A Free–Boundary Problem in Dermal Drug Delivery

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Abstract

In this paper we study a free–boundary problem in a multicomponent domain. Our study was motivated by the mathematical modelling of dermal and transdermal drug delivery, where the multilayered skin model was considered. At the interface connecting two components the conservation of the flux and Nernst's distribution law hold and it is supposed that in any component there is a positive minimum concentration at which the diffusion front can proceed. The existence of a solution and uniqueness in special cases are shown.

Keywords: Free boundary problem, nonlinear diffusion, multicomponent domain, existence, uniqueness.
AMS Subject Classification: 35K50, 35K55.

1 Introduction

Let Ω^i, $i = 1,\dots,k$, $k \geq 2$ be bounded Lipschitz domains in $\mathbb{R}^N$, $N \geq 1$ such that for $i \in \{1,\dots,k-1\}$, $j \in \{1,\dots,k\}$, $i \neq j$,

$$\begin{aligned} \bar\Omega^i \cap \bar\Omega^j &= \emptyset \qquad \text{if } |j-i| \neq 1 \ , \\ \bar\Omega^i \cap \bar\Omega^{i+1} &= \Gamma^i \end{aligned}$$

and let Γ^i, $i \in \{1,\dots,k-1\}$ be nonempty and relatively open sets in $\partial\Omega^i$.
In this paper we study a system of singular parabolic equations

$$\partial_t b^i(u^i) = \Delta u^i \qquad \text{in} \quad \Omega^i \times (0,T) \tag{1.1}$$

for $i = 1,\dots,k$, $\partial_t = \partial/\partial t$, where $b^i(\cdot)$ represents a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ given by

$$b^i(u) \equiv \begin{cases} \beta^i(u) + \Lambda^i & u > 0 \\ [0,\Lambda^i] & u = 0 \\ 0 & u < 0 \end{cases} . \tag{1.2}$$

Here $\Lambda^i \geq 0$ are given constants and β^i are continuous monotone increasing functions, a.e. differentiable and such that

$$0 < \iota \leq (\beta^i)'(u) < \infty \qquad \text{if } u > 0 \ , \tag{1.3}$$

for a given positive constant ι, $\beta^i(0) = 0$, $i = 1,\dots,k$. For $u < 0$ we set $\beta^i(u) \equiv 0$. Equations (1.1) are endowed with the following boundary conditions

$$\nabla u^i \cdot \nu_i + \nabla u^{i-1} \cdot \nu_{i-1} = 0 \qquad \text{and} \qquad u^i = \psi^{i-1}(u^{i-1}) \tag{1.4}$$

on $\Gamma^{i-1} \times (0,T)$, $i = 2,\dots,k$, where ν_i denotes the outer unit normal to $\partial\Omega^i$ and $\psi^i(\cdot)$ are again monotone increasing functions, a.e. differentiable and

$$0 < \kappa \leq (\psi^i)' \leq K \tag{1.5}$$

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for two positive constants κ, K and $\psi^i(0) = 0$. On $\mathcal{T}^i \equiv \partial\Omega^i \setminus \Gamma^i \cup \Gamma^{i-1}$, if nonempty, we impose zero flux condition, i.e. $\nabla u^i \cdot \nu_i = 0$, where for the sake of completeness we set $\Gamma^0 = \emptyset$. On a part of $\partial\Omega^k \setminus \Gamma^{k-1}$, that we denote by Γ^k, we set $u^k = 0$. We do not exclude the case $\Gamma^k = \emptyset$, however. Moreover, the initial conditions

$$b^i(u^i) = b_0^i \qquad \text{on} \qquad \Omega^i \times \{t = 0\} \tag{1.6}$$

are prescribed, $i = 1, \ldots, k$.

Our study was motivated by the mathematical modelling of dermal and transdermal drug delivery, where the multilayered skin model in $\mathbb{R}^2$ was considered, see [10]. In this model the whole diffusion area is divided into four open subdomains Ω^i representing the vehicle and the layers of skin, connected by the interfaces Γ^i (see Figure 1 below), $i = 1, \ldots, 4$.

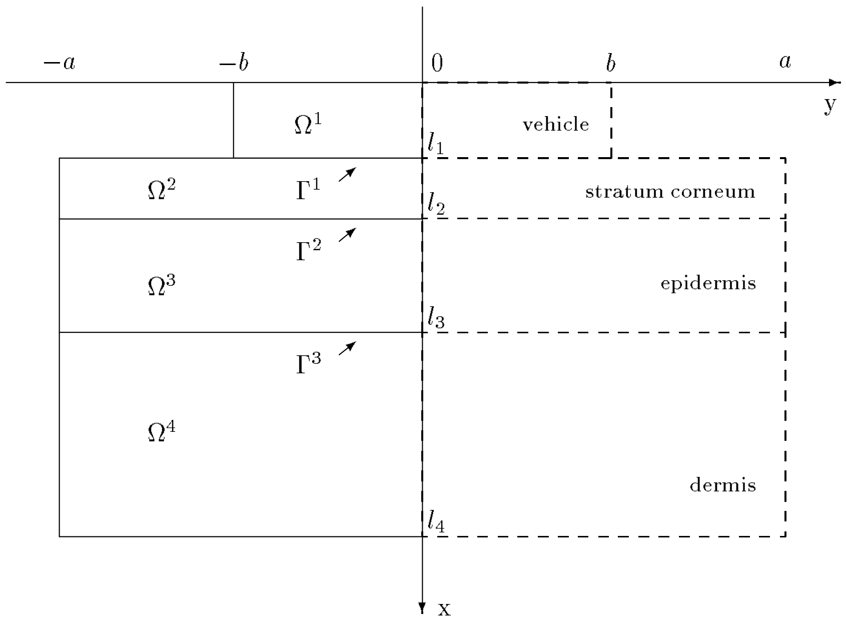

Fig. 1 – Multilayer skin model: $l_2, \ldots, l_4$ are chosen in accordance with the thickness of the skin layers; l_1 represents the application thickness and $2b$ its length. a either marks the position of the real isolating boundaries in ex vivo experiments or is chosen to be large enough such that the fluxes across $y = \pm a$ are negligible.

The drug concentration in the subdomains Ω^i is denoted by $c^i = c^i(x, y, t)$. The penetration is described by the nonlinear diffusion equation

$$\partial_t c^i = \text{div}\left(D^i(c^i)\,\nabla c^i\right) \tag{1.7}$$

which relates the change of the concentration c^i in time t to the substance flux $-D^i\,\nabla c^i$ with diffusivity D^i due to Fick's laws. At the interfaces Γ^i connecting two subdomains Ω^i and Ω^{i+1}, the conservation of flux and Nernst's distribution law must hold. These relations are given by

$$D^i\,\partial_x c^i = D^{i+1}\partial_x c^{i+1} \tag{1.8}$$

and

$$c^{i+1} = K^{i+1}\,c^i \tag{1.9}$$

with the partition coefficient being $K^{i+1} > 0$.

Since penetrantes can move into and through human skin only at a finite penetration velocity, it is supposed that a regular curve $\Phi(t)$ in $\Omega \equiv \bigcup \Omega^i$ appears separating Ω into two time–dependent subareas $\mathcal{D}_0(t)$ with zero and $\mathcal{D}(t)$ with nonzero concentration, as illustrated in Figure 2.

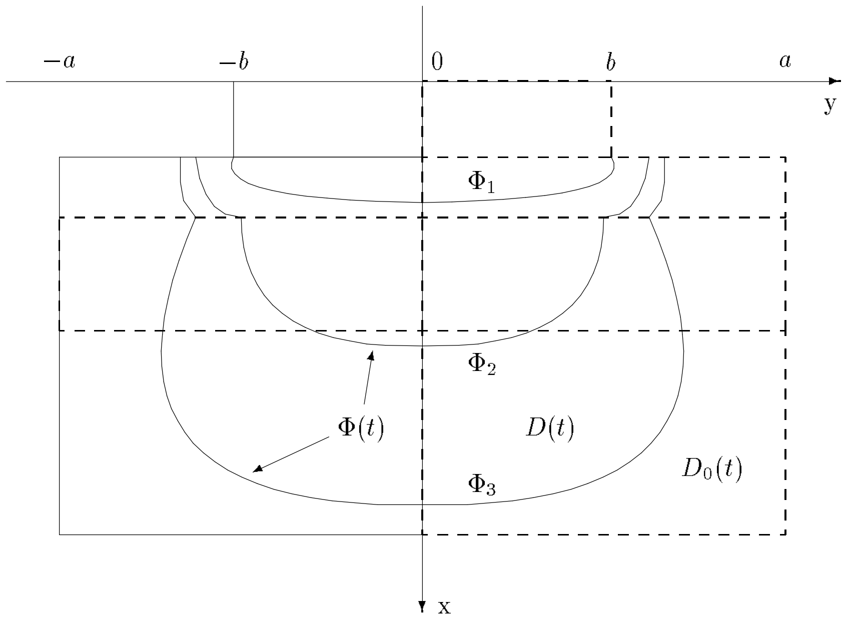

Fig. 2 – Time-dependent penetration depths. The penetration boundaries Φ_1, Φ_2, Φ_3 at three time levels $0 < t_1 < t_2 < t_3$ are shown; at the corresponding time the concentration is zero outside these boundaries.

Until $\Phi(t)$ reaches the boundary of the Ω^4, the whole mass of drug is inside of $\mathcal{D}(t)$. We require the concentration of the drug to have a fixed positive value λ at the boundary $\Phi(t)$ of $\mathcal{D}(t)$. In this context λ describes the minimum concentration at which any diffusion front can proceed. So we set

$$c^i(x,y,t) = \lambda^i \quad \text{for} \quad (x,y) \in \Phi(t) \cap \Omega^i \, . \tag{1.10}$$

Moreover, on $\Phi(t)$ we assume that the penetration velocity of the drug is proportional to the concentration gradient at $\Phi(t)$, i.e.

$$\lambda^i \, V_\nu = -D^i \, \nabla c^i \cdot \nu \, , \tag{1.11}$$

where V_ν means the normal velocity of the moving surface $\Phi(t)$ and ν represents its outer normal. To complete the problem formulation it is necessary to prescribe the initial position of the free boundary and the initial concentration distribution.

With the usual Kirchhoff transformation

$$u^i \; = \; \int_{\lambda^i}^{c^i} D^i(z) \, dz$$

denoting

$$c^i - \lambda^i = \beta^i(u^i) \, ,$$

the relations (1.7)–(1.11) above yield

$$\partial_t \beta^i(u^i) = \Delta u^i$$

in $\left(\mathcal{D}(t) \cap \Omega^i\right) \times (0,T)$,

$$\partial_x u^i = \partial_x u^{i+1} \quad \text{and} \quad \beta^{i+1}(u^{i+1}) = K^{i+1}\left(\beta^i(u^i) + \lambda^i\right) - \lambda^{i+1} \tag{1.12}$$

on $(\mathcal{D}(t)\cap\Gamma_i)\times(0,T)$,

$$\beta^i(u^i)=0 \quad \text{and} \quad \lambda^i V_\nu=-\nabla u^i\cdot\nu$$

on $\left(\Phi(t)\cap\Omega^i\right)\times(0,T)$ and

$$\beta^i(u^i)=\beta_0^i$$

on $\mathcal{D}(0)\cap\Omega^i$, $i=1,\ldots,4$.

We shall consider the weak or enthalpy formulation of the above problem, where all references to the free (unknown) boundary $\Phi(t)$ disappear and a problem on a given fixed domain Ω will be considered (see e.g. [11] and references therein). Nernst's distribution law (1.12) is, however, well defined only on $\mathcal{D}\cap\Gamma^i$, therefore we first modify this law setting

$$\beta^{i+1}(u^{i+1})=K^{i+1}\beta^i(u^i)+(K^{i+1}\lambda^i-\lambda^{i+1})\,\chi(\beta^i(u^i))\,,$$

$$\chi(\sigma)=\begin{cases}1 & \sigma>0\\ 0 & \sigma\le 0\end{cases} \tag{1.13}$$

and then, due to the noncontinuity of χ, we propose the following regularization

$$\beta^{i+1}(u^{i+1})=K^{i+1}\beta^i(u^i)+(K^{i+1}\lambda^i-\lambda^{i+1})\,\chi_\delta(\beta^i(u^i))\,,$$

where

$$\chi_\delta(\sigma)=\begin{cases}0 & \sigma\le 0\\ \sigma/\delta & 0\le\sigma\le\delta\\ 1 & 0<\delta\le\sigma\end{cases}\;. \tag{1.14}$$

Here δ is a small parameter, as far as we know, without a physical meaning and we keep it fix. In our opinion, however, this regularization may be interpreted as transition stage for small drug concentrations until Nernst's equilibrium is reached.

In this way we have arrived at Problem (1.1)–(1.6). Our original aim was to prove the existence and uniqueness for this problem. Nevertheless, uniqueness for the original problem turns out to be difficult and we are able to show uniqueness only for a problem, where regular $b^i(\cdot)$ are considered. The plan of the paper is as follows. Section 2 contains the existence and uniqueness result for the regularized problem, where the graph $b^i(\cdot)$ is replaced by a monotone Lipschitz continuous regularization $b^i_\varepsilon(\cdot)$ and boundary conditions (1.4) are regularized by the natural way as

$$\begin{aligned}&\partial_{\nu_{i-1}}u^{i-1}+n\left(\psi^{i-1}(u^{i-1})-u^i\right)=0 \quad \text{on} \quad \Gamma_T^{i-1}\,,\\ &\partial_{\nu_i}u^i+n\left(u^i-\psi^{i-1}(u^{i-1})\right)=0 \quad \text{on} \quad \Gamma_T^{i-1}\,,\end{aligned}$$

where $0<\varepsilon\ll 1$ and $1\ll n$. Here we adopt the following notations:

$$\partial_\nu u\equiv\nabla u\cdot\nu \quad \text{and} \quad \Gamma_T\equiv\Gamma\times(0,T)\;.$$

Assumptions and the statement of the existence theorem of our original Problem (1.1)–(1.6) are given in Section 3. Finally in Section 4 the uniqueness result for related nonlinear diffusion problem is discussed.

We finish this section by introducing some notation. In the sequel, if necessary, we shall consider any function $u(x)$ defined almost everywhere on $\Omega\subset\mathbb{R}^N$, $u\in H^1(\Omega)$ to be extended outside of Ω (and denoted again by u) such that $\|u\|_{H^1(\mathbb{R}^N)}\le C\|u\|_{H^1(\Omega)}$ with C independent of u. Due to the result of Calderon-Stein (see e.g. [12]) this is possible if $\partial\Omega$ is Lipschitz. By $\langle\,\cdot\,\rangle$ we denote the duality between $H^1(\Omega)$ and $H^{-1}(\Omega)$. The function spaces we use are rather familiar and we omit the definition (see e.g. [12]).

2 Auxiliary problems

Assume for this section that for given $0 < \varepsilon \ll 1$

$$b_\varepsilon^i(u) = \beta_\varepsilon^i(u) + \Lambda^i \chi_\varepsilon(u), \tag{2.1}$$

$\chi_\varepsilon(u) = \max\{0, \min\{\varepsilon^{-1}u, 1\}\}$, $\beta_\varepsilon^i(\cdot)$ are monotone increasing functions, a.e. differentiable and

$$0 < \varepsilon \le (\beta_\varepsilon^i)' \le K_\varepsilon \tag{2.2}$$

for a positive constant K_ε, $\beta_\varepsilon^i(0) = 0$, $i = 1, \dots, k$. Moreover, let

$$\beta_\varepsilon^i \longrightarrow \beta^i \qquad \text{as } \varepsilon \longrightarrow 0 \tag{2.3}$$

uniformly on compact subsets of $\mathbb{R}$ and assume that $0 < \iota \le (\beta_\varepsilon^i)' \le K_\varepsilon$ on $\mathbb{R}_+$ (c.f. (1.3)). Throughout this section we suppose

$$u_{0\varepsilon}^i \in H^1(\Omega^i) \cap L^\infty(\Omega^i), \quad u_{0\varepsilon}^i \ge 0\,, \tag{2.4}$$

$i = 1, \dots, k$ and

$$b_\varepsilon^i(u_{0\varepsilon}^i) \longrightarrow b_0^i \qquad \text{in } L^1(\Omega^i) \tag{2.5}$$

as $\varepsilon \to 0$.

Given a positive integer n, consider now the system

$$\begin{array}{ll}
\partial_t b_\varepsilon^i(u_\varepsilon^i) = \Delta u_\varepsilon^i & \text{in } \Omega_T^i\,, \\
\partial_{\nu_i} u_\varepsilon^i + n\left(u_\varepsilon^i - \psi^{i-1}(u_\varepsilon^{i-1})\right) = 0 & \text{on } \Gamma_T^{i-1}\,, \\
\partial_{\nu_i} u_\varepsilon^i + n(\psi^i(u_\varepsilon^i) - u_\varepsilon^{i+1}) = 0 & \text{on } \Gamma_T^i\,, \\
\partial_{\nu_i} u_\varepsilon^i = 0 & \text{on } \mathcal{T}_T^i\,, \\
u_\varepsilon^i = u_{0\varepsilon}^i & \text{on } \Omega^i \times \{t = 0\}\,,
\end{array} \tag{2.6}$$

for $i = 1, \dots, k$, where ψ^i satisfies (1.5), $\Gamma^0 = \emptyset$, $u^{k+1} \equiv 0$ and $\psi^k(u) \equiv u$.

Note that our intention in next section is to send $\varepsilon \to 0$ and $n \to \infty$. Throughout the rest of this section, however, ε and n are fixed and for the sake of convenience we shall not indicate the dependence of the appeared quantities on them.

Definition 2.1. *We say that $u \equiv (u^1, \dots, u^k) \in L^2(0,T; H^1(\Omega^1)) \times \ldots \times L^2(0,T; H^1(\Omega^k))$ is a weak solution of the system (2.6) provided $b^i(u^i) \in L^2(\Omega_T^i)$, $\partial_t b^i(u^i) \in L^2(0,T; H^{-1}(\Omega^i))$, that is*

$$\int_0^T \langle \partial_t b^i(u^i), v^i \rangle dt = -\int_{\Omega_T^i} (b^i(u^i) - b^i(u_0^i))\, \partial_t v^i \; dxdt \tag{2.7}$$

for any $v = (v^1, \dots, v^k) \in L^2(0,T; H^1(\Omega^1)) \times \ldots \times L^2(0,T; H^1(\Omega^k))$ with $\partial_t v^i \in L^2(\Omega_T^i)$, $v^i(T) = 0$, and if the following identity is fulfilled

$$\sum_{i=1}^k \left(\int_0^T \langle \partial_t b^i(u^i), \phi^i \rangle dt \;+\; \int_0^T \int_{\Omega^i} \nabla u^i \nabla \phi^i \; dxdt \right) \tag{2.8}$$
$$+ \sum_{i=1}^k n \int_0^T \int_{\Gamma^i} \left(u^{i+1} - \psi^i(u^i)\right)\; \left(\phi^{i+1} - \phi^i\right)(x,t)\; d\sigma dt = 0$$

for any $\phi = (\phi^1, \dots, \phi^k) \in L^2(0,T; H^1(\Omega^1)) \times \ldots \times L^2(0,T; H^1(\Omega^k))$.

A subsolution (supersolution) is defined by (2.8) with equality replaced by $\le$ ($\ge$) and $\phi \ge 0$, i.e. $\phi^i \ge 0$ on Ω^i for each $i \in \{1, \dots, k\}$.

Theorem 2.1. *A weak solution of problem (2.6) exists if conditions (2.1)–(2.5) are fulfilled. Moreover,*

$$u^i \geq 0 \qquad a.e.\ on \quad \Omega_T^i\ . \tag{2.9}$$

In addition, if

$$\left\| u_0^1 \right\|_{L^\infty(\Omega^1)} \leq c^1 \quad \text{ and } \quad \left\| u_0^i \right\|_{L^\infty(\Omega^i)} \leq c^i\ , \qquad c^i \equiv \psi^{i-1}(c^{i-1})$$

for $i = 2, \ldots, k,$ *then*

$$\left\| u^i \right\|_{L^\infty(\Omega_T^i)} \leq c^i \tag{2.10}$$

for each $i = 1, \ldots, k.$

Let us postpone the proof of Theorem 2.1 to the end of this section.

Theorem 2.2. *(i) Let* u *and* v *be weak solutions of the problem (2.6) with initial functions* u_0 *and* v_0*, respectively. Then for almost all* $t \in [0, T]$*,*

$$\sum_{i=1}^{k} \int_{\Omega^i} \left| b^i(u^i) - b^i(v^i) \right| (x,t)\, dx \ \leq\ \sum_{i=1}^{k} \int_{\Omega^i} \left| b^i(u_0^i) - b^i(v_0^i) \right| (x)\, dx\ . \tag{2.11}$$

(ii) Let u *be a subsolution and* v *a supersolution of the problem (2.6) with initial data* u_0 *and* v_0*, respectively. Then if* $u_0 \leq v_0$*, i.e.* $u_0^i \leq v_0^i$ *a.e. on* Ω^i *for any* $i \in \{1, \ldots, k\}$ *it follows that*

$$u^i(x,t) \leq v^i(x,t)$$

a.e. on Ω_T^i *for any* $i \in \{1, \ldots, k\}$.

Proof: We first prove the comparison principle *(ii)*. Let $0 < t \leq T$ be fixed. Take $\phi^i(x,\tau) = \chi_{[0,t]}(\tau)\varphi^i(x, t-\tau)$, where the choice of appropriate functions $\varphi^i = \varphi^i(x,s)$ will be determined later and χ approaches the characteristic function of the interval $[0,t]$. Recalling the definition of u, v one can see that

$$\begin{aligned}
&\int_{\Omega^i} \left(b^i(u^i) - b^i(v^i)\right)(x,t)\ \varphi^i(x,0)\ dx \\
&+ \int_0^t \int_{\Omega^i} \left(A^i(x,\tau)\ \partial_s \varphi^i(x,t-\tau)\ (u^i - v^i)(x,\tau) + \nabla\varphi^i(x,t-\tau)\ \nabla(u^i(x,\tau) - v^i(x,\tau))\right)\ dx d\tau \\
&+ n \int_0^t \int_{\Gamma^{i-1}} (u^i - v^i)(x,\tau)\ \varphi^i(x,t-\tau)\ d\sigma d\tau +\ n \int_0^t \int_{\Gamma^i} c^i(x,\tau)\ (u^i - v^i)(x,\tau)\ \varphi^i(x,t-\tau)\ d\sigma d\tau \\
&\leq \int_{\Omega^i} \left(b^i(u_0^i) - b^i(v_0^i)\right) \varphi^i(x,t)\ dx \\
&+ n \int_0^t \int_{\Gamma^{i-1}} \left(\psi^{i-1}(u^{i-1}) - \psi^{i-1}(v^{i-1})\right)(x,\tau)\ \varphi^i(x,t-\tau)\ d\sigma d\tau \\
&+ n \int_0^t \int_{\Gamma^i} (u^{i+1} - v^{i+1})(x,\tau)\ \varphi^i(x,t-\tau)\ d\sigma d\tau\ ,
\end{aligned} \tag{2.12}$$

where

$$A^i(x,\tau) \equiv \frac{b^i(u^i) - b^i(v^i)}{u^i - v^i}(x,\tau)$$

and

$$c^i(x,\tau) \equiv \frac{\psi^i(u^i) - \psi^i(v^i)}{u^i - v^i}(x,\tau)\ .$$

From the assumed hypotheses one can see that $\varepsilon \le A^i \le K_\varepsilon$ and $0 < \kappa \le c^i \le K$. Formally, if we insert $\varphi = \varphi(x, t-\tau)$ into (2.12), $0 \le \varphi^i \le 1$ such that $\varphi^i(x,s)$, $s \in [0,t]$, satisfies

$$
\begin{aligned}
& A^i(x,t-s)\,\partial_s \varphi^i - \Delta \varphi^i = 0 && \text{in } \Omega^i_t,\\
& \partial_{\nu_i}\varphi^i + n\,\left(\varphi^i - \chi((u^i - v^i)(x,t-s))\right) = 0 && \text{on } \Gamma^{i-1}_t,\\
& \partial_{\nu_i}\varphi^i + n\,c^i(x,t-s)\left(\varphi^i - \chi((u^i - v^i)(x,t-s))\right) = 0 && \text{on } \Gamma^{i}_t,\\
& \partial_{\nu_i}\varphi^i = 0 && \text{on } \mathcal{T}^i_t,\\
& \varphi^i = \chi(u^i(x,t) - v^i(x,t)) && \text{on } \Omega^i \times \{s=0\},
\end{aligned}
\tag{2.13}
$$

χ being given by (1.13), we get

$$
\begin{aligned}
&\int_{\Omega^i} \left(b^i(u^i) - b^i(v^i)\right)_+(x,t)\,dx \\
&+ n\int_0^t\int_{\Gamma^{i-1}} (u^i - v^i)_+(x,\tau)\,d\sigma d\tau + n\int_0^t\int_{\Gamma^i}\left(\psi^i(u^i) - \psi^i(v^i)\right)_+(x,\tau)\,d\sigma d\tau \\
&\le \int_{\Omega^i}\left(b^i(u^i_0) - b^i(v^i_0)\right)_+(x)\,dx \\
&+ n\int_0^t\int_{\Gamma^{i-1}}\left(\psi^{i-1}(u^{i-1}) - \psi^{i-1}(v^{i-1})\right)_+(x,\tau)\,d\sigma d\tau + n\int_0^t\int_{\Gamma^i}\left(u^{i+1} - v^{i+1}\right)_+(x,\tau)\,d\sigma d\tau
\end{aligned}
\tag{2.14}
$$

where

$$w_+ \equiv \max\{0, w\}\ .$$

If we add (2.14) through $i = 1, \dots, k$, we arrive at

$$\sum_{i=1}^k \int_{\Omega^i}\left(b^i(u^i) - b^i(v^i)\right)_+(x,t)\,dx \le \sum_{i=1}^k \int_{\Omega^i}\left(b^i(u^i_0) - b^i(v^i_0)\right)_+(x)\,dx \tag{2.15}$$

recalling that we have put $\Gamma^0 = \emptyset$, $u^{k+1} = v^{k+1} = 0$ and $\psi^k(u) \equiv u$.

Nevertheless, coefficients in (2.13) are only bounded functions and we can not therefore expect the required regularity of solutions to (2.13). Therefore, instead of (2.13) we shall consider the problem

$$
\begin{aligned}
& A^i_\rho(x,t-s)\,\partial_s \varphi^i - \Delta \varphi^i = 0 && \text{in } \Omega^i_t,\\
& \partial_{\nu_i}\varphi^i + n\,\left(\varphi^i - \omega^i_\delta(x,t-s)\right) = 0 && \text{on } \Gamma^{i-1}_t,\\
& \partial_{\nu_i}\varphi^i + n\,c^i_\delta(x,t-s)\left(\varphi^i - \omega^i_\delta(x,t-s)\right) = 0 && \text{on } \Gamma^{i}_t,\\
& \partial_{\nu_i}\varphi^i = 0 && \text{on } \mathcal{T}^i_t,\\
& \varphi^i = \varpi^i_\eta(x) && \text{on } \Omega^i \times \{s=0\},
\end{aligned}
\tag{2.16}
$$

where

$$
\begin{gathered}
A^i_\rho(x,t-s) \equiv (R_\rho * A^i)(x,t-s), \quad \omega^i_\delta(x,t-s) \equiv \left(R_\delta * \chi(u^i - v^i)\right)(x,t-s)\\
c^i_\delta(x,t-s) \equiv \left(R_\delta * c^i\right)(x,t-s),
\end{gathered}
$$

R_ρ, R_δ are the standard mollifier in the t-variable and

$$\varpi^i_\eta(x) \equiv \left(R_\eta * \chi(u^i(\cdot,t) - v^i(\cdot,t))\right)(x)\ ,$$

R_η is the standard mollifier in the $x-$variable.

(2.16) is uniformly parabolic problem with sufficiently smooth data and we can apply the classical results of [9] to get a unique weak solution φ^i such that

$$\varphi^i \equiv \varphi^i_{\rho\delta\eta} \in L^\infty(0,t;H^1(\Omega^i)) \cap H^1(0,t;L^2(\Omega^i))$$

and

$$0 \le \varphi^i \le 1 \qquad \text{a.e on } \Omega^i_t\ . \tag{2.17}$$

To prove (2.17), note that the weak formulation of (2.16) easily gives

$$\int_0^t \int_{\Omega^i} \left(A^i_\rho \partial_s(\varphi^i - 1)\,\xi + \nabla(\varphi^i - 1)\nabla\xi\right)\,dxds \;+\; n\int_0^t \int_{\Gamma^{i-1}} (\varphi^i - 1)\xi\,d\sigma ds$$
$$+\; n\int_0^t \int_{\Gamma^i} c^i_\delta\,(\varphi^i - 1)\,\xi\,d\sigma ds \;\le\; 0$$

for any $\xi \in L^2(0,T;H^1(\Omega^i))$, $\xi \ge 0$. If we insert $\xi = \chi_\varepsilon(\varphi^i - 1)$ and if $\varepsilon \to 0$, we arrive at

$$\int_0^t \int_{\Omega^i} \partial_s\left(A^i_\rho\,(\varphi^i - 1)_+\right)\,dxds \;\le\; \int_0^t \int_{\Omega^i} \left|\partial_s A^i_\rho\right|(\varphi^i - 1)_+\,dxds\;.$$

This, due to Gronwall's lemma, yields

$$\int_{\Omega^i} (\varphi^i - 1)_+(x,t)\,dx \le 0\;, \qquad \text{i.e.}\;\; (2.17)_2\;.$$

Analogously one gets

$$\int_{\Omega^i} (-\varphi^i)_+(x,t)\,dx \le 0\;, \qquad \text{i.e.}\;\; (2.17)_1\;.$$

Moreover, it is easy to see (see e.g. [9]), that

$$\max_{0\le s\le t} \int_{\Omega^i} \left|\nabla\varphi^i\right|^2(x,s)\,dx + \int_0^t \int_{\Omega^i} A^i_\rho(x,t-s)\left|\partial_s\varphi^i\right|^2 dxds \le C, \tag{2.18}$$

where the positive constant $C = C(\eta,\delta)$ does not depend on ρ.

Inserting now $\varphi^i_{\rho\delta\eta}$ into (2.12) we get

$$\int_{\Omega^i} \left(b^i(u^i) - b^i(v^i)\right)(x,t)\,\varpi^i_\eta(x)\,dx + \int_0^t \int_{\Omega^i} \left(A^i - A^i_\rho\right)(x,\tau)\,\partial_s\varphi^i_{\rho\delta\eta}(x,t-\tau)\,(u^i - v^i)(x,\tau)\,dxd\tau$$
$$+\;n\int_0^t \int_{\Gamma^{i-1}} (u^i - v^i)(x,\tau)\,\omega^i_\delta(x,\tau)\,d\sigma d\tau$$
$$+\;n\int_0^t \int_{\Gamma^i} (c^i - c^i_\delta)(x,\tau)\,\varphi^i_{\rho\delta\eta}(x,t-\tau)\,(u^i - v^i)(x,\tau)\,d\sigma d\tau$$
$$+\;n\int_0^t \int_{\Gamma^i} c^i_\delta(x,\tau)\,\omega^i_\delta(x,t-\tau)\,(u^i - v^i)(x,\tau)\,d\sigma d\tau$$
$$\le\; \int_{\Omega^i} \left(b^i(u^i_0) - b^i(v^i_0)\right)_+\,dx$$
$$+\;n\int_0^t \int_{\Gamma^{i-1}} \left(\psi^{i-1}(u^{i-1}) - \psi^{i-1}(v^{i-1})\right)_+(x,\tau)\,d\sigma d\tau + \;n\int_0^t \int_{\Gamma^i} \left(u^{i+1} - v^{i+1}\right)_+(x,\tau)\,d\sigma d\tau\;.$$

First we go with $\rho \to 0$, then with $\delta \to 0$ and afterwards we let $\eta \to 0$. Hence we arrive at (2.15) and the desired comparison principle follows.

To prove assertion *(i)*, note that u, v are both, sub– and supersolutions, and therefore one analogously gets

$$\sum_{i=1}^k \int_{\Omega^i} \left(b^i(v^i) - b^i(u^i)\right)_+(x,t)\,dx \le \sum_{i=1}^k \int_{\Omega^i} \left(b^i(v^i_0) - b^i(u^i_0)\right)_+(x)\,dx\;.$$

This proves *(i)*. □

Proof: (*of Theorem 2.1*) 1. We intend to build a weak solution of the system (2.6) by first constructing solutions of certain simpler approximations to (2.6) and then passing to limits. More precisely, fix now a positive integer ℓ. For given

$$u_{\ell-1} = (u^1_{\ell-1}, \ldots, u^k_{\ell-1})\;, \quad u^i_0(x,t) \equiv u^i_0(x)\;,$$

we will look for a function $u_\ell \equiv (u_\ell^1, \dots, u_\ell^k)$ so that

$$\begin{aligned}
&\partial_t b^i(u_\ell^i) = \Delta u_\ell^i && \text{in } \ \Omega_T^i \ , \\
&\partial_{\nu_i} u_\ell^i + n\left(u_\ell^i - \psi^{i-1}(u_\ell^{i-1})\right) = 0 && \text{on } \ \Gamma_T^{i-1} \ , \\
&\partial_{\nu_i} u_\ell^i + n\left(\psi^i(u_\ell^i) - u_{\ell-1}^{i+1}\right) = 0 && \text{on } \ \Gamma_T^{i} \ , \\
&\partial_{\nu_i} u_\ell^i = 0 && \text{on } \ \mathcal{T}_T^i \ , \\
&u_\ell^i = u_0^i && \text{on } \ \Omega^i \times \{t=0\} \ ,
\end{aligned} \tag{2.19}$$

$i = 1, \dots, k$. Recall again, that $\Gamma^0 \equiv \emptyset$, $\psi^k(u) \equiv u$ and $u^{k+1} \equiv 0$.
Assume the nonnegative functions $u_{\ell-1}^i$ are sufficiently smooth, say,

$$u_{\ell-1}^i \in L^2(0,T;H^1(\Omega^i)) \cap L^\infty(\Omega_T^i)$$

for any $i \in \{1, \dots, k\}$ and let a positive constant c^1 be taken in such a way that

$$0 \le u_{\ell-1}^1 \le c^1$$

a.e. on Ω_T^1 and at the same time

$$0 \le u_{\ell-1}^i \le c^i \equiv \psi^{i-1}(c^{i-1}) \tag{2.20}$$

a.e. on Ω_T^i for each $i \in \{2, \dots, k\}$.
2. According to standard existence theory (Galerkin's method or implicit time discretization method), see e.g. [5] for existence and [6] for uniqueness, there exists a unique weak solution u_ℓ^1 satisfying (2.19) and then repeatedly u_ℓ^i for $i = 2, \dots, k$ such that

$$u_\ell^i \in L^2(0,T;H^1(\Omega^i)) \quad \text{and} \quad \partial_t b^i(u_\ell^i) \in L^2(0,T;H^{-1}(\Omega^i)) \ .$$

3. Due to the comparison principle we show now that

$$(2.20) \qquad \text{implies} \qquad 0 \le u_\ell^i \le c^i$$

for any $i \in \{1, \dots, k\}$. First, observe formally, that

$$\begin{aligned}
&\partial_t b^i(u_\ell^i) = \Delta u_\ell^i && \text{in } \ \Omega_T^i \ , \\
&\partial_{\nu_i} u_\ell^i + n u_\ell^i = n\psi^{i-1}(u_\ell^{i-1}) \ge 0 && \text{on } \ \Gamma_T^{i-1} \ , \\
&\partial_{\nu_i} u_\ell^i + n\psi^i(u_\ell^i) = n u_{\ell-1}^{i+1} \ge 0 && \text{on } \ \Gamma_T^{i} \ , \\
&\partial_{\nu_i} u_\ell^i = 0 && \text{on } \ \mathcal{T}_T^i \ , \\
&u_\ell^i = u_0^i \ge 0 && \text{on } \ \ \Omega^i \times \{t=0\} \ ,
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t \left(b^i(u_\ell^i) - b^i(c^i)\right) = \Delta\left(u_\ell^i - c^i\right) && \text{in } \ \Omega_T^i, \\
&\partial_{\nu_i}(u_\ell^i - c^i) + n\left(u_\ell^i - c^i\right) = n\left(\psi^{i-1}(u_\ell^{i-1}) - \psi^{i-1}(c^{i-1})\right) \le 0 && \text{on } \ \Gamma_T^{i-1}, \\
&\partial_{\nu_i}(u_\ell^i - c^i) + n\left(\psi^i(u_\ell^i) - \psi^i(c^i)\right) = n\left(u_{\ell-1}^{i+1} - c^{i+1}\right) \le 0 && \text{on } \ \Gamma_T^{i}, \\
&\partial_{\nu_i}(u_\ell^i - c^i) = 0 && \text{on } \ \mathcal{T}_T^i \ , \\
&u_\ell^i - c^i \le 0 && \text{on } \ \Omega^i \times \{t=0\} \ .
\end{aligned}$$

Consequently,

$$0 \le u_\ell^i \le c^i \tag{2.21}$$

a.e. on Ω_T^i for each $i \in \{1, \dots, k\}$. It is not difficult to make it precise as in the proof of Theorem 2.2 (ii) above and we omit further details.

4. We propose now to send ℓ to infinity and to show a subsequence of our solutions u_ℓ of the approximate problems (2.19) converges to a weak solution of (2.6). For this we will need some uniform estimates. First we obtain an a priori estimate by testing (2.19) with u^i_ℓ. Denoting

$$\Phi^i(z) \equiv \int_0^z b^i(\xi)d\xi \quad \text{and} \quad B^i(z) \equiv b^i(z)z - \Phi^i(z), \tag{2.22}$$

the parabolic term can be treated with the use of [1, Lemma 1.5]. Performing some calculations we find

$$\begin{aligned}\sum_{i=1}^{k}\Big\{\int_{\Omega^i} B^i(u^i_\ell(x,t))\,dx - \int_{\Omega^i} B^i(u^i_0(x))\,dx + \int_0^t\int_{\Omega^i}\left|\nabla u^i_\ell\right|^2 dx d\tau \\ + n\int_0^t\int_{\Gamma^i}\left(\psi^i(u^i_\ell)u^i_\ell + |u^{i+1}_\ell|^2\right) d\sigma d\tau\Big\} \\ = \sum_{i=1}^{k-1} n\int_0^t\int_{\Gamma^i}\left(\psi^i(u^i_\ell)u^{i+1}_\ell + u^{i+1}_{\ell-1}u^i_\ell\right) d\sigma d\tau \;.\end{aligned}$$

Since $u^i_\ell \in L^2(0,T;H^1(\Omega^i))$ the estimate (2.21) holds on Γ^i_T, too. Hence, the right-hand side is uniformly bounded w.r.t. ℓ and we arrive at

$$\sum_{i=1}^{k}\Big\{\int_{\Omega^i} B^i(u^i_\ell(x,t))\,dx + \left\|\nabla u^i_\ell\right\|^2_{L^2(\Omega^i_t)}\Big\} \le C_n \tag{2.23}$$

for any ℓ and for a.e. $t \in [0,T]$. In order to go to the limit in the nonlinear terms we need strong convergence. To arrive at an estimate concerning the time variable we test our problem with $\varphi^i = \chi_{[t,t+h]}(\tau)w^i$ for $w^i \in H^1(\Omega^i)$, $i = 1,\dots,k$, $w^{k+1} \equiv 0$,

$$\begin{aligned}\sum_{i=1}^{k}\Big\{\int_{\Omega^i}\left(b^i(u^i_\ell(t+h)) - b^i_\ell(u^i_\ell(t))\right)w^i dx + \int_t^{t+h}\int_{\Omega^i}\nabla u^i\nabla w^i\; dx d\tau \\ + n\int_t^{t+h}\int_{\Gamma^i}\Big((u^{i+1}_\ell - \psi^i(u^i_\ell))w^{i+1} + (\psi^i(u^i_\ell) - u^{i+1}_{\ell-1})w^i\Big)d\sigma d\tau\Big\} \;=\; 0\;,\end{aligned}$$

and put then $w^i \equiv u^i_\ell(t+h) - u^i_\ell(t)$. Since the items on Γ^i are bounded almost everywhere, after integration over $[0, T-h]$ one gets the estimate

$$\begin{aligned}\sum_{i=1}^{k}\int_0^{T-h}\int_{\Omega^i}\left(b^i(u^i_\ell(t+h)) - b^i(u^i_\ell(t))\right)\left(u^i_\ell(t+h) - u^i_\ell(t)\right)\; dx dt \\ \le c_1 h\sum_{i=1}^{k}\int_0^T \left\|[\nabla u^i_\ell]_h\right\|_{L^2(\Omega^i)}\left\|\nabla u^i_\ell\right\|_{L^2(\Omega^i)} dt \;+\; c_2 h \;\le\; ch\end{aligned} \tag{2.24}$$

with the constant c independent of ℓ. In (2.24) $[\cdot]_h$ denotes the Steklov average, i.e.

$$[f]_h \equiv \frac{1}{h}\int_t^{t+h} f(\tau)d\tau$$

and the estimate (2.23) was applied. Now, passing to the limit $\ell \to \infty$, the estimates (2.23) and (2.24) together with the weak formulation of problem (2.19) yield existence of a subsequence of $\{u^i_\ell\}$ such that

$$\begin{aligned} b^i(u^i_\ell) &\to b^i(u^i) && \text{in} \quad L^1(\Omega^i_T)\,, \\ u^i_\ell &\rightharpoonup u^i && \text{in} \quad L^2(0,T;H^1(\Omega^i))\,, \\ \partial_t b(u^i_\ell) &\rightharpoonup \partial_t b(u^i) && \text{in} \quad L^2(0,T;H^{-1}(\Omega^i))\end{aligned}$$

for $i = 1,\dots,k$ (cf. [1]). Note, that due to (2.1) and (2.2), u^i_ℓ converges strongly in $L^1(\Omega^i_T)$ and thus, by interpolation w.r.t. space variables also strongly in $L^2(\Gamma^i_T)$. Now, it is not difficult to see that u^i, $i = 1,\dots,k$, is a weak solution of (2.6). □

3 Free–boundary value problems

In this section we intend to prove existence of a solution to our original problem (1.1)–(1.6) with noncontinuous $b(u)$ by approximating it by problem (2.6). Since we let ε go to zero and n to infinity in (2.6) our main task is to derive a priori estimates that do not depend on ε, n. Recall our convention $u_\varepsilon^{k+1} \equiv 0$, $\psi^k(u) \equiv u$.

To this end we want to test our system (2.6) with

$$\phi^i(u) = \psi^k \circ \ldots \circ \psi^i(u), \qquad i = 1, \ldots, k.$$

Since $\partial_t b^i(u_\varepsilon^i)$ only belongs to the dual $L^2(0,T;H^{-1}(\Omega^i))$ we have to formulate a "formula of partial integration" for that case. Define

$$G_\varepsilon^i(s) = \int_0^s \phi^i((b_\varepsilon^i)^{-1}(r))\, dr\ , \tag{3.1}$$

then we get (cf. Carrillo [2])

Lemma 3.1. *Let $\phi \in C^{0,1}(\mathbb{R})$ be monotone, let $b_0 = b(u_0) \in L^1(\Omega)$ such that $G(b_0) \in L^1(\Omega)$, let $u \in L^2(0,T;H^1(\Omega))$ and $b(u) \in L^1(Q_T)$ with derivative $\partial_t b(u) \in L^2(0,T;H^{-1}(\Omega))$. Then*

$$G(b(u)) \in L^\infty(0,T;L^1(\Omega))$$

and, for almost every $t \in [0,T]$,

$$\int_\Omega G\big(b(u(x,t))\big)\, dx - \int_\Omega G\big(b_0(x)\big)\, dx = \int_0^t \langle b(u)_t, \phi(u)\rangle\, d\tau.$$

Proof: The lemma is an adaption of [2, Lemma 4] to our problem. □

In view of our assumptions due to (1.5) we may estimate G_ε^i below by

$$G_\varepsilon^i(b_\varepsilon^i(u)) \geq \kappa^{k-i} \int_0^{b_\varepsilon^i(u)} (b_\varepsilon^i)^{-1}(r)\, dr = \kappa^{k-i} \int_0^u z\, db_\varepsilon^i(z) = \kappa^{k-i} B_\varepsilon^i(u), \tag{3.2}$$

where $B_\varepsilon^i(u)$ is defined by (2.22), and above by

$$G_\varepsilon^i(b_{0\varepsilon}^i) \leq K^{k-i} \int_0^{b_{0\varepsilon}^i} (b_\varepsilon^i)^{-1}(r)\, dr \leq \frac{K^{k-i}}{\iota} \int_0^{b_{0\varepsilon}^i} r\, dr = \frac{K^{k-i}}{2\iota} |b_{0\varepsilon}^i|^2 \tag{3.3}$$

for nonnegative $b_{0\varepsilon}^i$ since $(b_\varepsilon^i)^{-1}(r) \leq \frac{1}{\iota} r$ for $r \geq 0$.

Now we test problem (2.6) with the above mentioned $\phi^i(u_\varepsilon^i)$. According to (1.5), for $i = 1, \ldots, k-1$,

$$\int_0^t \int_{\Gamma^i} (u_\varepsilon^{i+1} - \psi^i(u_\varepsilon^i))\, (\phi^{i+1}(u_\varepsilon^{i+1}) - \phi^i(u_\varepsilon^i))\, d\sigma d\tau \ \geq\ \kappa^{k-1-i} \int_0^t \int_{\Gamma^i} (u_\varepsilon^{i+1} - \psi^i(u_\varepsilon^i))^2\, d\sigma d\tau\, .$$

Consequently,

$$\sum_{i=1}^k \left\{ \int_{\Omega^i} \left(G_\varepsilon^i(b_\varepsilon^i(u_\varepsilon^i(t))) - G_\varepsilon^i(b_\varepsilon^i(u_{0\varepsilon}^i)) \right) dx + \int_0^t \int_{\Omega^i} D\phi^i(u_\varepsilon^i) \left|\nabla u_\varepsilon^i\right|^2 dx d\tau \right\}$$
$$+\ n \sum_{i=1}^k \kappa^{(k-1-i)_+} \int_0^t \int_{\Gamma^i} \left(u_\varepsilon^{i+1} - \psi^i(u_\varepsilon^i)\right)^2 d\sigma d\tau\ \leq\ 0\, .$$

For bounded initial data $u_{0\varepsilon}^i$ the function $G_\varepsilon^i(b_\varepsilon^i(u_{0\varepsilon}^i))$ is bounded in Ω^i uniformly with respect to ε, too. Thus, regarding (3.2),(3.3), and (1.5) again, we have proved the following a priori estimates:

Lemma 3.2. *Suppose assumptions (2.1)–(2.5) and let $b^i_{0\varepsilon} = b^i_\varepsilon(u^i_{0\varepsilon})$ be uniformly bounded in $L^2(\Omega^i)$ for $i = 1, \dots, k$. Then the solution u_ε from Theorem 2.1 fulfills the estimates*

$$\int_{\Omega^i} B^i_\varepsilon(u^i_\varepsilon(x,t))\,dx \leq C_1 \tag{3.4}$$

$$\|\nabla u^i_\varepsilon\|_{L^2(\Omega^i_T)} \leq C_2 \tag{3.5}$$

$$n\,\|u^{i+1}_\varepsilon - \psi^i(u^i_\varepsilon)\|^2_{L^2(\Gamma^i_T)} \leq C_3 \tag{3.6}$$

for all $\varepsilon > 0$ and $i = 1, \dots, k$.

In order to obtain strong convergence of u^i_ε as $\varepsilon \to 0$ we need an a priori estimate with respect to the time variable (cf. [1, Lemma 1.9]).

Lemma 3.3. *Let the assumptions of Lemma 3.2 be fulfilled and $0 < h < T$. Then the estimate*

$$\int_0^{T-h}\int_{\Omega^i} \left(b^i_\varepsilon(u^i_\varepsilon(x,t+h)) - b^i_\varepsilon(u^i_\varepsilon(x,t))\right)\left(u^i_\varepsilon(x,t+h) - u^i_\varepsilon(x,t)\right)\,dxdt \leq Ch \tag{3.7}$$

holds for all $\varepsilon > 0$ and $i = 1, \dots, k$.

Proof: Inserting $\chi_{(t,t+h)}w^i$, $w^i \in V_i$, as a test function into (2.8) we obtain by means of partial integration in time

$$\sum_{i=1}^k \left(\int_{\Omega^i} \left(b^i_\varepsilon(u^i_\varepsilon(t+h)) - b^i_\varepsilon(u^i_\varepsilon(t))\right) w^i\,dx + \int_t^{t+h}\int_{\Omega^i} \nabla u^i_\varepsilon \nabla w^i\,dxd\tau \right) =$$
$$- \sum_{i=1}^k n \int_t^{t+h}\int_{\Gamma^i} \left(u^{i+1}_\varepsilon - \psi^i(u^i_\varepsilon)\right)\left(w^{i+1} - w^i\right)(x,\tau)\,d\sigma d\tau\ .$$

Now we choose $w^i = \phi^i(u^i_\varepsilon(t+h)) - \phi^i(u^i_\varepsilon(t))$, $(w^{k+1} \equiv 0)$, integrate over $t \in [0, T-h]$, and estimate by means of (1.5),

$$\begin{aligned}
&\sum_{i=1}^k \kappa^{k-i} \int_0^{T-h}\int_{\Omega^i} \left(b^i_\varepsilon(u^i_\varepsilon(t+h)) - b^i_\varepsilon(u^i_\varepsilon(t))\right)\left(u^i_\varepsilon(t+h) - u^i_\varepsilon(t)\right)\,dxdt \\
&\quad \leq h \sum_{i=1}^k K^{k-i} \int_0^{T-h}\int_{\Omega^i} \left|\nabla[u^i_\varepsilon]_h\right| \left(|\nabla u^i_\varepsilon(t+h)| + |\nabla u^i_\varepsilon(t)|\right)\,dxdt \\
&\quad + nh \sum_{i=1}^k K^{(k-i-1)_+} \int_0^{T-h}\int_{\Gamma^i} \left|[u^{i+1}_\varepsilon - \psi^i(u^i_\varepsilon)]_h(x,t)\right| \\
&\qquad \times \Big(|u^{i+1}_\varepsilon(t+h) - \psi^i(u^i_\varepsilon(t+h))| + |u^{i+1}_\varepsilon(t)) - \psi^i(u^i_\varepsilon(t))|\Big)\,d\sigma dt,
\end{aligned}$$

where $[u]_h(t) = h^{-1}\int_t^{t+h} u(\tau)\,d\tau$ denotes the Steklov average of $u(t)$. Observe that if X is a normed space, then the Steklov average has the property

$$\|\,[u]_h\,\|_{L^p(0,T;X)} \leq \|u\|_{L^p(0,T;X)}.$$

Hence, we finish our calculation using the a priori estimates of Lemma 3.2,

$$\begin{aligned}
&\sum_{i=1}^k \kappa^{k-i} \int_0^{T-h}\int_{\Omega^i} \left(b^i_\varepsilon(u^i_\varepsilon(t+h)) - b^i_\varepsilon(u^i_\varepsilon(t))\right)\left(u^i_\varepsilon(t+h) - u^i_\varepsilon(t)\right)\,dxdt \\
&\quad \leq\ h \sum_{i=1}^k K^{k-i}\, \|\nabla u^i_\varepsilon\|^2_{L^2(\Omega^i_T)} + nh \sum_{i=1}^k K^{(k-i-1)_+}\, \|u^{i+1}_\varepsilon - \psi^i(u^i_\varepsilon)\|^2_{L^2(\Gamma^i_T)} \\
&\quad \leq\ Ch,
\end{aligned}$$

which proves (3.7). $\square$

Now we are in the position to formulate the existence result for our original problem (1.1)–(1.6). For this end we give the exact definition of a weak solution of this problem. Here we use the notation $V = \{v = (v^1, \dots, v^k) \in V_1 \times \dots \times V_k : v^{i+1} = v^i$ a.e. on Γ_i for $i = 1, \dots, k-1,\ v^k = 0$ a.e. on $\Gamma_k\}$, $V_i \equiv H^1(\Omega^i)$.

Definition 3.1. *A function $u \equiv (u^1, \dots, u^k) \in L^2(0,T;V_1) \times \dots \times L^2(0,T;V_k)$ is a weak solution of problem (1.1)–(1.6) provided there is a function $w \in b(u)$ with $w^i \in L^2(\Omega^i_T)$, $\partial_t w \in L^2(0,T;V^*)$, and initial values b^i_0 in the sense that*

$$\int_0^T \langle \partial_t w, v\rangle\, dt = -\sum_{i=1}^k \int_0^T \int_{\Omega^i} (w^i - b^i_0)\, \partial_t v^i \; dxdt \tag{3.8}$$

holds for any $v = (v^1, \dots, v^k) \in L^2(0,T;V)$ with $\partial_t v^i \in L^\infty(\Omega^i_T)$, $v^i(T) = 0$, the following identity

$$\int_0^T \langle \partial_t w, \phi\rangle\, dt \; + \sum_{i=1}^k \int_0^T \int_{\Omega^i} \nabla u^i \nabla \phi^i \; dxdt = 0 \tag{3.9}$$

is fulfilled for any $\phi \in L^2(0,T;V)$, and

$$u^{i+1} = \psi^i(u^i) \qquad a.e. \;\; on \; \Gamma^i_T. \tag{3.10}$$

Theorem 3.1. *Let $b^i(v)$, $\psi^i(v)$, $i = 1, \dots, k$, be as defined in Section 1 fulfilling assumptions (1.2), (1.3) and (1.5). Then for given nonnegative $b^i_0 \in L^2(\Omega^i)$ there is a nonnegative weak solution u of the system (1.1),(1.4),(1.6) in the sense of Definition 3.1.*

If, moreover, $0 \le b^i_0 \le C^i$ for $i = 1, \dots, k$ where $c^i = (b^i)^{-1}(C^i)$ fulfills $c^{i+1} = \psi^i(c^i)$, then the solution is bounded a.e. on Ω_T by $0 \le u^i \le c^i$.

Proof: We approximate our problem (1.1)–(1.6) by the regularized problem (2.6). As initial condition we choose $b^i_\varepsilon(u^i_{0\varepsilon}) = b^i_{0\varepsilon} := R_\varepsilon * b^i_0$, where R_ε is the standard mollifier with respect to x. For $b^i_0 \ge 0$ the regularization is nonnegative again, thus also $u^i_{0\varepsilon}$ will be nonnegative for all $i = 1, \dots, k$. Since $(b^i_\varepsilon)^{-1}$ is Lipschitz continuous we have $u^i_{0\varepsilon} \in V_i \cap L^\infty(\Omega^i)$, hence there is a solution u_ε to the approximate problem due to Theorem 2.1.

Our aim is to let $\varepsilon \to 0$ and $n \to \infty$ simultaneously, e.g. choosing $n = \varepsilon^{-1}$. Assuming such connection between ε and n we omit indication of the dependence of u^i_ε on n. First from (3.5) immediately follows existence of a subsequence (we write u^i_ε again for all subsequences) with

$$u^i_\varepsilon \rightharpoonup u^i \qquad \text{in } L^2(0,T;V_i) \quad \text{as } \varepsilon \to 0 \tag{3.11}$$

($i = 1, \dots, k$). Because of [1, Lemma 4.4], the estimate (3.4) together with Lemma 3.3 yields for a subsequence

$$b^i_\varepsilon(u^i_\varepsilon) \rightharpoonup w^i \in b^i(u^i) \qquad \text{in } L^1(\Omega^i_T) \quad \text{as } \varepsilon \to 0. \tag{3.12}$$

Now, consider relation (2.8) with test functions $\phi \in L^2(0,T;V)$. Then the integrals over Γ^i disappear. Since V is a subspace of $V_1 \times \dots \times V_k$ we have $\partial_t b_\varepsilon(u_\varepsilon) := \partial_t b^1_\varepsilon(u^1_\varepsilon) \times \dots \times \partial_t b^k_\varepsilon(u^k_\varepsilon) \in L^2(0,T;V^*)$ and obtain from (2.8) regarding (3.5)

$$\|\partial_t b_\varepsilon(u_\varepsilon)\|_{L^2(0,T;V^*)} \le C.$$

This yields for a subsequence

$$\partial_t b_\varepsilon(u_\varepsilon) \rightharpoonup \partial_t w \qquad \text{in } L^2(0,T;V^*) \quad \text{as } \varepsilon \to 0. \tag{3.13}$$

Moreover, we need strong convergence of a subsequence $\{u^i_\varepsilon\}$ in order to go to the limit in the items defined on the interfaces Γ^i. Recalling construction (2.1) of $b^i_\varepsilon(u)$, thanks to monotonicity of χ_ε the estimate (3.7) for $b^i_\varepsilon(u)$ also holds for its first item,

$$\int_0^{T-h} \int_{\Omega^i} \left(\beta^i_\varepsilon(u^i_\varepsilon(x,t+h)) - \beta^i_\varepsilon(u^i_\varepsilon(x,t))\right) \left(u^i_\varepsilon(x,t+h) - u^i_\varepsilon(x,t)\right) dxdt \;\le\; Ch$$

for all $h > 0$, $\varepsilon > 0$ and $i = 1, \ldots, k$. Analogously, the estimate (3.4) remains true if $B^i_\varepsilon(z)$ is defined by (2.22) with $\beta^i_\varepsilon(z)$ instead of $b^i_\varepsilon(z)$. Then, in view of (3.11), [1, Lemma 1.9] yields

$$\beta^i_\varepsilon(u^i_\varepsilon) \longrightarrow \beta^i(u^i) \qquad \text{in } L^1(\Omega^i_T) \quad \text{as } \varepsilon \to 0.$$

Since $\beta^i_\varepsilon \to \beta^i$ as $\varepsilon \to 0$ uniformly on compact sets in $\mathbb{R}$, and due to (1.3), the above strong convergence of $\beta^i(u^i_\varepsilon)$ implies strong convergence of u^i_ε itself,

$$\begin{aligned} \iota \|u^i_\varepsilon - u^i\|_{L^1(\Omega^i_T)} &\le \|\beta^i_\varepsilon(u^i_\varepsilon) - \beta^i(u^i)\|_{L^1(\Omega^i_T)} + \|\beta^i(u^i) - \beta^i_\varepsilon(u^i)\|_{L^1(\Omega^i_T)} \\ &\to 0 \qquad \text{as } \varepsilon \to 0. \end{aligned} \tag{3.14}$$

Strong convergence on the interfaces Γ^i we obtain by application of the interpolation inequality

$$\|u\|_{L^1(\Gamma^i_T)} \le C \, \|u\|^{1-\theta}_{L^1(0,T;V_i)} \, \|u\|^{\theta}_{L^1(\Omega^i_T)}$$

(see [5, Proposition 2]) to the difference $u^i_\varepsilon - u^i$. Regarding of (3.5) and (3.14), this inequality yields

$$u^i_\varepsilon \to u^i \qquad \text{in } L^1(\Gamma^i_T) \quad \text{as } \varepsilon \to 0 \ , \tag{3.15}$$

$(i = 1, \ldots, k)$. Since we arranged that $n \to \infty$ if $\varepsilon \to 0$ the estimate (3.6) shows that the limit u^i fulfills our jump condition (3.10).

To show that the limit function is a solution, let $\varepsilon \to 0$ in relations (2.7) and (2.8) for test functions as fixed in Definition 3.1 using (3.11)–(3.13), which yields relations (3.8) and (3.9), respectively.

Finally, consider bounded initial values. Then the regularized initial values are also bounded by $b^i_{0\varepsilon} = b^i_\varepsilon(u^i_{0\varepsilon}) \le C^i$. Since $(b^i_\varepsilon)^{-1}(s) \to (b^i)^{-1}(s)$ as $\varepsilon \to 0$ for every fixed $s \ge 0$ we find bounds c^i_ε such that

$$0 \le u^i_{0\varepsilon} \le c^i_\varepsilon, \qquad c^{i+1}_\varepsilon = \psi^i(c^i_\varepsilon), \qquad \text{and} \quad c^i_\varepsilon \to c^i \quad \text{as } \varepsilon \to 0.$$

Hence, the boundedness assertion $u^i_\varepsilon \le c^i_\varepsilon$ a.e. on Ω^i_T from Theorem 2.1 yields the boundedness assertion for u^i. □

All arguments in the preceding proof to obtain the convergence properties (3.11)–(3.13),(3.15) remain the same if we fix n, choose test functions from $V_1 \times \cdots \times V_k$ and let ε run to zero. This proves existence of a solution to the free boundary problem with conditions as formulated in (2.8) on the interfaces Γ^i.

Corollary 3.1. *Suppose assumptions of Theorem 3.1. Then there is a weak solution to the free boundary problem (2.6) with $b^i(u)$ given by (1.2), i.e. there are functions $u \in L^2(0,T;V_1) \times \cdots \times L^2(0,T;V_k)$ and $w^i \in L^2(\Omega^i_T)$ with $\partial_t w^i \in L^2(0,T;V^*_i)$ fulfilling relations (3.8) and*

$$\sum_{i=1}^k \left(\int_0^T \langle \partial_t w^i, \phi^i \rangle \, dt \ + \int_0^T \int_{\Omega^i} \nabla u^i \nabla \phi^i \, dx dt + n \int_0^T \int_{\Gamma^i} \left(u^{i+1} - \psi^i(u^i) \right) \left(\phi^{i+1} - \phi^i \right) \, d\sigma dt \right) = 0$$

for any $\phi^i \in L^2(0,T;V_i)$, $i = 1, \ldots, k$, $u^{k+1} \equiv \phi^{k+1} \equiv 0$.

Remark. For bounded initial values we obtain bounded solutions. DiBenedetto and Vespri prove in [3] that a bounded solution to (1.1) is locally continuous if it can be approximated in the topology of our approximation by a sequence of local smooth solutions to (1.1) for smooth $b^i_\varepsilon(\cdot)$. Hence, if our approximations u^i_ε were smooth on Ω^i_T, then u^i is continuous in Ω^i_T.

4 Nonlinear diffusion

Finally, let $b^i(u)$ and $\psi^i(u)$ be sufficiently smooth functions defined for $u \in \mathbb{R}$, say $C^2(\mathbb{R})$, which for some constants κ and K, satisfy

$$0 < \kappa \le (b^i)', \ (\psi^i)' \le K \tag{4.1}$$

for $i \in \{1, \dots, k\}$, $b^i(0) = \psi^i(0) = 0$ and $|(b^i)''|, |(\psi^i)''| \le K$ on $\mathbb{R}$.

The above can be viewed as monotone continuously differentiable regularization of the graph b^i and the function ψ^i from the beginning. See (1.3) and (1.5) above.

The main aim of this section is to show, at least in this regular case, that the nonlinear diffusion multicomponent system with the jump conditions $u^{i+1} = \psi^i(u^i)$ between two components is uniquely solvable.

Thus, let us now consider the problem

$$
\begin{aligned}
&\partial_t b^i(u^i) = \Delta u^i && \text{in } \Omega^i_T,\ i = 1, \dots, k\ , \\
&u^i = \psi^{i-1}(u^{i-1}) \quad \text{and} \quad \partial_{\nu_i} u^i + \partial_{\nu_{i-1}} u^{i-1} = 0 && \text{on } \Gamma^{i-1}_T,\ i = 2, \dots, k\ , \\
&\partial_{\nu_i} u^i = 0 && \text{on } \mathcal{T}^i_T,\ i = 1, \dots, k\ , \\
&u^k = 0 && \text{on } \Gamma^k_T\ , \\
&u^i = u^i_0 \ge 0 && \text{on } \Omega^i \times \{t = 0\},\ i = 1, \dots, k\ ,
\end{aligned}
\tag{4.2}
$$

where initial data u^i_0 are supposed to be bounded and sufficiently smooth.

Definition 4.1. *We say now that a k–tuple $u = (u^1, \dots, u^k) \in L^2(0, T; V_1) \times \dots \times L^2(0, T; V_k)$ is a weak solution of the system (4.2) provided*

(i)

$$
\sum_{i=1}^k \left(\int_{\Omega^i} b^i(u^i(x,t))\ \phi^i(x,t)\ dx - \int_0^t \int_{\Omega^i} \left(b^i(u^i)\partial_\tau \phi^i - \nabla u^i \nabla \phi^i \right)\ dx d\tau \right)
$$
$$
= \sum_{i=1}^k \int_{\Omega^i} b^i(u^i_0)\ \phi^i(x,0)\ dx \tag{4.3}
$$

is fulfilled for any $\phi \in L^2(0, T; V)$ (recall that the condition $\phi^{i+1} = \phi^i$ on Γ^i is required above, see Definition 3.1), and a.e. $t \in [0, T]$,

(ii)

$$
u^{i+1} = \psi^i(u^i) \qquad \text{a.e. on } \Gamma^i_T \tag{4.4}
$$

$i = 1, \dots, k$, and finally,

(iii) there exists a positive constant C such that

$$
\sum_{i=1}^k \int_0^{T-h} \int_{\Omega^i} \left| u^i(x, t+h) - u^i(x,t) \right|\ dx dt\ \le\ Ch \tag{4.5}
$$

for $h > 0$.

A subsolution (supersolution) is defined by (4.3) with equality replaced by $\le$ ($\ge$) and $\phi^i \ge 0$ on Ω^i_T for each $i \in \{1, \dots, k\}$.

Theorem 4.1. *Let u be a subsolution and v a supersolution of the problem (4.2) with initial data u_0 and v_0, respectively. Then if $u_0 \le v_0$, i.e. $u^i_0 \le v^i_0$ a.e. on Ω^i for any $i \in \{1, \dots .k\}$ it follows that*

$$
u^i(x,t) \le v^i(x,t) \tag{4.6}
$$

a.e. on Ω^i_T for any $i \in \{1, \dots, k\}$.

Moreover, if u and v are weak solutions of the problem (4.2) with initial functions u_0 and v_0, respectively. Then for almost all $t \in [0, T]$ the inequality (2.11) holds.

Proof: Let $0 < t \le T$ be fixed. Take the difference of the integral inequalities satisfied by u and

v for sufficiently smooth $\phi^i \geq 0$. Then

$$\sum_{i=1}^{k} \Big\{ \int_{\Omega^i} \left(b^i(u^i(x,t)) - b^i(v^i(x,t))\right) \ \phi^i(x,t)\ dx$$

$$- \int_0^t \int_{\Omega^i} \left(\psi^i(x,u^i(x,\tau)) - \psi^i(x,v^i(x,\tau))\right) \left[\alpha^i(x,\tau)\partial_t\phi^i(x,\tau) + \ \mu^i(x,\tau)\Delta\phi^i(x,\tau)\right]\ dxd\tau \tag{4.7}$$

$$+ \int_0^t \int_{\Gamma^i} \left(\psi^i(x,u^i(x,\tau)) - \psi^i(x,v^i(x,\tau))\right) \left[\mu^i(x,\tau)\partial_{\nu_i}\phi^i + \mu^{i+1}(x,\tau)\partial_{\nu_{i+1}}\phi^{i+1}\right]\ d\sigma d\tau \Big\}$$

$$\leq \ \sum_{i=1}^{k} \int_{\Omega^i} \left(b^i(u_0^i(x)) - b^i(v_0^i(x))\right) \ \phi^i(x,0)\ dx \ .$$

Recall that we require $\phi^i = \phi^{i+1}$ on Γ^i. In (4.7),

$$\psi^i(x,u) \equiv \begin{cases} u & \text{if } x \in \Gamma^{i-1} \\ \psi^i(u) & \text{if } x \in \Gamma^i \end{cases}$$

and for $x \in \Omega^i$, say

$$\psi^i(x,u) \equiv u\ \omega^i(x) + \psi^i(u)\ \varpi^i(x)\ ,$$

where nonnegative functions ω^i, $\varpi^i \in C^\infty(\bar{\Omega}^i)$ are such that

$$\omega^i(x) + \varpi^i(x) \equiv 1$$

for $x \in \bar{\Omega}^i$, $\omega^i \equiv 1$ on a neighbourhood of Γ^{i-1} and $\omega^i \equiv 0$ on a neighbourhood of Γ^i, $\psi^k(x,u) = u$ for $x \in \bar{\Omega}^k$ and $\psi^1(x,u) = \psi^1(u)$ on $\bar{\Omega}^1$. Note that due to (4.4)

$$\psi^{i+1}(x,u^{i+1}(x,t)) = \psi^i(x,u^i(x,t)) \tag{4.8}$$

for any $x \in \Gamma^i$ and $t \in (0,T)$ and similarly for v^i. Finally,

$$\alpha^i(x,\tau) \equiv \frac{b^i(u^i(x,\tau)) - b^i(v^i(x,\tau))}{\psi^i(x,u^i(x,\tau)) - \psi^i(x,v^i(x,\tau))}$$

and

$$\mu^i(x,\tau) \equiv \frac{u^i(x,\tau) - v^i(x,\tau)}{\psi^i(x,u^i(x,\tau)) - \psi^i(x,v^i(x,\tau))}\ .$$

It follows from (4.1) that

$$0 < \kappa \leq \alpha^i(x,\tau), \mu^i(x,\tau) \leq K$$

for any $(x,\tau) \in \Omega_T^i$, $i = 1,\ldots,k$. Now, let R_ε be the standard mollifier in (x,τ) variables and set

$$\alpha_\varepsilon^i(x,\tau) \equiv \frac{b^i(u_\varepsilon^i(x,\tau)) - b^i(v_\varepsilon^i(x,\tau))}{\psi^i(x,u_\varepsilon^i(x,\tau)) - \psi^i(x,v_\varepsilon^i(x,\tau))}\ ,$$

$$\mu_\eta^i(x,\tau) \equiv \frac{u_\eta^i(x,\tau) - v_\eta^i(x,\tau)}{\psi^i(x,u_\eta^i(x,\tau)) - \psi^i(x,v_\eta^i(x,\tau))}\ ,$$

where $u_\varepsilon^i(x,\tau) = (R_\varepsilon * u^i)(x,\tau)$, $u_\eta^i(x,\tau) = (R_\eta * u^i)(x,\tau)$ and analogously for v^i, $0 < \varepsilon, \eta \ll 1$.

Finally, let $\varphi_{\varepsilon\eta} = (\varphi_{\varepsilon\eta}^1,\ldots,\varphi_{\varepsilon\eta}^k)$, $\varphi_{\varepsilon\eta} = \varphi_{\varepsilon\eta}(x,s)$ be the solution of the regularized dual problem

$$\begin{array}{ll} \alpha_\varepsilon^i(x,t-s)\partial_s\varphi^i = \mu_\eta^i(x,t-s)\Delta\varphi^i & \text{in } \ \Omega_t^i\,,\ i = 1,\ldots,k, \\ \mu_\eta^i(x,t-s)\partial_{\nu_i}\varphi^i + \mu_\eta^{i+1}(x,t-s)\partial_{\nu_{i+1}}\varphi^{i+1} = 0 & \\ \text{and} \quad \varphi^i = \varphi^{i+1} & \text{on } \ \Gamma_t^i\,,\ i = 1,\ldots,k-1, \\ \varphi^k = 0 & \text{on } \ \Gamma_t^k, \\ \partial_{\nu_i}\varphi^i = 0 & \text{on } \ \mathcal{T}_t^i\,, \\ \varphi^i = \varphi_0^i, \ \ 0 \leq \varphi_0^i \leq 1 & \text{on } \ \Omega^i \times \{s = 0\}, \end{array} \tag{4.9}$$

assuming that $\varphi_0^{i+1} = \varphi_0^i$ on Γ^i. As we were not able to prove the existence of the solution $\varphi_{\varepsilon\eta}^i$ from $W_2^{2,1}(\Omega_t^i)$, we have to work in the class of weak solutions and we rewrite the inequality (4.7) into the form

$$\sum_{i=1}^k \Big\{ \int_{\Omega^i} \left(b^i(u^i(t)) - b^i(v^i(t))\right) \varphi^i(x,0)\, dx$$
$$+ \int_0^t \int_{\Omega^i} \left((\psi^i(x,u^i) - \psi^i(x,v^i)\right) \alpha^i(x,\tau)\, \partial_s \varphi^i(x,t-\tau)\, dx d\tau \tag{4.10}$$
$$+ \int_0^t \int_{\Omega^i} \nabla\varphi^i(x,t-\tau)\, \nabla\left[\mu^i(x,\tau)\left(\psi^i(x,u^i) - \psi^i(x,v^i)\right)\right] \Big) dx d\tau \Big\} \le 0$$

for any φ^i, $\varphi^{i+1} = \varphi^i$ on Γ_t^i, where we have put

$$\phi^i(x,\tau) = \varphi^i(x,t-\tau) \qquad i = 1,\dots,k\,.$$

By the weak formulation of the regularized dual problem (4.9) we understand the following identity

$$\sum_{i=1}^k \int_0^t \int_{\Omega^i} \left(\alpha_\varepsilon^i\, \partial_s\varphi^i\, \xi^i \;+\; \nabla\varphi^i\, \nabla\left(\mu_\eta^i\, \xi^i\right)\right) dx d\tau \;=\; 0 \tag{4.11}$$

for any $\xi = (\xi^1,\dots,\xi^k)$ such that $\xi^i \in L^2(0,t;V_i)$ and

$$\xi^{i+1} = \xi^i \qquad \text{on } \Gamma_t^i\,. \tag{4.12}$$

In (4.11) we expect to have

$$\varphi^i \in H^1(0,t;L^2(\Omega^i)) \cap L^2(0,t;V_i)\,, \quad 0 \le \varphi^i \le 1\,.$$

Indeed, the following assertion holds.

Lemma 4.1. *Let $0 < \varepsilon, \eta \ll 1$ be given. Then there exists a weak solution $\varphi_{\varepsilon\eta}$ of (4.9) such that*

$$\int_0^t \int_{\Omega^i} \left|\partial_s \varphi_{\varepsilon\eta}^i\right|^2 dx d\tau + \max_{0\le t\le T} \int_{\Omega^i} \left|\nabla\varphi_{\varepsilon\eta}^i\right|^2 dx \;\le\; C_1(\eta)\,, \tag{4.13}$$

where the positive constant C_1 depends on

$$\left\|\partial_t \mu_\eta^i\right\|_{L^\infty(\Omega_T^i)} \qquad \text{and} \qquad \left\|\nabla\mu_\eta^i\right\|_{L^\infty(\Omega_T^i)}\,,$$

and does not depend on ε. In addition,

$$\int_0^t \int_{\Omega^i} \left|\nabla\varphi_{\varepsilon\eta}^i\right|^2 dx d\tau \;\le\; C_2(\varepsilon,\eta)\,, \tag{4.14}$$

where the positive constant C_2 depends on

$$\left\|\nabla\mu_\eta^i\right\|_{L^2(\Omega_T^i)} \qquad \text{and} \qquad \left\|\partial_t \alpha_\varepsilon^i\right\|_{L^1(\Omega_T^i)}\,,$$

and

$$0 \le \varphi_{\varepsilon\eta}^i \le 1 \qquad \text{on} \qquad \Omega_T^i\,.$$

Proof: We shall approximate problem (4.9) by the following way:

$$\begin{aligned}
&\alpha_\varepsilon^i(x,t-s)\; \partial_s\varphi^i = \mu_\eta^i(x,t-s)\, \Delta\varphi^i && \text{in } \Omega_t^i\,,\\
&\mu_\eta^i(x,t-s)\partial_{\nu_i}\varphi^i + n(\varphi^i - \varphi^{i-1}) = 0 && \text{on } \Gamma_t^{i-1}\,,\\
&\mu_\eta^i(x,t-s)\partial_{\nu_i}\varphi^i + n(\varphi^i - \varphi^{i+1}) = 0 && \text{on } \Gamma_t^i\,,\\
&\mu_\eta^i(x,t-s)\partial_{\nu_i}\varphi^i = 0 && \text{on } \mathcal{T}_t^i\,,\\
&\varphi^i = \varphi_0^i && \text{on } \Omega^i \times \{s=0\}\,.
\end{aligned} \tag{4.15}$$

1. First of all, the existence of a weak solution $\varphi_n = \varphi_{\varepsilon\eta n}$ of (4.15) such that $0 \le \varphi_n^i \le 1$,

$$\varphi_n^i \in L^2(0,T;H^1(\Omega^i)) \qquad \text{and} \qquad \partial_s \varphi_n^i \in L^2(0,T;H^{-1}(\Omega^i))$$

$i = 1, \dots, k$ can be proved by the same way as it was done in the proof of Theorem 2.2 above and we omit further details. Now, testing (4.15) with φ_n^i and adding up through $i = 1, \dots, k$ we get

$$\begin{aligned} &\sum_{i=1}^k \left\{ \kappa \int_{\Omega^i} \left|\varphi_n^i(x,t)\right|^2 dx \; + \; 2\kappa \int_0^t \int_{\Omega^i} \left|\nabla \varphi_n^i\right|^2 dx\, ds + \; 2n \int_0^t \int_{\Gamma^i} \left|\varphi^{i+1} - \varphi^i\right|^2 d\sigma ds \right\} \\ &\le \; \sum_{i=1}^k \left\{ K \int_{\Omega^i} \left|\varphi_0^i(x)\right|^2 dx + \int_0^t \int_{\Omega^i} \left|\partial_s \alpha_\varepsilon^i\right| \, dx ds \; + \; 2 \left\|\nabla \varphi_n^i\right\|_{L^2(\Omega_t^i)} \left\|\nabla \mu_\eta^i\right\|_{L^2(\Omega_t^i)} \right\} \end{aligned}$$

and (4.14) for $\varphi_{\varepsilon\eta n}^i$ follows easily due to Gronwall's lemma.

2. Now, assume for a moment, that φ_n^i are such smooth that we can test (4.15) with $\partial_s \varphi_n^i$ and perform all necessary manipulations to arrive at

$$\begin{aligned} &\sum_{i=1}^k \int_0^t \int_{\Omega^i} \left(\alpha_\varepsilon^i \left|\partial_s \varphi_n^i\right|^2 + \partial_s \left(\mu_\eta^i |\nabla \varphi_n^i|^2\right) - |\nabla \varphi_n^i|^2 \partial_s \mu_\eta^i \right) dx\, ds \\ + &\sum_{i=1}^k \left\{ 2 \int_0^t \int_{\Omega^i} \nabla \varphi_n^i \; \partial_s \varphi_n^i \; \nabla \mu_\eta^i \; dx\, ds + n \int_0^t \int_{\Gamma^i} \partial_s (\varphi^{i+1} - \varphi^i)^2 \, d\sigma ds \right\} = \; 0 \; . \end{aligned}$$

Hence, (4.13) for $\varphi_{\varepsilon\eta n}^i$ follows easily. This can be made precise performing analogous manipulations with

$$\partial_s^h \varphi_n^i \equiv \frac{\varphi_n^i(x, s+h) - \varphi_n^i(x,s)}{h}$$

instead of $\partial_s \varphi_n^i$ and let us omit it.

Finally, as all estimates are independent of n, they remain to hold also for limit functions $\varphi_{\varepsilon\eta}^i$. □

Now, due to (4.8) we can insert

$$\xi^i = \psi^i(\cdot, u^i) - \psi^i(\cdot, v^i)$$

as a test function in (4.11) that together with (4.10) yield

$$\begin{aligned} \sum_{i=1}^k \Bigg\{ &\int_{\Omega^i} \left(b^i(u^i(t)) - b^i(v^i(t))\right) \; \varphi_{\varepsilon\eta}^i(x,0) \; dx \\ &+ \int_0^t \int_{\Omega^i} \left(\psi^i(x,u^i) - \psi^i(x,v^i)\right) \; (\alpha^i - \alpha_\varepsilon^i) \; \partial_s \varphi_{\varepsilon\eta}^i \, dx d\tau \\ &+ \int_0^t \int_{\Omega^i} \nabla \varphi_{\varepsilon\eta}^i \; \nabla \left((\mu^i - \mu_\eta^i)(\psi^i(x,u^i) - \psi^i(x,v^i)) \right) \; dx d\tau \Bigg\} \le \; 0 \; . \end{aligned} \tag{4.16}$$

Before letting $\varepsilon \to 0$ let us note that

$$(4.5) \qquad \text{yields} \qquad \int_0^T \int_{\Omega^i} \left|\partial_t u_\varepsilon^i(x,t)\right| \; dx dt \; \le \; C$$

where u_ε^i is defined above, cf. (4.8)–(4.9). The same holds for $\partial_t v_\varepsilon^i$. Therefore, due to (4.1) and what follows we have

$$\begin{aligned} \left\|\partial_t \alpha_\varepsilon^i\right\|_{L^1(\Omega_T^i)} \; &\le \; C \; \textstyle\int_0^T \int_{\Omega^i} \left(\left|\partial_t u_\varepsilon^i(x,t)\right| + \left|\partial_t v_\varepsilon^i(x,t)\right| \right) \; dx\, dt \\ &\le \; C \end{aligned}$$

and also

$$\|\nabla \mu_\eta^i\|_{L^2(\Omega_T^i)} \le C,$$

C being independent of ε, η. Hence, (4.14) remains uniformly bounded and we are ready to let $\varepsilon \to 0$ and afterwards $\eta \to 0$ in (4.16) to get

$$\sum_{i=1}^k \int_{\Omega^i} \left(b^i(u^i(x,t)) - b^i(v^i(x,t))\right) \varphi_0^i(x)\, dx \;\le\; 0\ .$$

As this holds for any smooth function φ_0^i, $0 \le \varphi_0^i \le 1$, it continuous to holds also for $\varphi_0^i = \chi(u^i(x,t) - v^i(x,t))$ and (4.6) follows easily. The rest of the proof is the same as in Theorem 2.3. □

We show now that the solution of (4.2) which we get as the limit of the sequence of solutions to (2.6) under the assumption (4.1) indeed satisfies (4.5).

Theorem 4.2. *In addition to the conditions of Section 3 we assume $u_0 \in V_i \cap W_1^2(\Omega^i)$, (4.1), and the compatibility condition*

$$u_0^{i+1} = \psi^i(u_0^i) \qquad on\ \Gamma^i, \quad i = 1, \dots, k. \tag{4.17}$$

Then the solution of problem (4.2) obtained by Theorem 3.1 is a solution in the sense of Definition 4.1. In particular, this solution satisfies condition (4.5).

Proof: Consider the approximate problem (2.6). Theorem 2.1 yields existence of a solution u_n. We want to show that $b^i(u_n^i)$ fulfills condition (4.5) with a constant C independent of n. Since b is continuous we obtain strong convergence $b^i(u_n^i) \to b^i(u^i)$ as $n \to \infty$ in $L_1(\Omega_T^i)$ where u is a solution of (4.2) (see proof of Theorem 3.1). Then (4.1) yields the assertion.

Hence, let us estimate the difference $b^i(u^i(x,t+h)) - b^i(u^i(x,t))$ in $L_1(\Omega^i)$ for a.e. $t \in [0, T-h]$ where u is a solution of (2.6). We follow the idea of the proof of Theorem 2.2 replacing $v(x,t)$ by $u(x, t+h)$. Then all manipulations remain the same if we are able to manage the item

$$b^i(u^i(x, t_0 + h)) - b^i(u^i(x, t_0))$$

at some initial time t_0. To this end we fix $h > 0$ and extend problem (2.6) to the interval $t \in [-h, T]$. Define

$$\tilde u(x,t) = \begin{cases} u(x,t) & \text{for } t \in [0,T] \\ u_0(x) & \text{for } t \in [-h, 0] \end{cases},$$

and note that $\tilde u$ is the solution of

$$\begin{aligned}
&\partial_t b^i(\tilde u^i) = \Delta \tilde u^i + F^i && \text{in } \Omega^i \times (-h,T)\ ,\\
&\partial_{\nu_i} \tilde u^i + n\left(\tilde u^i - \psi^{i-1}(\tilde u^{i-1})\right) = f^i && \text{on } \Gamma^{i-1} \times (-h,T)\ ,\\
&\partial_{\nu_i} \tilde u^i + n\left(\psi^i(\tilde u^i) - \tilde u^{i+1}\right) = f^i && \text{on } \Gamma^i \times (-h,T)\ ,\\
&\partial_{\nu_i} \tilde u^i = f^i && \text{on } \mathcal{T}^i \times (-h,T)\ ,\\
&\tilde u^i = u_0^i && \text{on } \Omega^i \times (-h, 0]\ ,
\end{aligned} \tag{4.18}$$

with

$$\begin{aligned}
F^i(x,t) &= -\chi_{[-h,0]}(t)\, \Delta u_0^i(x), && t \in [-h,T],\ x \in \Omega^i,\\
f^i(x,t) &= \chi_{[-h,0]}(t)\, \partial_{\nu_i} u_0^i(x), && t \in [-h,T],\ x \in \partial\Omega^i,
\end{aligned}$$

$i = 1, \dots, k$. Here we have used compatibility condition (4.17). Note that F^i and f^i do not depend on the approximation parameter n. Now we proceed as in the proof of Theorem 2.2 for problem

(4.18) instead of (2.6) with $u = \tilde{u}(x,t), v = \tilde{u}(x,t+h)$. Since we integrate over $\tau \in [-h,t]$, by our construction the item

$$\int_{\Omega^i} \left(b^i(u_0^i) - b^i(v_0^i)\right) \varphi^i(x,t)\; dx := \int_{\Omega^i} \left(b^i(\tilde{u}^i(x,-h)) - b^i(\tilde{u}^i(x,0))\right) \varphi^i(x,t)\; dx$$
$$= \int_{\Omega^i} \left(b^i(u_0^i) - b^i(u_0^i)\right) \varphi^i(x,t)\; dx$$

on the right hand side of (2.12) disappears. On the other hand, the following additional items appear

$$r^i(\varphi^i,h) = \int_{-h}^{t}\int_{\Omega^i} \left(F^i(x,\tau) - F^i(x,\tau+h)\right)\; \varphi^i(x,t-\tau)\; dxd\tau$$
$$+ \int_{-h}^{t}\int_{\partial\Omega^i} \left(f^i(x,\tau) - f^i(x,\tau+h)\right)\; \varphi^i(x,t-\tau)\; d\sigma d\tau.$$

Again we choose a solution φ^i of the regularized dual problem (2.16) as test function. Since it is bounded by (2.17) by the definition of F^i, f^i we can estimate these additional items by

$$|r^i(\varphi^i,h)| \leq h\,(\|\Delta u_0^i\|_{L_1(\Omega^i)} + \|\partial_{\nu_i} u_0^i\|_{L_1(\partial\Omega^i)}) =: C_0^i\, h\,.$$

Therefore, performing the manipulations as in the proof of Theorem 2.2 we finally arrive at

$$\sum_{i=1}^{k} \int_{\Omega^i} |b^i(u^i(x,t+h)) - b^i(u^i(x,t))|\, dx \;\leq\; \sum_{i=1}^{k} C_0^i\, h \tag{4.19}$$

for a.e $t \in [0, T-h]$, which concludes the proof. □

Remark. Actually, we have proven a stronger condition than (4.5). Indeed, the solution u_n of the approximate problem (2.6) fulfills the Lipschitz condition (4.19) pointwise w.r.t. time. Since convergence of $b^i(u_n^i)$ in $L_1(\Omega_T^i)$ implies convergence of a subsequence in $L_1(\Omega^i)$ for a.e. $t \in [0, T-h]$ we even obtain

$$\sum_{i=1}^{k} \int_{\Omega^i} |u^i(x,t+h) - u^i(x,t)|\, dx \;\leq\; C\, h \qquad \text{for a.e. } t \in [0, T-h].$$

Acknowledgements

We thank Volkswagen-Stiftung for its support and hospitality by the RiP-program at Oberwolfach. The first author was partially supported also by VEGA grant 1/4195/97.

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