background and the present state of the art in this theory, we refer to the papers Clément, de Pagter, Sukochev, and Witvliet [2], Clément and Prüss [5], Kalton and Weis [12], and for surveys one should consult Denk, Hieber, and Prüss [7] and Weis [16].

It is the aim of this paper to discuss sectorial (or merely pseudo-sectorial or nonnegative) operators A in their associated real interpolation spaces $D_A(\alpha, p)$, where $\alpha \in (0,1)$ and $p \in [1, \infty]$. In Section 2 we show that the part $A_{\alpha,p}$ of A in $D_A(\alpha,p)$ is always $\mathcal{R}$-sectorial. This not surprising since $A_{\alpha,p}$ admits an $\mathcal{H}^\infty$-calculus, as was proved by Dore [8]. On the other hand, it was shown in Clément and Prüss [5] that such operators are $\mathcal{R}$-sectorial, if the space is UMD. Here we give a direct proof which is valid for nonnegative operators in arbitrary Banach spaces. Section 2 deals with the extension of the before mentioned result of Dore to the operator-valued case. Our functional calculus is strong enough to cover the theorem of Da Prato and Grisvard [6]. It is an analog in the setting of real interpolation spaces of a recent result due to Kalton and Weis [12]. Based on the counter-example of Kalton and Lancien [11] we show that our result is independent from that of Kalton and Weis. An application of Theorem 2 to a certain symbol arising in connection with Stefan problems is presented in Section 4. The corresponding theory for this symbol in B^s_{pq} and H^s_p is contained in Escher, Prüss and Simonett [9]. We conclude the paper with the extension of the famous Stein multiplier theorem for Symmetric Diffusion Semigroups on L_p-spaces to operator-valued multipliers in vector-valued setting.

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2. $\mathcal{R}$-Sectoriality in Interpolation Spaces.

Recall that a linear operator A in a Banach space X is called **nonnegative or pseudo-sectorial**, if $(-\infty, 0) \subset \rho(A)$, the resolvent set of A, and there is a constant $M_0 > 0$ such that

$$|t(t+A)^{-1}| \leq M_0, \quad t > 0.$$

This class of operators has been studied in detail by Komatsu [13], in particular in respect of fractional powers. Note that a pseudo-sectorial operator is always closed since it has a nonempty resolvent set. If in addition, the domain $\mathcal{D}(A)$ and the range $\mathcal{R}(A)$ of A are dense in X, then A is called **sectorial.** We define as usual the **spectral angle** ϕ_A as the infimum of all angles $\phi \in (0, \pi]$ such that

$$\Sigma_{\pi-\phi} := \{z \in \mathbb{C} : \ z \neq 0, |\arg(z)| < \pi - \phi\} \subset \rho(A),$$

and the resolvent estimate

$$|z(z+A)^{-1}| \leq M_{\pi-\phi}, \quad z \neq 0, |\arg(z)| < \pi - \phi,$$

is satisfied, for some constant $M_{\pi-\phi} > 0$ which is independent of z. If for a (pseudo)-sectorial operator A in addition a set $\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_{\pi-\phi}\}$ is $\mathcal{R}$-bounded for some angle ϕ, then we call A **(pseudo)-$\mathcal{R}$-sectorial**; pseudo-$\mathcal{R}$-sectorial will also be called **$\mathcal{R}$-nonnegative**. The infimum of such angles π is called the $\mathcal{R}$-angle of A, we denote it by ϕ^R_A. Note that we always have $\phi_A \leq \phi^R_A$.

It is well-known that $-A$ generates an analytic semigroup e^{-At} in X which is bounded on some sector Σ_ψ if and only if A is pseudo-sectorial with spectral angle $\phi_A < \pi/2$; e^{-At} is strongly continuous at 0 if and only if in addition $\mathcal{D}(A)$ is dense in X. A famous result due to Da Prato and Grisvard [6] states that in this case the abstract Cauchy problem

$$\dot{u} + Au = f, \quad t \in J = [0, T], \ u(0) = 0, \tag{2.1}$$

has maximal regularity in $L_p(J; D_A(\alpha, p))$, for each $\alpha \in (0, 1)$, $p \in (1, \infty)$. Here, as usual, $D_A(\alpha, p)$ denotes the real interpolation space $(X, \mathcal{D}(A))_{\alpha,p}$ between X and the domain of A equipped with the graph norm of order α and exponent p. Therefore by a recent result of Weis [16], the part $A_{\alpha,p}$ of A in $D_A(\alpha, p)$ is pseudo-$\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi^R_{A_{\alpha,p}} < \pi/2$; see also Clément and Prüss [5].

Despite its elegance, this argument has some apparent disadvantages. It only applies to the case $\phi_A < \pi/2$, it gives only the bounds $\phi^R_{A_{\alpha,p}} < \pi/2$, and it is rather indirect. The following result takes care of these shortcomings.

Theorem 1. *Let $\alpha \in (0,1)$, $p \in [1,\infty)$, and suppose that A is pseudo-sectorial in the Banach space X.*

Then the part $A_{\alpha,p}$ of A in $D_A(\alpha,p)$ is pseudo-$\mathcal{R}$-sectorial and $\phi^R_{A_{\alpha,p}} \le \phi_A$ holds.

Proof. Let A be pseudo-sectorial with spectral angle $\phi_A \in [0,\pi)$; choose $\psi > \theta > \phi_A$ and let Γ_θ denote the standard contour

$$\Gamma_\theta = (\infty, 0]e^{i\theta} \cup [0,\infty)e^{-i\theta}.$$

Then for each $\lambda \in \Sigma_{\pi-\psi}$ and $t > 0$, the Dunford calculus for A in X yields the representation

$$A(t+A)^{-1}(\lambda + A)^{-1} = \frac{1}{2\pi i}\int_{\Gamma_\theta} \frac{1}{\lambda + z} A(z-A)^{-1}\frac{dz}{z+t}.$$

Fixing finitely many $\lambda_j \in \Sigma_{\phi-\psi}$, $\varepsilon_j = \pm 1$, and $x_j \in D_A(\alpha,p)$, we obtain

$$t^\alpha A(t+A)^{-1}\sum_j \varepsilon_j\lambda_j(\lambda_j + A)^{-1}x_j = \frac{1}{2\pi i}\int_{\Gamma_\theta} z^\alpha A(z-A)^{-1}\sum_j \varepsilon_j \frac{\lambda_j}{\lambda_j + z}x_j \frac{t^\alpha dz}{(z+t)z^\alpha}.$$

Since

$$\int_{\Gamma_\theta} \frac{t^\alpha|dz|}{(|z|+t)|z|^\alpha} = \int_{\Gamma_\theta}\frac{d\zeta|}{(1+|\zeta|)|\zeta|^\alpha} < \infty,$$

Jensen's inequality yields

$$|t^\alpha A(t+A)^{-1}\sum_j \varepsilon_j\lambda_j(\lambda_j + A)^{-1}x_j|^p \le C\int_{\Gamma_\theta} |z^\alpha A(z-A)^{-1}\sum_j \varepsilon_j\frac{\lambda_j}{\lambda_j+z}x_j|^p\frac{t^\alpha|dz|}{(|z|+t)|z|^\alpha}.$$

By means of

$$\int_0^\infty \frac{t^\alpha dt}{t|z|^\alpha(|z|+t)} = \frac{1}{|z|}\int_0^\infty \frac{s^{\alpha-1}ds}{1+s} = \frac{c_\alpha}{|z|}$$

we continue

$$\int_0^\infty |t^\alpha A(t+A)^{-1}\sum_j \varepsilon_j\lambda_j(\lambda_j+A)^{-1}x_j|^p\frac{dt}{t} \le c_\alpha C\int_{\Gamma_\theta} |z^\alpha||A(z-A)^{-1}\sum_j\varepsilon_j\frac{\lambda_j}{\lambda_j+z}x_j|^p\frac{|dz|}{|z|}.$$

For independent symmetric random variables ε_j on a probability space (Ω,μ), integration w.r.t. μ and application of the contraction principle then yields

$$\begin{aligned}
&|\sum_j \varepsilon_j\lambda_j(\lambda_j+A)^{-1}x_j|^p_{L_p(\Omega;D_A(\alpha,p))} \le C\int_{\Gamma_\theta}\int_\Omega |\sum_j \varepsilon_j\frac{\lambda_j}{\lambda_j+z}z^\alpha A(z-A)^{-1}x_j|^p d\mu\frac{|dz|}{|z|}\\
&\le 2^pC\int_{\Gamma_\theta}[\int_\Omega|\sum_j\varepsilon_j z^\alpha A(z-A)^{-1}x_j|^p d\mu]\frac{|dz|}{|z|} \le C|\sum_j\varepsilon_j x_j|^p_{L_p(\Omega;D_A(\alpha,p))}.
\end{aligned}$$

This shows that the set of operators $\{\lambda(\lambda+A)^{-1} : \lambda \in \Sigma_{\pi-\psi}\}$ is $\mathcal{R}$-bounded in $D_A(\alpha,p)$, for each $\psi > \phi_A$, $\alpha \in (0,1)$, $p \in [1,\infty)$, which means that the part $A_{\alpha,p}$ of A in $D_A(\alpha,p)$ is pseudo-$\mathcal{R}$-sectorial and $\phi^R_{A_{\alpha,p}} \le \phi_A$. $\square$

3. Operator-Valued $\mathcal{H}^\infty$-calculus in Interpolation Spaces.

Recall that a sectorial operator A in a Banach space X is said to admit an $\mathcal{H}^\infty$**-calculus** if the usual Dunford calculus defined by

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma_\theta} f(z)(z-A)^{-1}dz$$

for $f \in \mathcal{H}_0^\infty(\Sigma_\phi)$ satisfies

$$|f(A)|_{\mathcal{B}(X)} \leq C|f|_{\mathcal{H}^\infty(\Sigma_\phi)},$$

for some constant $C > 0$ independent of f. The infimum of all such angles $\phi > \phi_A$ is called the $\mathcal{H}^\infty$**-angle** of A. It is well-known that in this case the Dunford calculus extends uniquely to $\mathcal{H}^\infty(\Sigma_\phi)$, for any $\phi > \phi_A^\infty$. Clément and Prüss [5] have shown that sectorial operators with $\mathcal{H}^\infty$-calculus are $\mathcal{R}$-sectorial with $\phi_A^R \leq \phi_A^\infty$, provided the underlying Banach space X has the *unconditional martingale difference property*, i.e. is UMD for short. The converse of this is not true, in general, since there are examples of sectorial operators in Hilbert spaces without $\mathcal{H}^\infty$-calculus; see Baillon and Clément [1] and McIntosh and Yagi [14].

This concept can easily be extended to the operator-valued case, where scalar $\mathcal{H}^\infty$-functions are replaced by functions in $\mathcal{H}_0^\infty(\Sigma_\phi;\mathcal{B}(X))$. We say that a pseudo-sectorial operator A in X admits an **operator-valued** $\mathcal{H}^\infty$**-calculus** if there is an angle $\phi > \phi_A$ and a constant $M > 0$ such that

$$|F(A)|_{\mathcal{B}(X)} \leq C|F|_{\mathcal{H}^\infty(\Sigma_\phi;\mathcal{B}(X))}$$

is valid, for all $F \in \mathcal{H}_0^\infty(\Sigma_\phi;\mathcal{B}(X)$ commuting with A in the sense that

$$F(z)(\lambda - A)^{-1} = (\lambda - A)^{-1}F(z), \quad z \in \Sigma_\phi,\ \lambda \in \rho(A).$$

Although for many important sectorial operators, in particular partial differential operators, the $\mathcal{H}^\infty$-calculus is known to exist, it is not an easy task to prove it. However, Dore [8] has obtained the striking result that sectorial operators A always admit $\mathcal{H}^\infty$-calculi in the corresponding real interpolation spaces $D_A(\alpha,p)$, where $\alpha \in (0,1)$ and $p \in [1,\infty]$. Here we present the extension of Dore's result to the operator-valued case.

Theorem 2. *Let $\alpha \in (0,1)$, $p \in [1,\infty]$, and suppose that A is pseudo-sectorial in the Banach space X with spectral angle ϕ_A.*

Then for any $\phi > \phi_A$ there is a constant C_ϕ such that

$$|F(A)|_{\mathcal{B}(D_A(\alpha,p))} \leq C_\phi |F|_{\mathcal{H}^\infty(\Sigma_\phi;\mathcal{B}(X))},$$

for each $F \in \mathcal{H}_0^\infty(\Sigma_\phi;\mathcal{B}(X))$ commuting with A.

Proof. Fix any $\phi > \phi_A$, choose $\theta \in (\phi_A,\phi)$ and let $\Gamma = \Gamma_\theta$ denote the standard path, as in the proof of Theorem 1. For $F \in \mathcal{H}_0^\infty(\Sigma_\phi;\mathcal{B}(X))$ commuting with A we have the Dunford integral

$$F(A) = \frac{1}{2\pi i}\int_\Gamma F(\lambda)(\lambda - A)^{-1}d\lambda.$$

Applying the resolvent formula as well as Cauchy's theorem this implies for $\alpha \in (0,1)$ and $t > 0$

$$t^\alpha A(t+A)^{-1}F(A) = \frac{1}{2\pi i}\int_\Gamma \lambda^\alpha F(\lambda)A(\lambda - A)^{-1}\frac{t^\alpha d\lambda}{\lambda^\alpha(\lambda+t)}.$$

Since

$$\int_\Gamma \frac{t^\alpha |dz|}{(|z|+t)|z|^\alpha} = \int_\Gamma \frac{d\zeta|}{(1+|\zeta|)|\zeta|^\alpha} < \infty,$$

Jensen's inequality yields for $1 \le p < \infty$

$$\begin{aligned}|t^\alpha A(t+A)^{-1}F(A)x|^p &\le C\int_\Gamma [|\lambda^\alpha||F(\lambda)A(\lambda - A)^{-1}x|]^p \frac{t^\alpha |d\lambda|}{|\lambda^\alpha|(|\lambda|+t)}\\ &\le C|F|_{\mathcal{H}^\infty(\Sigma_\phi;\mathcal{B}(X))}\int_\Gamma |\lambda^\alpha A(\lambda - A)^{-1}x|^p \frac{t^\alpha|d\lambda|}{|\lambda^\alpha|(|\lambda|+t)}.\end{aligned}$$

Integrating over $t > 0$ this implies

$$\int_0^\infty |t^\alpha A(t+A)^{-1}F(A)x|^p\frac{dt}{t} \le c_\alpha C|F|_{\mathcal{H}^\infty(\Sigma_\phi;\mathcal{B}(X))}\int_\Gamma |\lambda^\alpha A(\lambda - A)^{-1}x|^p\frac{|d\lambda|}{|\lambda|},$$

since

$$\int_0^\infty \frac{t^\alpha dt}{t|z|^\alpha(|z|+t)} = \frac{c_\alpha}{|z|}.$$

This implies the inequality

$$|F(A)x|_{D_A(\alpha,p)} \le Cc_\alpha|F|_{\mathcal{H}^\infty(\Sigma_\phi;\mathcal{B}(X))}|x|_{D_A(\alpha,p)}, \quad x \in D_A(\alpha,p),$$

for each $p \in [1,\infty)$.

For the case $p = \infty$ we use the change of variables $\lambda = tz$ to obtain

$$t^\alpha A(t+A)^{-1}F(A) = \frac{1}{2\pi i}\int_\Gamma (tz)^\alpha F(tz)A(tz-A)^{-1}\frac{dz}{z^\alpha(z+1)},$$

and estimate directly to the result

$$\begin{aligned}|t^\alpha A(t+A)^{-1}F(A)x| &\le C|F|_{\mathcal{H}^\infty(\Sigma_\phi;\mathcal{B}(X))}\int_\Gamma |(tz)^\alpha A(tz-A)^{-1}x|\frac{|dz|}{|z|^\alpha(|z|+1)},\\ &\le C|F|_{\mathcal{H}^\infty(\Sigma_\phi;\mathcal{B}(X))}|x|_{D_A(\alpha,\infty)}\int_\Gamma \frac{|dz|}{|z|^\alpha(|z|+1)}.\end{aligned}$$

This proves the theorem for all $\alpha \in (0,1)$ and $p \in [1,\infty]$. □

Having established a uniform $\mathcal{H}^\infty$-bound for the functional calculus of A, it is easy to extend it to functions $F \in \mathcal{H}^\infty(\Sigma_\phi;\mathcal{B}(X))$ which commute with A. In fact, if such F is given, set $F_\varepsilon(z) = F(z)z^\varepsilon/(1+z)^{2\varepsilon}$; then $F_\varepsilon(z) \to F(z)$ in $\mathcal{B}(X)$, locally uniformly in $z \in \Sigma_\phi$. Decomposing the path Γ into $\Gamma = \Gamma_1 \cup \Gamma_2$ where

$$\Gamma_1 = \{z \in \Gamma : |z| \le 1\}, \quad \Gamma_2 = \{z \in \Gamma : |z| \ge 1\},$$

and setting $\Gamma_0 = e^{i[-\phi,\phi]}$, by means of Cauchy's theorem we obtain the representation

$$\begin{aligned} F_\varepsilon(A)x &= \frac{1}{2\pi i}\int_{\Gamma_1} F_\varepsilon(z)z(z-A)^{-1}A^{-1}x\,dz + \frac{1}{2\pi i}\int_{\Gamma_2} F_\varepsilon(z)A(z-A)^{-1}x\frac{dz}{z} \\ &+ \Big(\frac{1}{2\pi i}\int_{\Gamma_0} F_\varepsilon(z)dz\Big)A^{-1}x + \Big(\frac{1}{2\pi i}\int_{\Gamma_0} F_\varepsilon(z)\frac{dz}{z}\Big)x, \end{aligned} \tag{3.1}$$

for each $x \in \mathcal{R}(A)$. (3.1) shows that $F(A)x := \lim_{\varepsilon\to 0} F_\varepsilon(A)x$ exists in X, for each $x \in \mathcal{R}(A) \cap D_A(\alpha,p)$. Therefore, in case $0 \in \rho(A)$, the Banach-Steinhaus theorem implies $F_\varepsilon(A)x \to F(A)x$ in X for each $x \in D_A(\alpha,p)$. On the other hand, $F_\varepsilon(A)x$ is bounded in $D_A(\alpha,p)$ by Theorem 2, hence $F(A)x \in D_A(\alpha,p)$. This shows that $F(A) \in \mathcal{B}(D_A(\alpha,p))$ is well-defined, for each $\alpha \in (0,1$ and $p \in [1,\infty]$.

The same result is valid if instead of $0 \in \rho(A)$ the function F is holomorphic in a neighbourhood of $z = 0$. This can be seen in the standard way, modyfying the path Γ near $z = 0$. Summarizing we have

Corollary 1. *Let $\alpha \in (0,1)$, $p \in [1,\infty]$, and suppose that A is pseudo-sectorial in the Banach space X with spectral angle ϕ_A, and let $\phi > \phi_A$. Assume $F \in \mathcal{H}^\infty(\Sigma_\phi;\mathcal{B}(X))$ commutes with A and either $0 \in \rho(A)$ or F is holomorphic in a neighborhood of $z = 0$.*

Then $F(A)$ is well-defined and belongs to $\mathcal{B}(D_A(\alpha,p))$.

This is a natural extension of the theorem of Da Prato and Grisvard [6]. In fact, let B be pseudo-sectorial in X commuting with A in the resolvent sense and let the *parabolicity condition*

$$\phi_A + \phi_B < \pi \tag{3.2}$$

be satisfied. Set $F_t(z) = B(tz+B)^{-1}$, $t > 0$; then because of (3.2), we have $F_t \in \mathcal{H}^\infty(\Sigma_\phi)$ for some $\phi > \phi_A$, and F_t commutes with A. Hence if either $0 \in \rho(A)$ or F_t holomorphic at $z = 0$, i.e. $0 \in \rho(B)$ then $F_t(A) = B(tA+B)^{-1}$ is well-defined and bounded in $D_A(\alpha,p)$, with bound independent of $t > 0$, i.e. AB^{-1} is sectorial in $D_A(\alpha,p)$.

It is interesting to compare Corollary 1 with a recent result due to Kalton and Weis [12], which states that whenever A is sectorial in X and admits an $\mathcal{H}^\infty$-calculus, $F \in \mathcal{H}^\infty(\Sigma_\phi;\mathcal{B}(X))$ commutes with A, $\phi > \phi_A$, and in addition $F(\Sigma_\phi) \subset \mathcal{B}(X)$ is $\mathcal{R}$-bounded, the $F(A)$ is well-defined and bounded in X. The following example shows that this result does not imply Corollary 1.

Example 1. Consider $L_p(G)$ for some open subset $G \subset \mathbb{R}^n$, $p \neq 2$. According to the paper of Kalton and Lancien [11] there is a sectorial operator A on $L_p(G)$ such that $-A$ generates an exponentially stable analytic C_0-semigroup in $L_p(G)$ such that problem (2.1) does not have maximal L_p-regularity. Therefore by the result of Weis mentioned in Section 2, A is not $\mathcal{R}$-sectorial resp. $\phi_A^R \geq \pi/2$. The result of Da Prato and Grisvard (or Corollary 1) shows that (2.1) has maximal regularity in $\mathrm{buc}^\alpha(\mathbb{R};\mathrm{L_p(G)})$ where $\alpha \in (0,1)$. Theorem 1 (or the result of Dore mentioned above) shows that $B = d/dt$ admits an $\mathcal{H}^\infty$-calculus in $\mathrm{buc}^\alpha(\mathbb{R};L_p(G))$, however, the result of Kalton and Weis does not apply here, since A is not $\mathcal{R}$-sectorial with $\phi_A^R < \pi/2$ in $L_p(G)$, hence also not in $\mathrm{buc}^\alpha(\mathbb{R};L_p(G))$.

4. Application to Symbols of Stefan Problems.

In a recent paper due to Escher, Prüss and Simonett [9] it was found that the principal symbol of the localized and linearized Stefan problem with Gibbs-Thomson correction is given by

$$s(\lambda,|\xi|^2)=\lambda+|\xi|^2\sqrt{\lambda+|\xi|^2},\quad \operatorname{Re}\lambda>0,\ \xi\in\mathbb{R}^n.$$

Setting $D=-\Delta$, the Laplacian on $\mathbb{R}^n$, this corresponds to the abstract Volterra integro-differential problem

$$\frac{d}{dt}[u(t)+\int_0^t De^{-Ds}u(t-s)\frac{ds}{\sqrt{\pi s}}]+\int_0^t D^2e^{-Ds}u(t-s)\frac{ds}{\sqrt{\pi s}}=f(t),\quad t>0, u(0)=0. \tag{4.1}$$

In that paper the maximal regularity properties of this problem have been studied by means of the Dore-Venni theorem in an L_p-setting, where $p\in(1,\infty)$. This result was used as the basis for the proof of well-posedness of the underlying nonlinear Stefan problem with surface tension. Here we want to apply Theorem 2 to the study of the symbol $s(\lambda,\xi)$ in interpolation spaces, in particular, in spaces of Hölder-continuous functions.

Note that $s(\lambda,|\xi|^2)$ is of first order in λ, of third order in ξ, but in addition there is the mixed order $1/2$ in λ and 2 in ξ. To show maximal regularity for this symbol we therefore have to study the three symbols

$$r_1(\lambda,|\xi|^2)=\lambda s(\lambda,|\xi|^2)^{-1},\ r_2(\lambda,|\xi|^2)=|\xi|^3s(\lambda,|\xi|^2)^{-1},\ r_3(\lambda,|\xi|^2)=\sqrt{\lambda}|\xi|^2s(\lambda,|\xi|^2)^{-1},$$

in particular, that these symbols give rise to bounded linear operators on a certain space of functions.

We begin with $D=-\Delta$ on $\mathbb{R}^n$. It is well-known that D sectorial in each of the Bessel-potential spaces $H_r^\beta(\mathbb{R}^n)$ and admits an $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle $\phi_D^\infty=0$, provided $\beta\in\mathbb{R}$ and $p\in(1,\infty)$. This is an easy consequence of e.g. the Fourier-multiplier theorem of Mikhlin. The same result is valid in the scale of Besov-spaces $B_{r,\rho}^\beta(\mathbb{R}^n)$ for all $\beta\in\mathbb{R}$ and $r,\rho\in[1,\infty]$. For $1<r<\infty$ the latter can be obtained by real interpolation between Bessel-potential spaces, but instead one may use Dore's result, i.e. Theorem 2 in the case of scalar-valued functions, to see this for all $r\in[1,\infty]$. Let Y denote any of these spaces.

Then we use the $\mathcal{H}^\infty$-calculus to obtain operator-valued functions $F_i(\lambda):=r_i(\lambda,D)$, $i=1,2,3$. These functions are holomorphic except for $\lambda\in(-\infty,0]$, they are also bounded on each sector Σ_ϕ, $\phi<\pi$.

Next fix a finite interval $J=[0,T]$ and let $X=C(J;Y)$. Define the operator A as $Au=du/dt$ with domain $\mathcal{D}(A)=\{u\in C^1(J;Y):\ u(0)=0\}$. It is also well-known -and in fact easy to prove- that A is pseudo-sectorial with spectral angle $\phi_A=\pi/2$ and $0\in\rho(A)$, however $\mathcal{D}(A)$ is not dense in X. Theorem 2 yields $F_i(A)\in\mathcal{B}(D_A(\alpha,p))$, for each $\alpha\in(0,1)$, $p\in[1,\infty]$ and $i=1,2,3$. Note that for $p=\infty$ we have $D_A(\alpha,\infty)={}_0C^\alpha(J;Y)$, where the subscript 0 means trace 0 at $t=0$. Without going into more details these considerations imply the following maximal regularity result for (4.1).

Theorem 3. *Let Y^β, $\beta\in\mathbb{R}$, denote any of the spaces $H_r^\beta(\mathbb{R}^n)$, $r\in(1,\infty)$, or $B_{r,\rho}^\beta(\mathbb{R}^n)$, $r,\rho\in[1,\infty]$, let $\alpha\in(0,1)$ and $J=[0,T]$.*

Then for each right hand side $f\in{}_0C^\alpha(J;Y^\beta)$, there is a unique solution $u:J\to Y^\beta$ for (4.1) of maximal regularity type. This means that u belongs to the class

$$u\in{}_0C^{1+\alpha}(J;Y^\beta)\cap{}_0C^{1/2+\alpha}(J;Y^{2+\beta})\cap{}_0C^\alpha(J;Y^{3+\beta}).$$

The solution map $f \mapsto u$ is bounded in the corresponding spaces.

5. The Vector-Valued Stein Multiplier Theorem.

Let (Ω, μ) and (Σ, ν) be measure spaces, $p, q \in (1, \infty)$. Assume that $\mathcal{T}(t)$ is a positive C_0-semigroup of contractions in $L_p(\Omega)$ which is in addition analytic, and let $-\mathcal{A}$ denote its generator. Then $\mathcal{T}(t)$ extends to a C_0-semigroup $T(t)$ on $X = L_p(\Omega, L_q(\Sigma))$ by the definition

$$T(t)[\varphi \otimes g] = [\mathcal{T}(t)\varphi]g, \quad t \geq 0,\ \varphi \in L_p(\Omega),\ g \in L_p(\Sigma);$$

we call it the **tensor extension** of $\mathcal{T}(t)$; cf. Clément and Egberts [3]. Let A denote the negative generator of $T(t)$, the tensor extension of $\mathcal{A}$. From the papers of Clément and Prüss [4], Hieber and Prüss [10] and Clément and Prüss [5] we know that A admits an $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle $\phi_A^\infty \leq \pi/2$ and also an operator-valued $\mathcal{H}^\infty$-calculus for functions $F \in \mathcal{H}^\infty(\Sigma_\phi; \mathcal{B}(X))$ with $F(\Sigma_\phi) \subset \mathcal{B}(X)$ $\mathcal{R}$-bounded, for each $\phi > \pi/2$. The question we address here is whether we may admit for some $\phi < \pi/2$. More precisely, we prove

Theorem 4. *Let (Ω, μ) and (Σ, ν) be measure spaces, $p, q \in (1, \infty)$. Suppose A on $X = L_p(\Omega; L_q(\Sigma))$ is the tensor extension of the negative generator $-\mathcal{A}$ of a positive analytic C_0-semigroup of contractions on $L_p(\Omega)$.*

Then there exists $\phi < \pi/2$ and $M > 0$ such that for each $F \in \mathcal{H}^\infty(\Sigma_\phi; \mathcal{B}(X))$ commuting with A and such that $F(\Sigma_\phi) \subset \mathcal{B}(X)$ is $\mathcal{R}$-bounded, $F(A) \in \mathcal{B}(X)$ is well-defined and $|F(A)|_{\mathcal{B}(X)} \leq M\mathcal{R}\{F(\Sigma_\phi)\}$, where $\mathcal{R}\{F(\Sigma_\phi)\}$ denotes the $\mathcal{R}$-bound of the set $F(\Sigma_\phi) \subset \mathcal{B}(X)$.

Proof. From Weis [16] we know $\phi_{\mathcal{A}}^\infty < \pi/2$. Therefore $\phi_A^\infty \leq \phi_{\mathcal{A}}^\infty < \pi/2$ in $L_p(\Omega, L_p(\Sigma))$. Interpolating this result with that of Hieber and Prüss we get

$$|A^{is}|_{\mathcal{B}(L_p(\Omega; L_q(\Sigma))} \leq M e^{(\theta\varphi + (1-\theta)\pi/2)|s|}, \quad s \in \mathbb{R},$$

where

$$\varphi \in (\phi_{\mathcal{A}}^\infty, \pi/2),\ 1 < r < \infty,\ \frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}.$$

Varying r from 1 to ∞ we cover all values of q, and since the exponent $\theta\varphi + (1-\theta)\pi/2$ is always strictly smaller than $\pi/2$ we see that the power angle θ_A of A in X is smaller than $\pi/2$. Clément and Prüss [5] implies $\phi_A^R \leq \theta_A$ and Kalton and Weis [12] have shown $\phi_A^\infty \leq \phi_A^R$, hence we conclude $\phi_A^\infty < \pi/2$. We may then apply the theorem of Kalton and Weis [12] mentioned in Section 3 to prove the assertion of Theorem 4. □

We obtain the following operator-valued version of the Stein Multiplier theorem [15] as a corollary.

Corollary 3. *In the situation of Theorem 4, let $E = L_q(\Sigma)$, and suppose that $M : (0, \infty) \to \mathcal{B}(E)$ is strongly measurable, $M(t)$ commutes with A for a.a. $t > 0$, and $M(0, \infty) \subset \mathcal{B}(E)$ is $\mathcal{R}$-bounded.*

Then with $F(z) = z\widehat{M}(z)$, the operator defined by

$$F(A) = A\widehat{M}(A) = \int_0^\infty M(t) A e^{-At} dt$$

is well-defined and bounded in X.

For a proof we apply Theorem 4 to $F(z)$. Note that $F(z)$ is holomorphic in $\Sigma_{\pi/2}$, commutes with A, and is bounded in each subsector Σ_ϕ, $\phi < \pi/2$. Finiteness of the $\mathcal{R}$-bounds $\mathcal{R}\{F(\Sigma_\phi)\}$ follow from convexity of $\mathcal{R}$-bounds and the contraction principle.

It is an open question whether in Theorem 4 the space $E = L_q(\Sigma)$ can be replaced by a general UMD-space.

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