Phelps' Lemma, Daneš' Drop Theorem and Ekeland's Principle in Locally Convex Topological Vector Spaces

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Abstract

We present a generalization of the Phelps lemma to locally convex topological vector spaces and show the equivalence of this theorem, Ekeland's principle and Daneš' drop theorem in locally convex spaces to their Banach space counterparts and to a Pareto efficiency theorem due to Isac. Concerning the drop theorem this solves a problem proposed by G. Isac in 1997.

We show that another formulation of Ekeland's principle in locally convex spaces using a family of topology generating semi-norms as perturbation functions rather than a single (in general discontinuous) Minkowski functional turns out to be equivalent to the original version.

Keywords and phrases. Locally convex space, Phelps' lemma, Ekeland's variational principle, Daneš' drop theorem

Mathematical Subject Classification. 49J40, 46A03

1 Introduction

The famous 1972 theorem of Ekeland called the variational principle is one of the most frequently applied results of nonlinear functional analysis. For example, the progress of variational calculus and optimal control over the last 25 years is unthinkable without Ekeland's principle.

The importance of it is emphasized by the fact that nearly at the same time but independently of each other a list of theorems were discovered which all charaterize the completeness of the underlying space and are equivalent to Ekeland's principle in metric spaces, namely the Krasnoslski/Zabrjeko theorem on normal solvability of operator equations [20], the Kirk-Caristi fixed point theorem [3], and the drop theorem of Daneš [5].

Over the last three decades a great deal of effort has been done to look for another equivalent formulation or generalization of Ekeland's principle.

In this paper we continue this story by showing that a 1974 result of Phelps (Lemma 1.2 of [17] with precursor Lemma 1 in [2], called Phelps lemma in the following) can be generalized to sequentially closed sets of locally convex spaces, moreover that the locally convex variant can be proved using the corresponding Banach space variant only and that this procedure is also possible for Ekeland's principle and the drop theorem of Daneš.

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Beyond this we show the equivalence of these theorems among each other, surprisingly to their Banach space counterparts and to a second variant of Ekeland's principle due to Fang [7] specialized to locally convex spaces.

Finally, solving a problem stated in [12], we point out the equivalence of the drop theorem in locally convex spaces to a theorem due to Isac which relates Ekeland's principle to Pareto efficiency.

We close the paper with some conclusions concerning the relations of different versions of the represented theorems. The appendix contains the Banach space versions of the Phelps lemma, Ekeland's principle and Daneš' drop theorem.

Let us start with some notational arrangements.

Let X be a vector space. A functional $p: X \to (-\infty, +\infty)$ is called a seminorm if it satisfies

(i) $p(\alpha x) = |\alpha| \, p(x)$ for all $x \in X$, $\alpha \in (-\infty, +\infty)$,
(ii) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

Following Treves [19] a pair $\left(X, \{p_i\}_{i\in I}\right)$ is said to be a locally convex topological vector space (short: locally convex space) if $\{p_i\}_{i\in I}$ is a family of seminorms satisfying
(iii) $i \in I$, $\alpha > 0$ implies the existence of $j \in I$ such that $p_j = \alpha p_i$,
(iv) $i \in I$, q is a seminorm such that $q(x) \leq p_i(x)$ for every $x \in X$ implies the existence of $j \in I$ such that $p_j = q$.
(v) $i_1, i_2 \in I$ implies the existence of $j \in I$ such that $p_j = \sup\{p_{i_1}, p_{i_2}\}$.

The locally convex space X is said to be Hausdorff (or separated) if
(vi) $x \neq y$ for $x, y \in X$ implies the existence of $j \in I$ such that $p_j(x) \neq p_j(y)$.

Throughout the paper, X always denotes a Hausdorff locally convex topological vector space. The family $\{p_i\}_{i\in I}$ of topology generating seminorms is called the spectrum of X and is denoted by $spec(X)$. A subfamily $\{p_\lambda\}_{\lambda\in\Lambda}$ of $spec(X)$ is said to be a base of continuous seminorms if for every $p \in spec(X)$ on X there are $\alpha > 0$ and p_λ, $\lambda \in \Lambda$ such that $p(x) \leq \alpha p_\lambda(x)$ for every $x \in X$. Every Hausdorff locally convex topological vector space has always a base of continuous seminorms. It can be constructed in the following way. Starting with $spec(X) = \{p_i\}_{i\in I}$ we define $\Lambda = \{\lambda \subset I, \lambda \text{ finite}\}$ and

$$p_\lambda(x) = \sup_{j\in\lambda} p_j(x) \quad \text{for } x \in X,\ \lambda \in \Lambda.$$

As usual, we define the Minkowski functional of a set $A \subset X$ to be

$$\mu_A(x) := \inf\{\alpha > 0 : \ x \in \alpha A\}.$$

2 The Phelps lemma in locally convex spaces

Let X be a locally convex Hausdorff space. For a given set $B \subset X$ we define the cone

$$K := \mathbb{R}_+ B = \{x \in X : \ \exists \alpha \geq 0, b \in B \text{ such that } x = \alpha b\}.$$

The following theorem is a generalization of Phelps' lemma 1.2 in [17] (see theorem 11 of the appendix) to locally convex spaces.

Theorem 1 *Let X be a Hausdorff sequentially complete locally convex topological vector space. Let $M \subset X$ a nonempty sequentially closed subset of X $B \subset X$ a nonempty sequentially closed bounded and convex set such that $0 \notin B$.*

Then, for each $x_0 \in X$ such that $M \cap (x_0 + K)$ is bounded and nonempty there exists $x^* \in X$ such that

$$x^* \in M \cap (x_0 + K) \quad \textit{and} \quad \{x^*\} = M \cap (x^* + K).$$

Proof. Define the set

$$T := s - cl\, co \{B \cup -B \cup \{\pm x_0\}\}$$

where $s - cl\ co$ denotes the sequentially closure of the convex hull and let $X_1 = span\, T$. Let μ_T be the Minkowski functional of T. We claim that $\mu_T(\cdot) := \|\cdot\|_1$ is a norm on X_1 and $(X_1, \|\cdot\|_1)$ is a Banach space.

Note that μ_T is positive homogenous and convex which implies the triangle inequality. The implication $\|x\|_1 = 0 \Rightarrow x = 0$ is a consequence of the boundedness of T. It remains to show the completeness of X_1 with respect to $\|\cdot\|_1$. Let $\{x_n\}$ be a Cauchy sequence in X_1 with respect to $\|\cdot\|_1$. Then it is Cauchy with respect to the locally convex topology because of the boundedness of T for every $i \in I$ there exists $\alpha_i > 0$ such that $T \subset \alpha_i U_i$ where $U_i := \{x \in X : \ p_i(x) \le 1\}$ and

$$\frac{1}{\alpha_i} p_i (x_m - x_n) = \frac{1}{\alpha_i} \mu_{U_i} (x_m - x_n) \le \mu_T (x_m - x_n).$$

Hence there exists $x_0 \in X$ such that $x_n \to x_0$ in X. Moreover, $x_0 \in X_1$ since X_1 is sequentially closed in X. Finally, we shall show $x_n \to x_0$ with respect to $\|\cdot\|_1$. Fix $\varepsilon > 0$. Since $\{x_n\}$ is Cauchy in X_1 we have $x_m - x_n \in \varepsilon T$ for all m, n sufficiently large. Letting $n \to \infty$ we conclude $x_m - x_0 \in \varepsilon T$ for all m sufficiently large since T is sequentially closed in X. This implies

$$\mu_T (x_m - x_0) = \|x_m - x_0\|_1 \le \varepsilon$$

for all m sufficiently large. Hence $(X_1, \|\cdot\|_1)$ is a Banach space as claimed.

Define $M_1 := M \cap X_1$ which is nonempty and closed in the Banach space topology as well as B. Moreover, $x_0 \in M_1$, $K \subset X_1$ and $M_1 \cap (x_0 + K)$ is bounded in X_1.

The Phelps lemma in Banach spaces (theorem 11 of the appendix) yields a point $x^* \in X_1$ such that

$$x^* \in M_1 \cap (x_0 + K) \quad \text{and} \quad \{x^*\} = M_1 \cap (x^* + K).$$

We claim that even $\{x^*\} = M \cap (x^* + K)$. Otherwise there would be $\overline{x} \in M \cap (x^* + K)$, $x^* \neq \overline{x}$. This would imply $\overline{x} \in M \backslash M_1 \subset X \backslash X_1$ and $\overline{x} \in x^* + K \subset X_1$, a contradiction. ■

The Phelps lemma in Banach spaces turns out to be equivalent to Ekeland's variational principle in metric spaces as it has been proved by Georgiev [8] and later on by Attouch/Riahi [1], see also [18]. Moreover, it is equivalent to Daneš' drop theorem and a long list of theorems all equivalent to Ekeland's principle as well, see [15], [8], [14], for example. Concerning the drop theorem and "linear" variants of Ekeland's principle we show this equivalences in the next sections. The relations for the "nonlinear" (for uniform spaces) case can be found in [11].

3 Ekeland's principle in locally convex spaces

In this section we shall present two versions of Ekeland's principle in locally convex spaces which turn out to be equivalent.

Regardless of their equivalence there is a significant difference. The classical approach due to Phelps from the beginning of the 60s uses a Minkowski gauge discontinuous in general of a bounded set as perturbation function.

The recent approach inspired by Fang's paper [7] employs not a single perturbation function but a family of uniformly continuous seminorms.

We start with "the grandfather of it all" (Ekeland [6]), the following theorem due to Phelps [16]. Note that our formulation is slightly more general then the original version because we refer to sequentially completeness and sequentially lower semicontinuity rather then completeness and lower semicontinuity in the sense of the locally convex topology.

Theorem 2 *Let X be a Hausdorff sequentially complete locally convex topological vector space. Let $f : X \to (-\infty, +\infty]$ be an extended-valued proper sequentially lower semicontinuous function, bounded from below. Let $S \subset X$ be a sequentially closed bounded convex and starshaped (i.e., $x \in S$, $0 \leq \alpha \leq 1$ implies $\alpha x \in S$) set and μ_S the corresponding Minkowski functional.*

Then, for each $\gamma > 0$, $x_0 \in dom f$ there exists $x^* \in X$ such that

$$f(x^*) + \gamma \mu_S (x^* - x_0) \leq f(x_0), \tag{1}$$

and for all $x \in X$, $x \neq x^*$ we have

$$f(x^*) < f(x) + \gamma \mu_S (x - x^*). \tag{2}$$

Proof. Define the set

$$T := s - cl\, co \{S \cup -S \cup \{\pm x_0\}\}$$

Let $X_1 = span\, T$. Let μ_T be the Minkowski functional of T. Then it is a norm on X_1 and (X_1, μ_T) is a Banach space, cf. the proof of theorem 1. Next, we define the set

$$C := \{x \in X_1 : \ f(x) + \gamma \mu_S (x - x_0) \leq f(x_0)\}$$

which is closed in X_1 and the function

$$g(x) := \begin{cases} f(x) & : \ x \in C \\ +\infty & : \ \text{otherwise} \end{cases}$$

which is lower semicontinuous with respect to the Banach space topology.

Now, we can apply Ekeland's variational principle in Banach spaces (theorem 10 of the appendix) getting a point $x^* \in C$ such that for $x \in X_1$, $x \neq x^*$

$$g(x^*) < g(x) + \gamma \|x - x^*\|_1 . \tag{3}$$

We shall consider different cases.

First, if $x \in C$ (3) implies

$$f(x^*) < f(x) + \gamma \mu_S (x - x^*)$$

since, because of $S \subset T$, $\|y\|_1 = \mu_T(y) \leq \mu_S(y)$.

Second, if $x \in X_1 \backslash C$ we have

$$f(x) + \gamma\mu_S(x - x_0) > f(x_0)$$

as well as

$$f(x^*) + \gamma\mu_S(x^* - x_0) \leq f(x_0),$$

hence

$$f(x^*) < f(x) + \gamma(\mu_S(x - x_0) - \mu_S(x^* - x_0)) \leq f(x) + \gamma\mu_S(x - x_0)$$

since μ_S is positive homogenous and convex, hence it satisfies the triangle inequality.

Third, if $x \in X \backslash X_1$ (2) is trivially satisfied because $\mu_S(y) = +\infty$ for all $y \in X \backslash X_1$. ∎

The Minkowski functional of a bounded set in locally convex spaces is discontinuous in general. A recent result of Fang [7] gives rise to replace the possibly discontinuous perturbation function μ_S by uniformly continuous seminorms. The price we have to pay for is that we can not use a single perturbation function but the family $\{p_\lambda\}_{\lambda\in\Lambda}$. The result reads as follows.

Theorem 3 *Let X be a Hausdorff sequentially complete locally convex topological vector space. Let $f : X \to (-\infty, +\infty]$ an extended valued proper and sequentially lower semicontinuous function, bounded from below. Let $\{p_\lambda\}_{\lambda\in\Lambda}$ be a base of continuous seminorms generating the topology on X and $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ a family of positive numbers. Then, for every $x_0 \in domf$ there exists some $x^* \in X$ such that*

$$f(x^*) + \gamma_\lambda p_\lambda(x^* - x_0) \leq f(x_0) \tag{4}$$

for all $\lambda \in \Lambda$ and for all $x \neq x^*$ there exists $\mu \in \Lambda$ such that

$$f(x^*) < f(x) + \gamma_\lambda p_\mu(x - x^*). \tag{5}$$

Moreover, we shall prove the following

Theorem 4 *Theorem 1 and theorem 3 are mutually equivalent.*

Proof. First we show that theorem 1 implies theorem 3. In order to do this we define the locally convex space $Y = X \times \mathbb{R}$ where the elements of the spectrum are given by $q_i((x, r)) = \max\{p_i(x), |r|\}$ for $(x, r) \in Y$, $i \in I$. Let $M = \mathrm{epi} f$ and

$$B := \{(x, -1) \in Y : \gamma_\lambda p_\lambda(x) \leq 1 \text{ for all } \lambda \in \Lambda\}.$$

Then M is nonempty and sequentially closed since f is proper and sequentially lower semicontinuous and B is nonempty closed convex and bounded. We note that the cone $K = \mathbb{R}_+ B$ is given by

$$K = \{(y, r) \in Y : s + \gamma_\lambda p_\lambda(y) \leq 0 \text{ for all } \lambda \in \Lambda\}.$$

Let $(x_0, f(x_0))$ be an element of M and $m = \inf f$. Then $M_0 = M \cap ((x_0, f(x_0)) + K)$ is bounded because for $(x, r) \in M_0$ and for each $\lambda \in \Lambda$ we have

$$m - f(x_0) + \gamma_\lambda p_\lambda(x - x_0) \leq r - f(x_0) + \gamma_\lambda p_\lambda(x - x_0) \leq 0$$

hence

$$p_\lambda(x - x_0) \le \frac{f(x_0) - m}{\gamma_\lambda}.$$

Applying theorem 1 to Y, M and K as defined above we get $(x^*, r^*) \in M_0$ such that

$$\{(x^*, r^*)\} = M \cap ((x^*, r^*) + K).$$

It is an easy task to derive (4) and (5) from these relationships.

Now, we prove the converse, namly theorem 3 implies theorem 1. The first step is to prove the existence of a linear continuous functional $l \in X^*$ and numbers $0 < \gamma_\lambda < 1$ such that

$$K \subset K_\Lambda := \{x \in X : \gamma_\lambda p_\lambda(x) \le l(x) \text{ for all } \lambda \in \Lambda\}.$$

Since $0 \notin B$ there exists a neighborhood U of $0 \in X$ such that $U \cap B = \emptyset$. The separation theorem (theorem 13 of the appendix) yields $l \in X^*$, $\|l\| = 1$ such that

$$\delta := \sup l(U) \le \inf l(B).$$

Since B is bounded there exists for each $\lambda \in \Lambda$ a constant $\eta_\lambda > 0$ such that $p_\lambda(x) \le \eta_\lambda$ for all $x \in B$. Then

$$\frac{\delta}{\eta_\lambda} p_\lambda(x) \le \delta \le l(x) \quad \text{for all } x \in B,\ \lambda \in \Lambda,$$

hence taking $\gamma_\lambda = \frac{\delta}{\eta_\lambda}$ we see that

$$\gamma_\lambda p_\lambda(x) \le l(x) \quad \text{for all } x \in B,\ \lambda \in \Lambda$$

which means $B \subset K_\Lambda$. For the next step, we define the function $f = l + \delta_{M_0}$ where $M_0 = M \cap (x_0 + K)$ and δ_{M_0} denotes the indicator function of the set M_0. Then f is bounded below and sequentially lower semicontinuous. Applying theorem 3 we get $x^* \in M_0$ such that

$$l(x^*) + \gamma_\lambda p_\lambda(x^* - x_0) \le l(x_0) \quad \text{for all } \lambda \in \Lambda \tag{6}$$

and for all $x \in M_0$, $x \ne x^*$ there exists $\mu \in \Lambda$ such that

$$l(x^*) < l(x) + \gamma_\mu p_\mu(x - x^*). \tag{7}$$

Note that since $x^* \in M_0$ we have $x^* \in x_0 + K$. Next, we claim that $\{x^*\} = M_0 \cap (x^* + K)$. This is true since from (7) we conclude that, for $x \in M_0$, $x \ne x^*$,

$$l(x - x^*) + \gamma_\mu p_\mu(x - x^*) > 0$$

which implies $x \notin x^* + K_\Lambda$ and further $x \notin x^* + K$. At last we show by contradiction $\{x^*\} = M \cap (x^* + K)$. Assume there exists $z \in M \cap (x^* + K)$, $z \notin M_0 \cap (x^* + K) = M \cap (x_0 + K) \cap (x^* + K)$. This would imply $z \notin x_0 + K$. Since $x^* \in x_0 + K$ we have $z \in x^* + K \subset x_0 + K + K = x_0 + K$, a contradiction. The proof is complete. ■

Almost the same proof can be used to show the equivalence of Phelps' lemma and the variational principle with Minkowski functional.

Theorem 5 *Theorem 1 and theorem 2 are mutually equivalent.*

Proof. We indicate the minor modifications of the proof of theorem 4.

First, let theorem 1 be valid. Define $Y = X \times \mathbb{R}$, $M = \operatorname{epi} f$, $B = \{(x, -1) \in Y : \gamma \mu_S(x) \leq 1\}$ and $K = \mathbb{R}_+ B = \{(y, s) \in Y : s + \gamma \mu_S(y) \leq 0\}$. Then the set $M_0 = M \cap ((x_0, f(x_0)) + K)$ is bounded since S is bounded. The remaining part of the proof goes along the lines of the first part of the proof of theorem 4.

Second, let theorem 2 be true. Define $S = \{y \in X : y = \alpha x, \ 0 \leq \alpha \leq 1, \ x \in B\} = co\{\{0\}, B\}$. Find by means of the separation theorem (theorem 13 of the appendix) $l \in X^*$ and $0 < \gamma < 1$ such that $K \subset \{x \in X : \gamma \mu_S(x) \leq l(x)\}$ and proceed as in the second part of the proof of theorem 4. ∎

Since both versions of Ekeland's principle in locally convex spaces are equivalent to the Phelps lemma they are equivalent to each other:

Theorem 6 *Theorem 3 and theorem 2 are mutually equivalent.*

Phelps' lemma in the product space $X \times \mathbb{R}$ is often called maximal point lemma because of the underlying ordering argument, see [18]. Very general versions of the maximal point lemma (in spaces $X \times Z$, X a complete metric space, Z a locally convex Hausdorff space) can be found for example in [9], [10].

Note that it is also possible to derive theorem 3 from Fang's theorem 3.2 in [7].

4 The drop theorem in locally convex spaces

Mizoguchi [13] as well as Cheng et al. [4] extended the drop theorem to locally convex spaces. We shall give a reformulation of Cheng's version which is at the same time an improvement of Mizoguchi's result. We prove it using the drop theorem in Banach spaces (theorem 12 of the appendix) only.

The drop $D(x, B)$ generated by $x \in X$ and $B \subset X$ is defined to be the set

$$D(x, B) := co\{\{x\}, B\} = \{tx + (1-t)b : b \in B, t \in [0,1]\}.$$

Note that the simple definition of drops does not require any topological notion. The crucial point in formulating the drop theorem is to say when two closed sets are of positive distance. Here we use a single element of the base of seminorms to make available a substitute for the distance of sets in metric spaces.

Theorem 7 *Let X be a Hausdorff sequentially complete locally convex topological vector space. Let $A \subset X$ be a nonempty sequentially closed set and $B \subset X$ a nonempty sequentially closed bounded and convex set. Let $\{p_\lambda\}_{\lambda \in \Lambda}$ be a base of continuous seminorms on X and $\mu \in \Lambda$, $\delta > 0$ such that for all $a \in A$, $b \in B$ we have*

$$p_\mu(a - b) \geq \delta. \tag{8}$$

Then, for each $x_0 \in A$ there exists $x^* \in X$ such that

$$x^* \in A \cap D(x_0, B) \quad \textit{and} \quad \{x^*\} = A \cap D(x^*, B).$$

Proof. Without loss of generality we may assume $0 \in B$. Define the set

$$T := s - cl\, co \{B \cup -B \cup \{\pm x_0\}\}$$

and let $X_1 = span\, T$. Let μ_T be the Minkowski functional of T; then it is a norm on X_1 and $(X_1, \|\cdot\|_1)$ is a Banach space where $\|\cdot\|_1 = \mu_T(\cdot)$, cf. the proof of theorem 1. Define $A_1 := A \cap X_1$ which is nonempty and closed in the Banach space topology as well as B.

We shall apply the drop theorem in Banach spaces, theorem 12 of the appendix. This is possible because the sets A_1 and B are of positive distance in X_1: Since T is bounded there exists $\alpha > 0$ such that $T \subset \alpha U$ where $U = \{x \in X : p_\mu(x) \leq 1\}$. This implies $p_\mu(x) \leq \mu_{\alpha U}(x) \leq \mu_T(x)$ for all $x \in X$. Hence

$$\mu_T(a-b) \geq \frac{1}{\alpha}\mu_U(a-b) \geq \frac{\delta}{\alpha} \quad \forall a \in A, b \in B$$

and it follows

$$\|a-b\|_1 := \mu_T(a-b) \geq \frac{\delta}{\alpha} \quad \forall a \in A_1, b \in B.$$

According to the drop theorem in Banach spaces there exists $x^* \in X_1$ such that

$$x^* \in A_1 \cap D(x_0, B) \quad \text{and} \quad \{x^*\} = A_1 \cap D(x^*, B).$$

We claim that $\{x^*\} = A \cap D(x^*, B)$. Otherwise there would exist $\overline{x} \in A \cap D(x^*, B)$ such that $x^* \neq \overline{x}$. This would imply $\overline{x} \in A \backslash A_1 \subset X \backslash X_1$ and $\overline{x} \in D(x^*, B) \subset X_1$, a contradiction. ■

Next, we clarify the relation of the above theorem to Cheng's variant of [4]. Cheng et al. introduced the following definition.

Definition 1 *Two nonempty subsets of the locally convex space X are said to be strongly Minkowski separated iff there exists a seminorm $p \in spec(X)$ and $z \in X$ such that either*

$$\inf\{p(a+z) :\ a \in A\} > \sup\{p(b+z) :\ b \in B\} \tag{9}$$

or

$$\sup\{p(a+z) :\ a \in A\} < \inf\{p(b+z) :\ b \in B\}. \tag{10}$$

The following proposition is an extension of proposition 2 of Cheng et al. [4]. It shows that our drop theorem 7 implies Cheng's version.

Proposition 1 *Let X be a Hausdorff sequentially complete locally convex topological vector space. Let $A \subset X$ a nonempty sequentially closed set and $B \subset X$ a nonempty sequentially closed bounded and convex set. If (8) is satisfied for some $\mu \in \Lambda$, $\delta > 0$ then A and B are strongly Minkowski separated.*

Proof. Define

$$\delta = \inf\{p_\mu(a-b) :\ a \in A, b \in B\},$$

$$S = \left\{s \in X :\ \inf\left\{p_\mu(s-a) \leq \frac{\delta}{2} :\ a \in A\right\}\right\}.$$

Then $\operatorname{int} S \neq \emptyset$ and

$$\inf \{p_\mu (s-b) : s \in S, b \in B\} \geq \frac{\delta}{2} \tag{11}$$

since

$$p_\mu (s-b) = p_\mu ((a-b)-(a-s)) \geq p_\mu (a-b) - p_\mu (a-s) \geq \frac{\delta}{2}.$$

Without loss of generality we may assume $0 \in \operatorname{int} S$. To complete the proof it suffices to show

$$r = \inf \{\mu_S (b) - \mu_S (s) : b \in B, \ s \in S\} > 0.$$

Suppose the contrary, i.e. $r = 0$. Then there exist, for each $\varepsilon > 0$, $b_\varepsilon \in B$, $s_\varepsilon \in S$ such that $0 < \mu_S (b_\varepsilon) - \mu_S (s_\varepsilon) < \varepsilon$. Since $\mu_S (b_\varepsilon) > 1$, $\mu_S (s_\varepsilon) \leq 1$ we get $1 < \mu_S (b_\varepsilon) \leq 1 + \varepsilon$, hence $\mu_S (b_\varepsilon) \to 0$ as $\varepsilon \to 0$. Let $\beta_\varepsilon = \mu_S (b_\varepsilon)^{-1}$. Then $\mu_S (\beta_\varepsilon b_\varepsilon) = 1$, i.e. $\beta_\varepsilon b_\varepsilon \in S$. Since B is bounded we have

$$p_\mu (b_\varepsilon - \beta_\varepsilon b_\varepsilon) = (1-\beta_\varepsilon) p_\mu (b_\varepsilon) \to 0$$

as $\varepsilon \to 0$. This contradicts (11) since $b_\varepsilon \in B$, $\beta_\varepsilon b_\varepsilon \in S$. ■

Mizoguchi [13] in 1990 presented a generalization of Daneš' drop theorem to locally convex spaces which turns out to be equivalent to Ekeland's principle and Caristi's fixed point theorem in uniform spaces. A stronger separation property is assumed, i.e. (8) has to be valid not for a single seminorm but for members of a base of seminorms. A careful analysis of the proof shows that it goes through assuming the separation property (8) only. We omit the details.

5 The Pareto efficiency theorem due to Isac

In a recent paper [12] G. Isac presented a version of the variational principle in locally convex spaces which relates it to vectorial optimization.

Following Isac, we shall call a point $x \in A \subset X$ an efficient point of A with respect to the pointed convex cone $K \subset X$ iff $A \cap (x+K) = \{x\}$. The set of efficient points of A with respect to the cone K is denoted by $\mathit{Eff}(A;K)$. The notion of efficiency replaces minimality if one deals with optimization of vector valued functions.

As before, let $\{p_\lambda\}_{\lambda\in\Lambda}$ be a base of continuous seminorms generating the topology on X. Let $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ be a family of positive numbers. We define the cone

$$K = \{(x,r) \in Y : \ r + \gamma_\lambda p_\lambda (x) \leq 0, \ \lambda \in \Lambda\}$$

which is a nuclear cone in the product space $Y = X \times \mathbb{R}$, see Isac [12].

The following theorem is a slightly simplified version of theorem 3 of [12] for sequentially complete locally convex spaces.

Theorem 8 *Let X be a Hausdorff sequentially complete locally convex topological vector space. Let $f : X \to (-\infty, +\infty]$ an extended-valued proper sequentially lower semicontinuous function, bounded from below. Then $\mathit{Eff}(\operatorname{epi} f; K)$ is nonempty and for each $x_0 \in X$ there exists some $x^* \in X$ such that*

$$(x^*, f(x^*)) \in (x_0, f(x_0)) + K \tag{12}$$

$$\left(x^*, f\left(x^*\right)\right) \in \mathit{Eff}\left(\mathit{epi}\, f; K\right). \tag{13}$$

Moreover, we prove the following

Theorem 9 *Theorem 9 and theorem 3 are mutually equivalent.*

Proof. First, let us note that the relations (12) and (4) are in fact equivalent.

Let $\left(x^*, f\left(x^*\right)\right) \in \mathit{Eff}(\mathrm{epi} f; K)$. Then, by definition

$$\mathrm{epi} f \cap \left(x^*, f\left(x^*\right)\right) + K = \left\{\left(x^*, f\left(x^*\right)\right)\right\}.$$

Thus, if $(x, f(x)) \in \mathrm{epi} f$, $(x, f(x)) \neq \left(x^*, f\left(x^*\right)\right)$ we have $(x, f(x)) - \left(x^*, f\left(x^*\right)\right) \notin K$, hence

$$f(x) - f\left(x^*\right) + \gamma_\lambda p_\mu\left(x - x^*\right) > 0$$

for some $\mu \in \Lambda$ which proves (5).

Conversely, let $(x, r) \in \mathrm{epi} f$, $(x, r) \neq \left(x^*, f\left(x^*\right)\right)$ and

$$(x, r) \in \left(x^*, f\left(x^*\right)\right) + K \tag{14}$$

Then, for each $\lambda \in \Lambda$

$$f(x) - f\left(x^*\right) + \gamma_\lambda p_\lambda\left(x - x^*\right) \leq r - f\left(x^*\right) + \gamma_\lambda p_\lambda\left(x - x^*\right) \leq 0$$

which contradicts (5). Hence an element $(x, r) \in \mathrm{epi} f$ such that $(x, r) \neq \left(x^*, f\left(x^*\right)\right)$ satisfying (14) does not exist which proves (13). ∎

6 Conclusion and comments

We summarize the considerations above to the following conclusions.

Conclusion 1 *Phelps' lemma (theorem 1), Ekeland's principle with Minkowski functional (theorems 2) and the drop theorem (theorem 7) in locally convex spaces are equivalent to their Banach space counterparts (theorem 10 - 12 of the appendix).*

Conclusion 2 *Phelps' lemma (theorem 1), both versions of Ekeland's principle (theorems 2, 3) and the drop theorem in locally convex spaces are equivalent to each other.*

Conclusion 3 *The drop theorem in locally convex spaces and Isac's efficiency theorem (theorem 8) are equivalent.*

This solves a problem stated by Isac in [12].

Conclusion 4 *Phelps' lemma (theorem 1), both versions of Ekeland's principle (theorems 2, 3) and the drop theorem in locally convex spaces are equivalent to Ekeland's principle in metric spaces.*

We conclude that Phelps' lemma 1 in [16] is in fact the first of a list of theorems equivalent to Ekeland's variational principle in metric spaces and to each other which characterize the property of completeness of the space they play in. We only mention Caristi's fixed point theorem, Penot's flower petal theorem and the equilibirum version of Ekeland's principle due to Oettli and Théra.

Of course, it is possible to put (and to show its equivalence) the mentioned theorems into the context of locally convex and even more general topological spaces, namely the $\mathcal{F}$-type spaces of Fang [7]. This has been done elsewhere [11].

7 Appendix

For the convenience of the reader we give the Banach space versions of Ekeland's principle, Phelps' lemma and Daneš' drop theorem.

Theorem 10 *Ekeland's variational principle in Banach spaces*
Let $(X, \|\cdot\|)$ *be a Banach space and* $f: X \to (-\infty, +\infty]$ *an extended-valued proper lower semicontinuous function, bounded from below. Then, for each* $\gamma > 0$, $x_0 \in domf$ *there exists* $x^* \in X$ *such that*

$$f(x^*) + \gamma \|x^* - x_0\| \le f(x_0),$$

$$f(x^*) < f(x) + \gamma \|x - x^*\| \quad \forall x \in X \setminus \{x^*\}.$$

Theorem 11 *Phelps' lemma in Banach spaces*
Let $(X, \|\cdot\|)$ *be a Banach space. Let Let* $A \subset X$ *a nonempty closed set and* $B \subset X$ *a nonempty closed bounded and convex set such that* $0 \notin B$. *Let* K *be the cone defined by*

$$K := \mathbb{R}_+ B = \{x = \alpha b : \alpha \ge 0, \ b \in B\}.$$

Then, for each $x_0 \in A$ *such that* $A \cap (x_0 + K)$ *is bounded and nonempty there exists* $x^* \in X$ *such that*

$$x^* \in A \cap (x_0 + K) \quad \textit{and} \quad \{x^*\} = A \cap (x^* + K).$$

Theorem 12 *Daneš' drop theorem in Banach spaces*
Let $(X, \|\cdot\|)$ *be a Banach space. Let* $A \subset X$ *a nonempty closed set and* $B \subset X$ *a nonempty closed bounded and convex set such that*

$$d(A, B) = \inf\{\|a - b\| : a \in A, \ b \in B\} > 0.$$

Then, for each $x_0 \in A$ *there exists* $x^* \in X$ *such that*

$$x^* \in A \cap D(x_0, B) \quad \textit{and} \quad \{x^*\} = A \cap D(x^*, B).$$

Finally, we quote the separation theorem (Bishop/Phelps [2]) used for the proof of theorems 4, 5.

Theorem 13 *Suppose that* A *and* B *are convex subsets of a real Hausdorff topological vector space, and that the interior of* B *is nonempty and disjoint from* A. *Then there exists a continuous linear functional* $l \neq 0$ *on* X *such that* $\sup l(A) \le \inf l(B)$.

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