Topologies for the Set of Disjunctive ω-words

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Abstract

An infinite sequence (ω-word) is referred to as disjunctive provided it contains every finite word as infix (factor). As Jürgensen and Thierrin [JT83] observed the set of disjunctive ω-words, D, has a trivial syntactic monoid but is not accepted by a finite automaton.

In this paper we derive some topological properties of the set of disjunctive ω-words. We introduce two non-standard topologies on the set of all ω-words and show that D fulfills some special properties with respect to these topologies:
In the first topology – the so-called topology of forbidden words – D is the smallest nonempty $\mathbf{G}_\delta$-set, and in the second one D is the set of accumulation points of the whole space as well as of itself.

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Contents

In 1983 two papers dealing with the ω-language of disjunctive ω-words appeared [JST83, JT83]. In the latter it was shown that this ω-language is a natural example of an ω-language having a trivial (finite) syntactic monoid but not being accepted by a finite automaton. For a more detailed account see [St83, JT86].
Subsequently, disjunctive ω-words became of interest in connection with random and Borel normal sequences (see, for instance, [Ca94, He96]). In contrast to Borel normality, "disjunctivity" is a natural qualitative property which is satisfied, in particular, by Borel normal and by random ω-words.
As in [JST83, JT83] we say that an ω-word is *disjunctive* if it contains any (finite) word as a subword. In this paper we are going to investigate topological properties of the set of all disjunctive sequences (ω-words).
Usually, one considers the space of all ω-words over a finite alphabet X as the infinite product space of the discrete space X. Introducing the Baire metric, this space can be considered as a metric space (Cantor space) (X^ω, ρ), that is, a compact totally disconnected space.
In this paper we consider topologies on the set of all ω-words over a finite alphabet X in which the set of all disjunctive ω-words has a special property:
First, we consider the topology of "forbidden words" in which the set of disjunctive ω-words is the smallest $\mathbf{G}_\delta$-set. The second topology is a special case of the topologies derived from formal languages (cf. [St87]). Here the set of disjunctive ω-words turns out to be the largest closed and dense in itself set.

1 Notation

By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers. Let X be our alphabet of cardinality $\# X = r$, $r \in \mathbb{N}$, $r \geq 2$.
By X^* we denote the set of finite strings (words) on X, including the *empty* word e. We consider the space X^ω of infinite sequences (ω-words) over X. For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$.
We extend the operations * and $^\omega$ to arbitrary subsets $W \subseteq X^*$ in the usual way :

$$W^* := \bigcup_{n \in \mathbb{N}} W^n \text{ where } W^0 := \{e\}, W^{n+1} := W^n \cdot W \text{ , and}$$

$$W^\omega := \{w_0 \cdot w_1 \cdot \ldots \cdot w_i \cdot \ldots : i \in \mathbb{N} \wedge w_i \in W \setminus \{e\}\}$$

is the set of ω-words in X^ω formed by concatenating members of W.
We will refer to subsets of X^* and X^ω as languages or ω-languages, respectively.
By "$\sqsubseteq$" we denote the prefix relation, that is, $w \sqsubseteq \eta$ if and only if there is an η' such that $w \cdot \eta' = \eta$, and $\mathbf{A}(\eta) := \{w : w \in X^* \wedge w \sqsubseteq \eta\}$ and $\mathbf{A}(B) := \bigcup_{\eta \in B} \mathbf{A}(B)$ are the languages of finite prefixes of η and B, respectively.
The set of subwords (infixes) of $\eta \in X^* \cup X^\omega$ will be denoted by $\mathbf{T}(\eta) := \{w : w \in X^* \wedge \exists v(vw \sqsubseteq \eta)\}$.
An ω-language F is called *regular* provided there is an $n \in \mathbb{N}$ and regular languages W_i, V_i $(1 \leq i \leq n)$ such that

$$F = \bigcup_{i=1}^{n} W_i V_i^\omega. \tag{1}$$

Similarly, an ω-language F is called *context-free* if F has the form of Eq. (1) where W_i and V_i are context-free languages.
Observe, that $V^\omega = \emptyset$, $V^\omega = \{v\}^\omega$ or $V^\omega \supseteq \{v, u\}^\omega$ for some words $v, u \in V^*$ with $|v| = |u| > 0$ and $v \neq u$. Thus, every at most countable context-free ω-language consists entirely of ultimately periodic ω-words (cf. [St97]).

2 Preliminary Considerations

In the study of ω-languages it is useful to consider X^ω as a metric space (Cantor space) with the following metric.

$$\rho(\eta, \xi) := \inf\{r^{-|w|} : w \sqsubset \eta \wedge w \sqsubset \xi\} \tag{2}$$

or an equivalent one[1].
In this paper, however, we will consider also a topology on X^ω which cannot be specified by a metric, that is, a so-called non-metrizable topology. To this end we introduce topologies on X^ω in the general way (cf. [Ku66, En77]).
A *topology* in X^ω is a family $\mathcal{O} \subseteq 2^{X^\omega}$ of subsets of X^ω such that $\emptyset, X^\omega \in \mathcal{O}$ and $\mathcal{O}$ is closed under finite intersection and arbitrary union. The sets in $\mathcal{O}$ are called *open* subsets of X^ω. The complements of open subsets are referred to as *closed*. Since an arbitrary intersection of closed

[1] For example, the Baire metric $\rho(\eta, \xi) := \inf\{\frac{1}{1+|w|} : w \sqsubset \eta \wedge w \sqsubset \xi\}$ generates the same topology.

sets is again closed, every set $F \subseteq X^\omega$ is contained in a minimal closed set, its *closure* $\mathcal{C}_\mathcal{O}(F)$.
Having defined open and closed sets for some topology in X^ω, we proceed to the next classes of the Borel hierarchy (cf. [Ku66]):

$\mathbf{G}_\delta$ is the set of countable intersections of open subsets of X^ω,

$\mathbf{F}_\sigma$ is the set of countable unions of closed subsets of X^ω.

A metric σ generates the set of open sets $\mathcal{O}_\sigma$ in the following way: First we define the *open balls* $B_\epsilon(\xi) := \{\eta : \sigma(\xi,\eta) < \epsilon\}$ for $\epsilon > 0$. Then a set is open in the space (X^ω, σ) if it is a union of open balls. In Cantor space, open balls are of the form $w \cdot X^\omega$, and, consequently, the set of open subsets of X^ω is $\mathcal{O}_C = \{W \cdot X^\omega : W \subseteq X^*\}$. From this it follows that a subset $F \subseteq X^\omega$ is closed in Cantor space if and only if $\mathbf{A}(\xi) \subseteq \mathbf{A}(F)$ implies $\xi \in F$, and the closure in Cantor space can be specified as $\mathcal{C}(F) := \{\xi : \mathbf{A}(\xi) \subseteq \mathbf{A}(F)\}$.
In Section 4 we shall consider the so-called topology of "forbidden" words which is specified by the set of open sets $\mathcal{O}_\mathcal{T} := \{X^* \cdot W \cdot X^\omega : W \subseteq X^*\}$.[2]
This topology is a subtopology of Cantor topology $\mathcal{O}_C \supset \mathcal{O}_\mathcal{T}$, or, equivalently, the Cantor space is a refinement of the topology of "forbidden" words.
Finally, we define for a language $W \subseteq X^*$ its *δ-limit* of W, W^δ, which consists of all infinite sequences of X^ω that contain infinitely many prefixes in W,

$$W^\delta = \{\xi \in X^\omega : \#(\mathbf{A}(\xi) \cap W) = \infty\}.$$

For $\mathbf{G}_\delta$-sets in Cantor space we have the following characterization via languages (cf. [Th90, St87, St97]). It explains also why we call W^δ the δ-limit of the language W.

2.1 Theorem *In Cantor space, a subset $F \subseteq X^\omega$ is a $\mathbf{G}_\delta$-set if and only if there is a language $W \subseteq X^*$ such that $F = W^\delta$.*

3 The ω-Language of Disjunctive Sequences

In this section we will present a few simple general properties of the ω-language of all disjunctive sequences over X, D, and its topological

[2] The term *forbidden* refers to the fact that closed subsets are specified by forbidding a certain set W of infixes.

properties in Cantor space. Some of the results in this section are reported in [CPS97, St01].
As in [JST83, JT83] an ω-word $\xi \in X^\omega$ is called *disjunctive* provided $\mathbf{T}(\xi) = X^*$. Thus $D = \{\xi : \mathbf{T}(\xi) = X^*\}$.
From this definition we obtain

$$D = \bigcap_{w \in X^*} X^* w X^\omega . \tag{3}$$

Our next lemma shows that D is an example of a ω-language which has a trivial finite syntactic congruence but is not context-free. The proof refers to the investigations of Jürgensen and Thierrin [JT83, JT86].
The *syntactic congruence* $\sim_F$ of an ω-language $F \subseteq X^\omega$ is defined as follows[3]

$$w \sim_F v :\Leftrightarrow \forall u \forall \xi (u \in X^* \wedge \xi \in X^\omega \rightarrow (uw\xi \in F \leftrightarrow uv\xi \in F)) .$$

As usual, we call $\sim_F$ of *finite index* iff its number of equivalence classes is finite.
Observe that $\mathbf{T}(uw\xi) = X^*$ iff $\mathbf{T}(\xi) = X^*$. Thus it is clear that $w \sim_D v$ for arbitrary $w, v \in D$, and $\sim_D$ has exactly one equivalence class which coincides with X^*. Thus we have proven the first part of the following.

3.1 Lemma ([JT83]) *The ω-language D has a syntactic congruence of finite index but is not context-free.*

Proof. As $\mathbf{T}(\prod_{w \in X^*} w) = X^*$ and $\mathbf{T}(wv^\omega) \neq X^*$, D is nonempty and does not contain an ultimately periodic ω-word wv^ω. Following Eq. (1) the ω-language D cannot be context-free. ❏

The representation of Eq. (3) verifies that D is a $\mathbf{G}_\delta$-set in Cantor space. Thus, in view of Theorem 2.1 it can be represented as the δ-limit of a language. In case of D we construct such a language W_D explicitly (cf. [St01]).

3.2 Proposition *Let* $W_D = \{wx : w \in X^* \wedge x \in X \wedge \exists n (n \leq |w| + 1 \wedge \mathbf{T}(wx) \supseteq X^n \wedge \mathbf{T}(w) \not\supseteq X^n)\}$. *Then* $D = W_D^\delta$.

Finally, we are going to show that the topological complexity of D in Cantor space cannot be decreased. To this end we quote Theorem 21 from [St83].

[3] There are other notions of syntactic congruences for ω-languages in use (cf. [MS97]).

3.3 Theorem ([St83]) *If $F \subseteq X^\omega$ has a syntactic congruence of finite index and is simultaneously an $\mathbf{F}_\sigma$- and a $\mathbf{G}_\delta$-set in Cantor space, then F is regular.*

Combining Theorem 3.3 with Lemma 3.1 and Eq. (3) we get:

3.4 Proposition *In Cantor space, D is not an $\mathbf{F}_\sigma$-set.*

4 The Topology of Forbidden Words

In this section we investigate the topology of forbidden words described above and its relations to the set of disjunctive sequences. It turns out that this topology is not a metric one.
Recall $\mathcal{O}_\mathcal{T} = \{X^*WX^\omega \mid W \subseteq X^*\}$ from Section 2. As $X^*VX^\omega \cap X^*WX^\omega = (X^*WX^* \cap X^*VX^*)X^\omega$ this family $\mathcal{O}_\mathcal{T}$ is closed under finite intersection. The closure under arbitrary union is obvious. Thus it defines a topology on X^ω.
An ω-language $F \subseteq X^\omega$ *avoids* words of a language $W \subseteq X^*$ provided $F \subseteq X^\omega \setminus X^*WX^\omega$, that is, no word $w \in W$ occurs as a subword (infix) of an ω-word $\xi \in F$. Therefore, the closed sets in the topology $\mathcal{O}_\mathcal{T}$ are characterized by the fact that their ω-words do not contain subwords from W. The following theorem gives a connection to closed sets in Cantor space.
To this end we define $F/w := \{\xi : w\xi \in F\}$.

4.1 Theorem *Let $F \subseteq X^\omega$. Then the following conditions are equivalent:*

1. *F is closed in the topology of forbidden words.*
2. *F is closed in Cantor space and $\forall w(w \in X^* \Rightarrow F \supseteq F/w)$.*
3. *F is closed in Cantor space and $\mathbf{A}(F) = \mathbf{T}(F)$.*
4. *$\forall \xi(\mathbf{A}(\xi) \subseteq \mathbf{T}(F) \Rightarrow \xi \in F)$.*

Proof. "*1.* $\Rightarrow$ *2*": As we noticed above, every ω-language closed in the topology of forbidden words is also closed in Cantor's topology. Let $w \in X^*$ and $F = X^\omega \setminus X^*WX^\omega$. Then $F/w = X^\omega \setminus (X^*WX^\omega)/w$, and the assertion follows from the obvious inclusion $(X^*WX^\omega)/w \supseteq X^*WX^\omega$.
"*2.* $\Rightarrow$ *3.*" follows from the identity $\mathbf{A}\left(\bigcup_{w\in X^*} F/w\right) = \mathbf{T}(F)$.
"*3.* $\Rightarrow$ *4.*": If F is closed in Cantor space we have $F = \{\xi : \mathbf{A}(\xi) \subseteq \mathbf{A}(F)\}$. Now the assertion *4.* follows from $\mathbf{A}(F) = \mathbf{T}(F)$.

Finally, we show that Condition 4 implies $F = X^\omega \setminus X^* \cdot (X^* \setminus \mathbf{T}(F)) \cdot X^\omega$. Since $X^* \setminus \mathbf{T}(F) = X^* \cdot (X^* \setminus \mathbf{T}(F)) \cdot X^*$ it suffices to prove that $F = X^\omega \setminus (X^* \setminus \mathbf{T}(F)) \cdot X^\omega$.
The inclusion $F \subseteq X^\omega \setminus (X^* \setminus \mathbf{A}(F)) \cdot X^\omega \subseteq X^\omega \setminus (X^* \setminus \mathbf{T}(F)) \cdot X^\omega$ follows from $\mathbf{A}(F) \subseteq \mathbf{T}(F)$. To prove the converse inclusion let $\xi \notin F$. Then in view of Condition 4 there is a prefix $w \sqsubset \xi$ such that $w \notin \mathbf{T}(F)$. Consequently, $\xi \in (X^* \setminus \mathbf{T}(F)) \cdot X^\omega$. ❑

In view of the equivalence "*1.* $\Leftrightarrow$ *4.*" we obtain the following representation of the closure operator $\mathcal{C}_\mathcal{T}$ defined by the topology of forbidden words:

$$\mathcal{C}_\mathcal{T}(F) = \{\xi : \mathbf{A}(\xi) \subseteq \mathbf{T}(F)\}.$$

Recall that the closure in Cantor space was definable as $\mathcal{C}(F) = \{\xi : \mathbf{A}(\xi) \subseteq \mathbf{A}(F)\}$.
The additional requirements $\forall w(w \in X^* \Rightarrow F \supseteq F/w)$ and $\mathbf{A}(F) = \mathbf{T}(F)$ in *2.* and *3.* are, however, not equivalent in general. The following example shows that there is an ω-language (necessarily not closed in Cantor space) which satisfies $\mathbf{A}(F) = \mathbf{T}(F)$, but not the condition $\forall w(w \in X^* \Rightarrow F \supseteq F/w)$.

4.2 Example Let $F = (X^2)^* bba^\omega \cup X(X^2)^* aab^\omega$. Then $\mathbf{A}(F) = \mathbf{T}(F) = X^*$, but $F/a \not\subseteq F$.

Since the family of regular ω-languages is closed under Boolean operations, the ω-language $F_W = X^\omega \setminus X^* W X^\omega$ is a regular if the language of forbidden patterns $W \subseteq X^*$ is regular. In connection with Eq. (1) and the considerations on V^ω immediately following it this yields as a consequence the following generalization of a result of El-Zanati and Transue [ET90].

4.3 Theorem *Let $W \subseteq X^*$ be a regular language. If F_W is uncountable, then F_W contains a subset of the form $w\{u, v\}^\omega$, where $u \neq v$ and $|u| = |v| > 0$.*

We continue with some more examples. The first is an example of a countable regular ω-language F_W which requires an infinite set of forbidden patterns.

4.4 Example Let $X = \{a, b\}$ and $W = ba^*b$. Then $F_W = X^\omega \setminus X^* W X^\omega = a^*ba^\omega \cup a^\omega$ is a countable ω-language. It is clear that $F_W \neq F_V$, for any finite language $V \subseteq X^*$.

Though the regularity of W implies the regularity of F_W this same relation is not true for context-free languages and ω-languages.

4.5 Example Let $X = \{a, b\}$ and $W = \{bb\} \cup \{ba^i ba^j b \mid j \neq i+1\}$. Clearly, W is a deterministic context-free language, and $F_W = a^*(\{\eta_i \mid i \in \mathbb{N}\} \cup \{\eta_{i,j} \mid i, j \in \mathbb{N} \wedge i \leq j\})$ where $\eta_i = ba^i ba^{i+1} b \cdots$ and $\eta_{i,j} = ba^i ba^{i+1} \cdots ba^j ba^\omega$. Since F_W is countable but does not consist entirely of ultimately periodic ω-words, Eq. (1) shows that F_W is not context-free.

Finally, we discuss a characterization of the ω-language of disjunctive sequences D by means of the topology of forbidden words. From Eq. (3) we obtain immediately

4.6 Proposition *In the topology of forbidden words,* D *is the smallest nonempty* $\mathbf{G}_\delta$*-set.*

A set $F \subseteq X^\omega$ is *dense* in X^ω in case X^ω is the smallest closed set containing F, that is, $X^\omega \setminus F$ does not contain a nonempty open set. Since $\xi \in X^\omega$ is disjunctive, we have $\mathbf{T}(\xi) = X^*$, and therefore $\{\xi\} \cap X^* w X^\omega \neq \emptyset$ for all $w \in X^*$. Thus we have shown the following.

4.7 Proposition *An ω-word $\xi \in X^\omega$ is disjunctive if and only if the set $\{\xi\}$ is dense in X^ω in the topology of forbidden words.*

This proposition shows that every closed set in the topology of forbidden words which contains some $\xi \in D$ must coincide with the whole space X^ω. Consequently, every $\mathbf{F}_\sigma$-set containing $\xi \in D$ equals X^ω.

4.8 Corollary D *is not an* $\mathbf{F}_\sigma$*-set in the topology of forbidden words.*

Above we mentioned, the topology of forbidden words is not a metrizable topology, that is, it is not definable by a metric. Proposition 4.7 gives evidence of this fact, because the sets $\{\xi\}$, $\xi \in D$ are not closed, and in a metrizable topology every finite set has to be closed.

5 A Metric Related to Languages

The definition of the topologies considered in this part is related to the well-known fact that every $\mathbf{G}_\delta$-set of a complete metric space is a complete metric space itself (cf. [Ku66]), possibly using a different metric. We use here the construction presented in [St87]. Related investigations were carried out in [DNPY92]

As we have seen in Theorem 2.1, in Cantor space a $\mathbf{G}_\delta$-set is of the form U^δ for some $U \subseteq X^*$. We use this language U to define a new metric ρ_U on X^ω which makes U^δ a closed set in the metric space (X^ω, ρ_U):

$$\rho_U(\xi, \eta) = \begin{cases} 0 & \text{, if } \xi = \eta \text{, and} \\ r^{1-\#\mathbf{A}(\xi)\cap\mathbf{A}(\eta)\cap U} & \text{, otherwise.} \end{cases} \tag{4}$$

This metric, in some sense, resembles the metric ρ in Cantor space; in fact, $\rho = \rho_{X^*}$. Moreover, since $\rho_U(\xi,\eta) \geq \rho(\xi,\eta)$, the U-topology refines the topology of the Cantor space. In particular, every closed set in cantor space is also closed in the U-topology.

We denote by $\mathcal{C}_U(F)$ the smallest closed (with respect to ρ_U) subset of X^ω containing F. A point $\xi \in \mathcal{C}_U(F)$ is called an *isolated point of* F provided $\exists\varepsilon(\varepsilon > 0 \wedge \forall\eta(\eta \in F \wedge \eta \neq \xi \Rightarrow \rho_U(\xi,\eta) > \varepsilon))$. It should be mentioned that an arbitrary set of isolated points of X^ω is open.

A point $\xi \in \mathcal{C}_U(F)$ which is not an isolated point of F is called an *accumulation point* of F.

5.1 Theorem *Let* $U \subseteq X^*$*. Then* U^δ *is the set of accumulation points of the whole space in* (X^ω, ρ_U).

Proof. It suffices to show that every point $\xi \notin U^\delta$ is isolated. Let $\#\mathbf{A}(\xi) \cap U = n < \infty$. From the definition of ρ_U we have $\rho_U(\xi,\eta) \geq r^{1-n}$ for all $\eta \in X^\omega$. ❑

As an immediate consequence we obtain the following property of U^δ in the space (X^ω, ρ_U) which explains that the U-topology may be indeed finer than the topology of cantor space.

5.2 Corollary *If* $F \supseteq U^\delta$ *then* F *is a closed subset of* (X^ω, ρ_U).

Proof. Theorem 5.1 shows that every point $\xi \in X^\omega \setminus F$ is an isolated point of X^ω. Consequently, $X^\omega \setminus F$ is open in (X^ω, ρ_U). ❑

It should be mentioned that, although U^δ is the set of accumulation points of the whole space (X^ω, ρ_U), it may contain isolated points with respect to itself.

5.3 Example Let $U := a^* \cup a^*ba^* \subseteq \{a,b\}^*$. Then every ω-word $\xi \in a^*ba^\omega$ is an isolated point of $U^\delta = a^\omega \cup a^*ba^\omega$.

In the case of the ω-language of disjunctive sequences, D, we can prove even more. To this end we derive the following relationship between accumulation points in U-topology and in Cantor space.

5.4 Lemma *Let* $U \subseteq X^*$ *and let* $F \subseteq U^\delta$. *Then* ξ *is an accumulation point of* F *in* (X^ω, ρ_U) *if and only if* ξ *is an accumulation point of* F *in* (X^ω, ρ).

Proof. Let ξ be an accumulation point of F in (X^ω, ρ_U). Then for every $\epsilon > 0$ there is an $\eta_\epsilon \in F \setminus \{\xi\}$ such that $\rho_U(\xi, \eta_\epsilon) < \epsilon$. Since $\rho(\xi, \eta_\epsilon) \leq \rho_U(\xi, \eta_\epsilon)$, ξ is also an accumulation point of F in Cantor space. Conversely, let ξ be an accumulation point of F in Cantor space, and let $\xi \in U^\delta$. Then $\mathbf{A}(\xi) \cap U$ is infinite. Hence there is a surjective function $\psi : \mathbf{A}(\xi) \to \mathbb{N}$ such that $\psi(w) := \# \mathbf{A}(\xi) \cap U$. If for every $n \in \mathbb{N}$ there is an $\eta_n \in F$ such that $\rho(\xi, \eta_n) < r^{-n}$ then $\rho_U(\xi, \eta_n) < r^{-\psi(w_n)}$ where $w_n \in \mathbf{A}(\xi) \cap \mathbf{A}(\eta_n)$ and $|w_n| = n$. Since the function ψ is surjective, for every $\epsilon > 0$ one can find an $\eta_n \in F$ such that $\rho_U(\xi, \eta_n) < \epsilon$. Hence, ξ is also an accumulation point of F in U-topology. ❑

In Proposition 3.2 we constructed a language W_D for which $D = W_D^\delta$. The following theorem shows that D is the set of its accumulation points, that is, D is closed in (X^ω, ρ_U) and is dense in itself.

5.5 Theorem *Let* $U^\delta = D$. *In the space* (X^ω, ρ_U) *the ω-language* D *equals the set of its accumulation points.*

Proof. From Corollary 5.2 we know that D is closed in U-topology. Thus no point $\eta \notin D$ is an accumulation point of D.
On the other hand, since $w \in X^*$ and $\zeta \in D$ imply $w\zeta \in D$, every point $\xi \in D$ is an accumulation point of D in Cantor space. The assertion follows with Lemma 5.4. ❑

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