# Study of a Coefficient Control Problem Using Nonsmooth Analysis

**By**

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**Abstract :** This report is concerned with a control problem associated to a semilinear second order ordinary differential equation with pointwise state constraints. The control acts as a coefficient of the state equation. The nonlinear part of the equation is governed by a Nemytskij operator defined by a nonsmooth function. We prove the existence of optimal controls and obtain a necessary optimality condition utilizing the notion of Clarke's generalized directional derivative.

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## 1. Introduction and position of the problem

The purpose of this report is to generalize and complete the results of the paper [1]. In a precise manner, for a given function $h$ in the Banach space $C([0,1])$ (of continuous functions on $[0,1]$), we are interested by finding

$$\inf \mathcal{J}(y), \qquad \mathcal{J}(y) = \int_0^1 (y(x) - h(x))^2\, dx, \tag{1}$$

where the state $y$ verifies the nonlinear boundary value problem :

$$\frac{d}{dx}\left(\frac{y'(x)}{u(x)}\right) + q(x)y(x) + \theta(y(x)) = 0, \quad x \in (0,1), \quad y(0) = 0, \quad y'(1) = 0, \tag{2}$$

under the constraint

$$0 \leq y(x) \leq a \quad \forall x \in [0, a], \tag{3}$$

the controls $u$ are Lipshitz continuous functions belonging to some compact subset $\mathcal{U}_{ad}$ of the Banach space $C([0,1])$ which is contained in $C^1([0,1])$. $q$ is a fixed nonnegative continuous function on $[0,1]$, and $\theta$ is a fixed Lipshitz continuous function on $[0,a]$, which is not belonging to $C^1([0,1])$. So, we can find two positive constants $b, l$ such that $0 \leq \theta(x) \leq b$ for all $x \in [0,a]$ and

$$|\theta(x_1) - \theta(x_2)| \leq l\,|x_1 - x_2|\,, \forall x_1, x_2 \in [0,a].$$

In all this paper, $\mathcal{U}_{ad}$ will be the closure, in the Banach space $C([0,1])$, of the following set :

$$\begin{aligned} \mathcal{S}_{\phi,k,r} := \Big\{ u \in C^2([0,1]) : \phi(x) \leq u(x) \leq \Omega \quad \forall x \in [0,1], \\ |u'(x)| \leq k, \quad \text{and} \quad |u''(x)| \leq r,\, \forall x \in [0,1] \Big\}, \end{aligned} \tag{4}$$

where $k, r, \Omega$ are fixed numbers in $]0, \infty[$, and $\phi$ is a fixed continuous function on $[0,1]$ such that $\phi(x) > 0$ for all $x \in [0,1]$.

Our control problem will be denoted by $\mathbf{P}_\theta$. We see that the controls are acting in this setting as coefficients for the state equation associated to $\mathbf{P}_\theta$. when $q$ is zero and when $\theta \in C^1([0,1])$, this problem was studied in [1]. In this paper, we are interested by the more general case where $q$ is not zero and $\theta$ is not in $C^1([0,1])$. In this case, we are led to use the notions of non smooth analysis. Similar problems were studied in [7], [13], and [12], however, using completely different methods. General remarks concerning coefficient control problems in both ordinary and partial differential equations can be found in [14]. This paper generalizes [1] and makes a sequel of the papers [2, 9, 10], where investigations were made for smooth and nonsmooth optimal Lipschitz control for problems governed by semilinear second order differential equations in which the nonlinear part is given by the action of a Nemytskij operator.

This report is organized as follows. In the second section, we establish existence of the states and optimal controls. The main result of this section is Theorem 2.2. In the section three, we establish necessary optimality conditions using tools and notions from nonsmooth analysis. The principal result of this section is Theorem 3.2.

## 2. Existence of the states and optimal controls.

In this section, we provide some sufficient conditions ensuring existence for solutions to our problem. These conditions are expressed by some inequalities between the fixed parameters $\Omega, a, b, l, k$ involved in the problem $\mathbf{P}_\theta$.

## 2.1 Preliminaries

**2.1.1** Let $u \in \mathcal{U}_{ad}$, and let $G_u = G_u(x, \xi)$ be the uniquely determined Green's function to the next boundary problem

$$\frac{d}{dx}\left[\frac{y'(x)}{u(x)}\right] = 0, \quad x \in (0,1), \quad y(0) = 0, \quad y'(1) = 0, \tag{5}$$

An easy computation shows that $G_u$ is given by

$$\begin{cases} G_u(x,\xi) = -\displaystyle\int_0^{\xi} u(s)\,ds & \text{for } 0 \le \xi \le x \le 1, \quad \text{and} \\ G_u(x,\xi) = -\displaystyle\int_0^{x} u(s)\,ds & \text{for } 0 \le x \le \xi \le 1. \end{cases}$$

$G_u$ is continuous and symmetric on $[0,1] \times [0,1]$ and the following estimation holds

$$0 \le -\int_0^1 G_u(x,\xi)\,d\xi \le \frac{\Omega}{2} \quad \forall x \in [0,1].$$

**2.1.2** We set $Q := \sup\{q(x) : x \in [0,1]\}$. In all this paper, we suppose that $Q < 2$, and that

$$\Omega < \min\left\{\frac{a(2-Q)}{b}, \frac{2}{l+Q}\right\}. \tag{6}$$

The Banach space $C([0,1])$ will be equipped with its usual norm denoted by $\|.\|_{C([0,1])}$. We consider the subset $B_+(a)$ of $C([0,1])$ defined by

$$B_+(a) := \{y \in C([0,1]) : 0 \le y(x) \le a \quad \forall x \in [0,1]\}.$$

For any arbitrary control $u \in \mathcal{U}_{ad}$, an element $y \in C^2([0,1]) \cap B_+(a)$ is a solution to the nonlinear boundary value problem (2) if, and only if, $y \in B_+(a)$ and $y$ is a solution to the Hammerstein integral equation

$$y(x) = -\int_0^1 G_u(x,\xi)\theta(y(\xi))\,d\xi - \int_0^1 G_u(x,\xi)q(\xi)y(\xi)\,d\xi \quad \forall x \in [0,1]. \tag{7}$$

**2.1.3** To each control $u \in \Sigma_{ad}$ we associate the unique solution $S(u) = y_u$ to the problem (2) (under condition (3)). One can see that $S$ is a Lipschitz continuous map from the compact convex subset $\Sigma_{ad}$ of $C([0,1])$ to the Banach space $C([0,1])$. Indeed, for all controls $u, v \in \mathcal{U}_{ad}$, an easy computation will give the following estimation

$$\|y_u - y_v\|_{C([0,1])} \le \frac{b + aQ}{2 - (l+Q)\Omega}\|u - v\|_{C([0,1])}. \tag{8}$$

With these preliminaries, we are in position to establish the existence of the solutions to our control problem. More precisely, we have the following theorem.

**2.2 Theorem :** (i) *For each control $u \in \mathcal{U}_{ad}$, the boundary value problem* (1) *has a unique solution $y_u$. Moreover this solution belongs to $C^2([0,1]) \cap B_+(a)$.*
(ii) *The optimal control problem $\mathbf{P}_\theta$ has (at least) an optimal solution $u_0 \in \Sigma_{ad}$.*

**Proof :** (i) Let $u \in \Sigma_{ad}$ be fixed and associate to it the map $T_u$ defined from $B_+(a)$ to $C([0,1])$ by

$$T_u(y)(x) := -\int_0^1 G_u(x,\xi)\theta(y(\xi))\,d\xi - \int_0^1 G_u(x,\xi)q(\xi)y(\xi)\,d\xi \quad \forall x \in [0,1]. \tag{9}$$

An easy computation will show that for all $y, z \in B_+(a)$, we have

$$\|T_u(y) - T_u(z)\|_{C([0,1])} \leq \frac{(l+Q)\Omega}{2}\|y - z\|_{C([0,1])}. \tag{10}$$

Since $0 \leq q(x) \leq Q < 2$ $(x \in [0,1])$ and since we have supposed that $b\Omega \leq a(2-Q)$, we see that $T_u(B_+(a)) \subset B_+(a)$. Moreover, according to the inequality (6), $T_u$ must be a contraction from $B_+(a)$ to itself. Since the set $B_+(a)$ is a closed (convex) subset of the Banach space $C([0,1])$, we deduce by using the Banach fixed point theorem that $T_u$ has a unique fixed point $y_u \in B_+(a)$. This proves (i). It remains to prove (ii)

(ii) For each control $u \in \Sigma_{ad}$ we set $J(u) := \mathcal{J}(y_u)$. We obtain by easy computation the following estimation

$$|J(u) - J(v)| \leq \frac{2(b+aQ)(a+\|h\|_{C([0,1]})}{2-(l+Q)\Omega}\|u - v\|_{C([0,1])}. \tag{11}$$

This inequality says that the map $J$ is Lipschitz continuous from the compact subset $\mathcal{U}_{ad}$ of the Banach space $C([0,1])$ to the set of real numbers. Hence, using the classical Weierstrass theorem, we deduce that there exists at least one optimal control to our problem $(\mathbf{P}_\theta)$. ∎

## 3. Necessary optimality condition

### 3.1. Preliminaries :

**3.1.1** Let $u_0 \in \mathcal{U}_{ad}$ be an optimal control to the problem $\mathbf{P}_\theta$ and $u \in \mathcal{U}_{ad}$ another admissible control. The respective states are denoted by $y_0 = S(u_0)$ and $y_u = S(u)$. For any $\lambda \in [0,1]$ we set $u_\lambda := u_0 + \lambda(u - u_0) \in \Sigma_{ad}$, and $y_\lambda := S(u_\lambda)$. ¿From the inequality (8) we obtain

$$\|y_\lambda - y_0\|_{C([0,1])} \leq \frac{\lambda(b+aQ)}{2-(l+Q)\Omega}\|u - u_0\|_{C([0,1])}, \quad \forall \lambda \in [0,1]. \tag{12}$$

**3.1.2** As in the paper [1], we can prove that if $\theta$ belongs to $C^1([0,1])$, then the quotient $\phi_\lambda^u := \frac{y_\lambda - y_0}{\lambda}$ $(\lambda \in ]0,1])$ converges in the Banach space $C([0,1])$, when $\lambda \longrightarrow 0^+$, to the unique fixed point $\tilde{y}(u)$ of the selfmapping $\Upsilon$ of $C([0,1])$ defined for all $z \in C([0,1])$, by

$$\Upsilon(z)(x) := -y_0(x) - m(x) - \int_0^1 G_{u_0}(x,\xi)\theta'(y_0(\xi))z(\xi)\,d\xi \quad \forall x \in [0,1],$$

where $m$ is the function defined for all $x \in [0,1]$, by

$$m(x) := \int_0^1 G_u(x,\xi)\theta(y_0(\xi))\, d\xi + \int_0^1 G_u(x,\xi)q(\xi)y_0(\xi)\, d\xi.$$

Indeed, it is easy to see that $\Upsilon(z)$ verifies the following inequality :

$$\|\Upsilon(z_1) - \Upsilon(z_2)\|_{C([0,1])} \leq \frac{(l+Q)\Omega}{2}\|z_1 - z_2\|_{C([0,1])}. \tag{13}$$

Therefore, the inequality (6) and the Banach fixed point theorem ensure the existence and uniqueness of a unique fixed point which we have denoted here by $\tilde{y}(u)$.

In our situation, we have no information about the convergence of the quotient $\phi_\lambda^u$ in the Banach space $C([0,1])$. Instead of this, we shall see next that we can always find at least a subsequence $(\lambda_n)_n$ of elements in $]0,1]$ converging to zero for which the quotient $((y_{\lambda_n} - y_0){\lambda_n}^{-1})_n$ converges in $C([0,1])$.

**3.1.3** As it was done in the papers [2, 9, 10], we shall use tools from nonsmooth analysis to derive our optimal conditions. We start by pointing out that since the map $J : \mathcal{U}_{ad} \to \mathbb{R}$ is Lipschitz continuous (but surely not convex), we have the optimality condition

$$0 \in \partial J(u_0) + N_{\mathcal{U}_{ad}}(u_0). \tag{14}$$

where $J$ is now a Lipschitz continuous extension of the map $J : \mathcal{U}_{ad} \to \mathbb{R}$ to the whole Banach space $C([0,1])$. As usual, the notation $\partial J(u_0)$ designates Clarke's subdifferential of $J$ at $u_0$ and $N_{\mathcal{U}_{ad}}(u_0)$ is the normal cone to $\mathcal{U}_{ad}$ at $u_0$ (see [6], p. 52). For each admissible control $u$ we introduce (as in [2, 9, 10]) the set $\Phi(u)$ given by

$$\Phi(u) := \Big\{\phi_\lambda \in C([0,1]) : \phi_\lambda = \frac{y_\lambda - y_0}{\lambda}, \quad \lambda \in ]0,1]\Big\},$$

where $y_\lambda = y_{u_\lambda}$ is the same as before, and define two new functions $\phi_-$ and $\phi_+$ by setting

$$\phi_-(x) := \liminf_{\lambda \to 0+} \phi_\lambda(x), \quad \phi_+(x) := \limsup_{\lambda \to 0+} \phi_\lambda(x), \quad \forall x \in [0,1].$$

By using the inequality (12), we see that $\Phi(u)$ is a bounded subset of $C([0,1])$. Also, it is easy to show that $\Phi(u)$ is equicontinuous. Therefore this set is relatively compact in the Banach space $C([0,1])$. As a consequence, we deduce that the functions $\phi_-$ and $\phi_+$ are continuous on $[0,1]$. Another consequence is that, if we introduce the new set $\Phi_0(u)$ given by

$$\begin{aligned}\Phi_0(u) := \Big\{\phi \in C([0,1]) : \phi = \lim_{n \to +\infty} \phi_{\lambda_n} \quad \text{in} \quad C([0,1]), \quad \text{where} \\ (\phi_{\lambda_n})_n \subset \Phi(u) \quad \text{and} \quad (\lambda_n)_n \subset [0,1] \quad \text{with} \quad \lambda_n \to 0+\Big\},\end{aligned}$$

then $\Phi_0(u)$ is nonempty and bounded. Now, let us denote by $\theta$ any given Lipschitz continuous extension of the function $\theta : [0,1] \to \mathbb{R}$ to the whole $\mathbb{R}$, and let $\theta^0(x;\xi)$ designates Clarke's directional derivative of $\theta$ at $x$ in the direction $\xi$ for all $x, \xi \in \mathbb{R}$. Then with these notations and preliminaries, we can state our result providing the optimality conditions.

**3.2. Theorem :** *Let $u_0 \in \mathcal{U}_{ad}$ be an optimal control to $P_\theta$ and $y_0 \in C([0,1])$ the related optimal state. Then for all $u \in \mathcal{U}_{ad}$ and all $\phi \in \Phi_0(\sigma)$ it holds*

(i) $\phi(x) \in [\phi_-(x), \phi_+(x)] \quad \forall x \in [0,1]$.

(ii) $\phi(x) \leq -\int_0^1 G_{u_0}(x,\xi)\theta^0(y_0(\xi);\phi(\xi))\,d\xi - \int_0^1 \Big(G_u(x,\xi) - G_{u_0}(x,\xi)\Big)\theta(\phi(\xi))\,d\xi$

$\qquad - \int_0^1 G_{u_0}(x,\xi)q(\xi)\phi(\xi)\,d\xi, \qquad$ *for all $x \in [0,1]$.*

(iii) $0 \leq \int_0^1 \phi(x)(y_0(x) - h(x))\,dx$.

**Proof :** Let $u \in \mathcal{U}_{ad}$. We verify first that the assertions (i), (ii) and (iii) hold if $u = u_0$. Indeed, in this case, we have $\Phi(u_0) = \{0\}$, and therefore $\Phi_0(u_0) = \{0\}$, and $\phi_- = \phi_+ = 0$. Thus we may suppose that $u \neq u_0$. Consider an element $\phi \in \Phi_0(u)$. Then by the definition of the set $\Phi_0(\sigma)$, one can find a zero sequence $(\lambda_n)_n \subset ]0,1]$ and a sequence $(\phi_{\lambda_n})_n \subset \Phi(u)$ converging to $\phi$ in the Banach space $C([0,1])$. Assertion (i) is obvious. Next, we prove assertion (ii).

According to the definition of $\phi_{\lambda_n}$, for all $x \in [0,1]$ we have

$$\begin{aligned}
\phi_{\lambda_n}(x) &= \frac{y_{\lambda_n}(x) - y_0(x)}{\lambda_n} \\
&= -\int_0^1 G_{u_0}(x,\xi)\frac{\theta(y_{\lambda_n}(\xi)) - \theta(y_0(\xi))}{\lambda_n}\,d\xi \\
&\quad - \int_0^1 \big[G_u(x,\xi) - G_{u_0}(x,\xi)\big]\theta(y_{\lambda_n}(x))\,d\xi \\
&\quad - \int_0^1 G_{u_0}(x,\xi)q(\xi)\phi_{\lambda_n}(\xi)\,d\xi \\
&\quad - \int_0^1 \big[G_u(x,\xi) - G_{u_0}(x,\xi)\big]q(\xi)y_{\lambda_n}(\xi)\,d\xi.
\end{aligned} \tag{15}$$

Because of (12) and the fact that $\theta$ is $l-$Lipshitz continuous, we have

$$\lim_{n\to+\infty} y_{\lambda_n} = y_0, \quad \text{and} \quad \lim_{n\to+\infty} \theta(y_{\lambda_n}) = \theta(y_0) \quad \text{in} \quad C([0,1]).$$

$\theta$ being $l-$Lipshitz continuous, for all $x \in [0,1]$ we have the inequality

$$\left|\frac{\theta(y_{\lambda_n}(\xi)) - \theta(y_0(\xi))}{\lambda_n}\right| \leq \frac{l}{\lambda_n}\left|y_{\lambda_n}(x) - y_0(x)\right| \leq \frac{b + aQ}{2 - (l+Q)\Omega}\|u - u_0\|_{C([0,1])}.$$

By taking in (15) the limit superior and applying Fatou's lemma we obtain

$$\begin{aligned}\phi(x) \leq &- \int_0^1 G_{u_0}(x,\xi) \limsup_{n\to+\infty} \frac{\theta(y_{\lambda_n}(\xi)) - \theta(y_0(\xi))}{\lambda_n}\, d\xi \\ &- \int_0^1 \big[G_u(x,\xi) - G_{u_0}(x,\xi)\big]\theta(y_0(x))\, d\xi \\ &- \int_0^1 G_{u_0}(x,\xi) q(\xi)\phi(\xi)\, d\xi \\ &- \int_0^1 \big[G_u(x,\xi) - G_{u_0}(x,\xi)\big] q(\xi) y_0(\xi)\, d\xi \quad \forall x \in [0,1].\end{aligned} \tag{16}$$

Since $\theta : [0,a] \to \mathbb{R}$ is $l-$Lipschitz continuous it can be extended to the whole $\mathbb{R}$ in a Lipschitz continuous function denoted again by $\theta$ having the same Lipschitz constant $l$. For this extension and for all $x, \xi \in \mathbb{R}$ we will denote Clarke's directional derivative of $\theta$ at $x$ in the direction $\xi$ by $\theta^0(x;\xi)$, and $\partial\theta(x)$ will be the corresponding subdifferential (see for example [6], [15]). By using the mean-value theorem for subdifferentials (see for example [15], p. 40) we get the following property : For any natural number $n \in \mathbb{N}$ and any $x \in [0,1]$ there are two real numbers $\zeta_n(x)$, $z_n(x) \in \mathbb{R}$ satisfying the following conditions

$$\zeta_n(x) \in \partial\theta(z_n(x)) \;\; \forall n \in \mathbb{N} \qquad \lim_{n\to+\infty}(z_n(x)) = y_0(x) \quad (\text{in}\, \mathbb{R}),$$
$$\left|\frac{\theta(y_{\lambda_n}(\xi)) - \theta(y_0(\xi))}{\lambda_n}\right| = \zeta_n(x)\phi_{\lambda_n}(x) \leq \theta^0(z_n(x);\phi_{\lambda_n}(x)).$$

According to the upper semicontinuity of Clarke's directional derivative we obtain

$$\limsup_{n\to+\infty}\left|\frac{\theta(y_{\lambda_n}(\xi)) - \theta(y_0(\xi))}{\lambda_n}\right| \leq \theta^0(y_0(x);\phi(x)) \quad \forall x \in [0,1]. \tag{17}$$

Now, by using the following estimate

$$\theta^0(y_0(x);\phi(x)) \leq l\,|\phi(x)| \quad \forall x \in [0,1],$$

we deduce that the function $\theta^0(y_0(.);\phi(.))$ is bounded. Furtheremore, since it is upper semicontinuous on $[0,1]$ it is measurable and, hence it is integrable on $[0,1]$. The property (ii) follows then from (16) and (17). It remains to prove (iii). To this end, we start by noticing that for all $\lambda \in ]0,1]$, we have the following inequality :

$$0 \leq \frac{j(u_\lambda) - j(u_0)}{\lambda} = 2\int_0^1 \frac{y_\lambda - y_0}{\lambda}(y_0 - h)\, dx + \frac{1}{\lambda}\int_0^1 (y_\lambda - y_0)^2\, dx. \tag{18}$$

Now, we use the inequalities (12), (18) and obtain (iii) by applying Lebesgue's theorem of dominated convergence to the sequence $\phi_{\lambda_n}$ which is converging to $\phi$ in the Banach space $C([0,1])$. $\square$

This theorem is a generalization of our theorem 3.1.3 stated in the paper [1] when $\theta$ is continuously differentiable.

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