STRIP INTEGRAL REPRESENTATIONS AND LAURENT EXPANSIONS FOR SOLUTIONS OF CAUCHY-RIEMANN SYSTEMS

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ABSTRACT. Strip integrals are constructed, by means of an averaging process, and applied to representing solutions of Cauchy-Riemann systems. The main result are (generalized) Laurent expansions for solutions of these systems in annuli, in particular in isolated singularities.

1. INTRODUCTION

The contour integral formulae of the classical function theory such as Cauchy integral theorem and formula also have counter parts for generalized Cauchy-Riemann systems (cf. [17], [5], [6], [8]), but require additional smoothness conditions on the coefficients of the respective systems. Thus it is quite natural, or even necessary, to ask for substitutes which yield the same as far as possible and remain in existence when the corresponding contour integrals might cease to exist. Such a substitute, called strip integral, is constructed here by means of an averaging process. This notion yields, in a simple manner, consistent versions of the Cauchy integral theorem and formula (theorem 2 and 3), a strip integral version for generalized Cauchy integrals (theorem 4 and 5), certain decompositions of solutions into simpler solutions (theorem 6 and 7) and (as the main result) generalized Laurent expansions for solutions of Cauchy-Riemann systems in annuli, in particular in isolated singularities (theorem 8). The formulae for the coefficients in these Laurent expansions, quite remarkably, possess the same structure as in the analytical case (which may be considered as another confirmation of the intrinsic strength of the equations $u_x = v_y$, $u_y = -v_x$).

The base for these expansions are generalized powers [7]; for certain classes of Cauchy-Riemann systems these generalized powers have been explicitly computed in [15], [16], [18], [19]. Generalized powers

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for Carleman-Vekua systems and, in more general terms, for pseudoanalytic functions of the first and second kind and corresponding Taylor expansions have been considered in [1], [2].

For technical reasons Cauchy-Riemann systems are called here again (ν,μ)-systems and their solutions (ν,μ)-solutions, cf. section 3 below.

2. Strip integrals

Let $\mathfrak{R}$ be the closure of a bounded two-fold connected domain in $\mathbb{C}$, $A(\cdot)$ be a schlicht mapping of $\mathfrak{R}$ onto an annulus

$$A(\mathfrak{R}) = \{w : R_1 \leq |w| \leq R_2\}, \quad 0 < R_1 < R_2 < \infty\,.$$

$A(z)$ is to possess continuous partial derivatives $A_z = \frac{\partial A(z)}{\partial z}$ and $A_{\bar z} = \frac{\partial A(z)}{\partial \bar z}$ in $\mathfrak{R}$ (i.e. A is to be continuously differentiable in a domain containing $\mathfrak{R}$), and to satisfy $|A_z|^2 - |A_{\bar z}|^2 > 0$ in $\mathfrak{R}$.

Let L_r be the image of the positively oriented circle

$$K_r = \{re^{i\varphi} : 0 \leq \varphi \leq 2\pi\}, \quad R_1 \leq r \leq R_2\,,$$

under the inverse mapping $A^{-1}(w)$ of $A(z)$ and let G_r denote the interior domain of the Jordan curve L_r. Let $A(\cdot)$ be monotone, which means here that $G_{r_1} \subset G_{r_2}$ if $R_1 \leq r_1 < r_2 \leq R_2$; then the L_r are also positively oriented.

Under these conditions on $\mathfrak{R}$ and $A(\cdot)$ we shall call $\mathbb{S} := (\mathfrak{R}, A)$ a (ring-)strip, $L_r = L_r(\mathbb{S})$ a strip line of $\mathbb{S}$, $\mathfrak{R}$ the strip region and A the strip mapping of $\mathbb{S}$. L_{R_1}, L_{R_2} are called the inner and outer boundary curve of $\mathbb{S}$, respectively.

Let $\mathbb{S}_j := (\mathfrak{R}_j, A_j)$, $j = 1, 2$, be strips in a domain G (i.e. $\mathfrak{R}_1 \cup \mathfrak{R}_2 \subset G$). $\mathbb{S}_1$ is said to be homotopic to $\mathbb{S}_2$ in G, $\mathbb{S}_1 \sim \mathbb{S}_2$ in G, if one strip line of $\mathbb{S}_1$ is homotopic in G to a strip line of $\mathbb{S}_2$. A strip $\mathbb{S}$ is called null-homologous in G, $\mathbb{S} \sim 0$ in G, if its strip lines are null-homologous in G.

Between the strips in domain G a half-ordering can be defined in an obvious manner. In particular $\mathbb{S}_1$ is said to be less than $\mathbb{S}_2$ (in G), $\mathbb{S}_1 \prec \mathbb{S}_2$, if $\mathbb{S}_1 \sim \mathbb{S}_2$ in G and the outer boundary curve of $\mathbb{S}_1$ is contained in the interior domain of the inner boundary curve of $\mathbb{S}_2$.

In the strip region $\mathfrak{R}$ of the strip $\mathbb{S} = (\mathfrak{R}, A)$ let two (for the present) continuous functions F, H be given, and let

$$c(r) := Re\frac{1}{2\pi i}\oint_{L_r(\mathbb{S})} Fdz + Hd\bar z \quad . \tag{1}$$

With

$$2\pi I(\mathbb{S}) := \int_{A(\mathfrak{R})} d\sigma = |A(\mathfrak{R})|$$

($|e|$ the Lebesgue measure of the measurable set e) let

$$C(\mathbb{S}) := \frac{1}{2\pi I(\mathbb{S})} \int_{A(\mathfrak{R})} c(|w|) d\sigma_w \quad . \tag{2}$$

An easy calculation (cf. [13]) yields

$$C(\mathbb{S}) = Re\ \frac{1}{2\pi I(\mathbb{S})} \int_{\mathfrak{R}} (F - \bar{H}) \frac{\partial |A(z)|^2}{\partial \bar{z}} d\sigma_z \quad . \tag{3}$$

In the following the functions F and H will be of the shape

$$F = f h_z \,, \quad H = f h_{\bar{z}} \quad . \tag{4}$$

In this case we have

$$C(\mathbb{S}) = Re\ \frac{1}{2\pi I(\mathbb{S})} \int_{\mathfrak{R}} f \left[h_z \frac{\partial |A(z)|^2}{\partial \bar{z}} - h_{\bar{z}} \frac{\partial |A(z)|^2}{\partial z} \right] d\sigma_z \,.$$

Putting, with $z = x + iy$,

$$h_z g_{\bar{z}} - h_{\bar{z}} g_z = \frac{i}{2}(h_x g_y - h_y g_x) =: \frac{\partial(h,g)}{\partial(z,\bar{z})} \tag{5}$$

and

$$\frac{1}{2\pi I(\mathbb{S})} \int_{\mathfrak{R}} f \frac{\partial(h, |A(z)|^2)}{\partial(z,\bar{z})} d\sigma_z =: \int_{\mathbb{S}} f[h] d\sigma \,, \tag{6}$$

then we have for $C(\mathbb{S})$ defined by (1), (2), (4)

$$C(\mathbb{S}) = Re \int_{\mathbb{S}} f[h] d\sigma \quad . \tag{7}$$

An expression as in (6) will be called a strip integral. Of course the symbol $[h]$ on the right-hand-side of (6) is to have only then its special meaning, if the "domain of integration" is a strip.

3. Strip integral formulae for (ν, μ)-solutions

Let a (ν, μ)-system, i.e. a linear uniformly elliptic system

$$f_{\bar{z}} = \nu f_z + \mu \overline{f_z} \tag{8}$$

be given in a domain G. The condition of uniform ellipticity of (8) means that

$$\nu, \mu \in L_\infty(\mathbb{C}) =: L_\infty \,, \quad \| |\nu| + |\mu| \|_{L_\infty} =: k < 1 \,, \tag{9}$$

and for the sake of simplicity we also assume that

$$\nu(z) = \mu(z) = 0 \quad \text{in} \quad \{|z| > R^*\} \tag{10}$$

with a fixed finite R^*.

Special cases of (ν, μ)-systems are the original Cauchy-Riemann system $f_{\bar{z}} = 0$, the Beltrami systems $f_{\bar{z}} = \nu f_z$, and the (anti-Beltrami) systems $f_{\bar{z}} = \mu \overline{f_z}$ defining, under certain additional smoothness conditions on μ, the (p, q)-analytic functions in the sense of Polozhii [5] as well as describing the Bers pseudoanalytic functions of the second kind [3]; if $\overline{\mu(z)}$ is an analytic function in G, then the schlicht solutions of $f_{\bar{z}} = \mu \overline{f_z}$ in G are harmonic mappings of G.

We will need a certain extension of the theorems 1 and 3 from [9], namely

Theorem 1. (I) *Let B, B_0 be bounded open subsets of $\mathbb{C}$, $B \subset\subset B_0$, and f be a (ν, μ)-solution in B_0, where ν, μ are to satisfy* (9), (10). *Then there exist $\nu_n(z)$, $\mu_n(z)$, $n = 1, 2, \cdots$, such that*

(a) $\nu_n(z) \to \nu(z), \quad \mu_n(z) \to \mu(z)$ *a.e. in* $\mathbb{C}$ *as* $n \to \infty$,
(b) $\| |\nu_n| + |\mu_n| \|_{L_\infty} \le \| |\nu| + |\mu| \|_{L_\infty} \quad \forall n = 1, 2, \cdots,$
(c) *for every* $\varepsilon > 0$ $\exists n_0(\varepsilon)$ *such that* $n_0(1) = 1$ *and* $supp(|\nu_n| + |\mu_n|) \subset supp(|\nu| + |\mu|) + \{z : |z| < \varepsilon\} \quad \forall n \ge n_0(\varepsilon),$
(d) ν_n , $\mu_n \in C^\infty(\mathbb{C}) \quad \forall n = 1, 2, \cdots$

and (ν_n, μ_n)-solutions f_n in B which tend uniformly to f in B as $n \to \infty$.

(II) *Let $H(t, z)$, $H_n(t, z)$ be the fundamental solutions of the (ν, μ)- and the (ν_n, μ_n)-systems, respectively, according to* [6], *chapter IV, and ν , μ , ν_n , μ_n are to satisfy the conditions mentioned in* (I). *Furthermore, let G_t, D_z be bounded domains in $\mathbb{C}$ of positive distance from each other and of the property that* $\arg(t - z)$ *can be defined uniquely, continuously and boundedly on $G_t \times D_z$, and D_t be an open set compactly contained in G_t. Then it holds*

(i) $|H_n(t, z)| \le K \quad \forall (t, z) \in D_t \times D_z$, *where K is a constant independent of n,*
(ii) $\|H_{nt}(\cdot, z) - H_t(\cdot, z)\|_{L_s(G_t)} \to 0$ *as* $n \to \infty$ *for each fixed* $s \in [1, 2)$ *uniformly* $\forall z \in \mathbb{C}$,
(iii) $\|H_t(\cdot, z)\|_{L_s(G_t)} \le K^* \quad \forall z \in \mathbb{C}$, *where the constant K^* depends only on k, R^*, G_t and $s \in [1, 2)$.*

We will postpone the proof of theorem 1 to section 6.

N.B. For certain classes of (ν, μ)-systems these fundamental solutions have been explicitly computed in [14], [16].

Let for the present $\nu, \mu \in C_0^\infty(\mathbb{C})$. Then every (ν, μ)-solution f in a domain G is in any case from $C^1(G)$. Let $\mathbb{S}_1$, $\mathbb{S}_2$ be strips homotopic in G. Then by the Cauchy integral theorem, cf. [6], for any strip lines

$L_{r_1}(\mathbb{S}_1)$, $L_{r_2}(\mathbb{S}_2)$ it holds

$$Re\,\frac{1}{i}\oint_{L_{r_1}(\mathbb{S}_1)} f dh = Re\,\frac{1}{i}\oint_{L_{r_2}(\mathbb{S}_2)} f dh \tag{11}$$

where h is a $(\nu,\overline{\mu})$-solution in G. Hence according to (1) - (7) we obtain

$$Re\int_{\mathbb{S}_1} f[h]d\sigma = Re\int_{\mathbb{S}_2} f[h]d\sigma\,. \tag{12}$$

Using theorem 1(I) and well-known compactness properties of (ν_n,μ_n)-solutions (e.g. cf. [6], theorem II.4.1), by means of a limiting process we obtain from (12) the following strip integral version of the Cauchy integral theorem for (ν,μ)-solutions :

Theorem 2. *Let ν, μ satisfy* (9)*, $\mathbb{S}_1$, $\mathbb{S}_2$ be strips homotopic in a domain G. Then for every (ν,μ)-solution f and every $(\nu,\overline{\mu})$-solution h in G it holds*

$$(i)\quad Re\int_{\mathbb{S}_1} f[h]d\sigma = Re\int_{\mathbb{S}_2} f[h]d\sigma\,;$$

if $\mathbb{S}$ is null-homologous in G then

$$(ii)\qquad Re\int_{\mathbb{S}} f[h]d\sigma = 0\ .$$

(Concerning the last assertion note that for null-homologous strip lines both sides of (11) are equal to zero.)

Let us say a set of points E were to lie within the strip $\mathbb{S}$, $E \prec \mathbb{S}$, if E is contained in the interior domain of the inner boundary curve of $\mathbb{S}$ (the latter called for short the interior domain of $\mathbb{S}$); accordingly we will say that E lies outside the strip $\mathbb{S}$, $\mathbb{S} \prec E$ ($\Longleftrightarrow E \succ \mathbb{S}$), if E is contained in the exterior domain of the outer boundary curve of $\mathbb{S}$ (the latter called for short the exterior domain of $\mathbb{S}$). If E consists only of a point z, then we will only write z in place of $\{z\}$ in the relations just mentioned. Moreover, $\mathbb{S}_1 \prec E \prec \mathbb{S}_2$ is to mean not only $\mathbb{S}_1 \prec E$ and $E \prec \mathbb{S}_2$ but also $\mathbb{S}_1 \prec \mathbb{S}_2$. If $\mathbb{S}_1 \prec E \prec \mathbb{S}_2$, then we will say that E lies between $\mathbb{S}_1$ and $\mathbb{S}_2$.

Let f be again a (ν,μ)-solution in a domain G, $M(t,z)$, $N(t,z)$ be the fundamental solutions of the $(\nu,\overline{\mu})$- and the $(\nu,-\overline{\mu})$-systems, respectively, and for the time being let ν, μ be from $C_0^\infty(\mathbb{C})$. Then $f(t)$, $M(t,z)$, $N(t,z)$ are also (in particular) C^1-functions in G, $\mathbb{C}\setminus\{z\}$, respectively. Let $\mathbb{S}^* \prec \mathbb{S}$ in G. Then, by the Cauchy integral formula, for every z between $\mathbb{S}^*$ and $\mathbb{S}$ we have

$$\begin{aligned} f(z) = {}& Re\frac{1}{2\pi i}\oint_{L_r(\mathbb{S})} f(t)d_tM(t,z) + iIm\frac{1}{2\pi i}\oint_{L_r(\mathbb{S})} f(t)d_tN(t,z) \\ & -Re\frac{1}{2\pi i}\oint_{L_{r^*}(\mathbb{S}^*)} f(t)d_tM(t,z) - iIm\frac{1}{2\pi i}\oint_{L_{r^*}(\mathbb{S}^*)} f(t)d_tN(t,z)\ . \end{aligned} \tag{13}$$

Analogously as with theorem 2 we obtain from (13) the following strip integral version of the Cauchy integral formula, which is independent of additional smoothness conditions on the coefficients ν, μ (apart from condition (10), which at this place only serves the purpose of a convenient definition of the fundamental solutions $M(\cdot, z)$, $N(\cdot, z)$).

Theorem 3. *Let the coefficients* ν, μ *satisfy* (9) *and* (10), f *be a* (ν, μ)-*solution in a domain* G, *and* $\mathbb{S}$, $\mathbb{S}^*$ *be strips with* $\mathbb{S}^* \prec \mathbb{S}$ *in* G. *Then*

$$(i) \quad f(z) = Re\int_{\mathbb{S}} f(t)[M(t,z)]d\sigma_t + iIm\int_{\mathbb{S}} f(t)[N(t,z)]d\sigma_t$$
$$-Re\int_{\mathbb{S}^*} f(t)[M(t,z)]d\sigma_t - iIm\int_{\mathbb{S}^*} f(t)[N(t,z)]d\sigma_t$$

for every z *between* $\mathbb{S}^*$ *and* $\mathbb{S}$.

If $\mathbb{S} \sim 0$ *in* G *then*

$$(ii) \quad Re\int_{\mathbb{S}} f(t)[M(t,z)]d\sigma_t + iIm\int_{\mathbb{S}} f(t)[N(t,z)]d\sigma_t = \begin{cases} f(z) & \forall z \prec \mathbb{S} \\ 0 & \forall z \succ \mathbb{S} \end{cases}$$

Proof. For smooth ν, μ we obtain (i) from (13) via the procedure (1)-(7). From this the validity of (i) for ν, μ satisfying only (9), (10) follows by means of a limiting process while using theorem 1 above.

The assertion (ii) follows from (i) in conjunction with theorem 2(ii) while observing the fact that ih is a $(\nu, \overline{\mu})$-solution if h is a $(\nu, -\overline{\mu})$-solution.

An analogue of theorem 3 of course also holds if z is situated in a domain which is “bounded” not only by two but by finitely many strips. Its formulation and proof are now obvious and may be omitted.

Furthermore, we will need a strip integral version of generalized Cauchy integrals. For this let u, v be real-valued functions continuous in the strip region $\mathfrak{R}$ of the strip $\mathbb{S} = (\mathfrak{R}, A)$, and let $f = u + iv$. Then

$$Re\int_{\mathbb{S}} f(t)[M(t,z)]d\sigma_t + iIm\int_{\mathbb{S}} f(t)[N(t,z)]d\sigma_t$$
$$= \int_{\mathbb{S}} u(t)[H^-(z,t)]d\sigma_t + i\int_{\mathbb{S}} v(t)[H(z,t)]d\sigma_t \tag{14}$$

where $H(t,z)$, $H^-(t,z)$ are the fundamental solutions of the (ν, μ)- and the $(\nu, -\mu)$-systems, respectively, according to [6], chapter IV. The relation (14) is a simple consequence of the interchange relations

$$H(z,t) = Re\, M(t,z) + iIm\, N(t,z) + i\pi \tag{15}$$

with the argument of $t - z$ suitably chosen, cf. [6], V.2.1, and

$$H^-(z,t) = Re\, N(t,z) + iIm\, M(t,z) + i\pi \tag{16}$$

(replace μ by$-\mu$ in (15)). This leads to

Theorem 4. *Let u, v be real-valued continuous functions in the strip region $\mathfrak{R}$ of the strip $\mathbb{S} = (\mathfrak{R}, A)$, and the coefficients ν, μ are to satisfy* (9), (10). *Then*

$$\int_{\mathbb{S}} u[H^-(z,\cdot)]d\sigma \quad \textit{and} \quad i\int_{\mathbb{S}} v[H(z,\cdot)]d\sigma$$

are (ν,μ)-solutions in $\mathbb{C}\setminus\mathfrak{R}$ with a zero at ∞.

Proof. First let additionally $\nu, \mu \in C^\infty(\mathbb{C})$. With an arbitrary $z_0 \in \mathbb{C}\setminus\mathfrak{R}$ we fix an open disk $\mathbb{K}$ with center z_0 of a positive distance from $\mathfrak{R}$. We divide $\mathfrak{R}$ into two simply connected closed regions B, B' with piece-wise smooth boundaries and such that $B \cap B'$ has no interior points and $B \cup B' = \mathfrak{R}$. There exists a bounded domain B_0 containing B and of positive distance to $\mathbb{K}$ such that (after a possible diminution of the radius of $\mathbb{K}$) B_0 and $\mathbb{K}$ satisfy the conditions of theorem 1(II) concerning $B_0(= G_t)$ and $\mathbb{K}(= D_z)$. Thus, for every $t \in B_0$, $H(z,t)$ can be defined uniquely in $\mathbb{K}$ and is a (ν,μ)-solution there. Hence every difference quotient

$$\frac{H(z,t+\Delta\xi) - H(z,t)}{\Delta\xi}, \quad \frac{H(z,t+i\Delta\eta) - H(z,t)}{\Delta\eta}$$

is a (ν,μ)-solution in $\mathbb{K}$, if $\Delta\xi$, $\Delta\eta$ are real, different from zero and of sufficiently small absolute value.

By [6], theorem II.5.2.(II) (under observing (15) as well as theorem 1(II) above), the limits of these difference quotients as $\Delta\xi \to 0$, $\Delta\eta \to 0$ exist and are uniformly bounded $\forall (t,z) \in B_0 \times \mathbb{K}$ (understand these difference quotients as corresponding expressions of $Re\, M(t,z)$, $Im\, N(t,z)$ and consider the set $\mathfrak{M}$ of theorem II.5.2 to be the set of $M(\cdot,z)$ or $N(\cdot,z)$, respectively, where z is considered as a parameter with range $\mathbb{K}$; the set $\mathfrak{P}$ mentioned there consists here only of the single pair (ν,μ)). Moreover, as the proof of theorem II.5.2 from [6] shows or as we see subsequently by means of the mean value theorem for derivatives, these difference quotients themselves are uniformly bounded $\forall (t,z) \in B \times \mathbb{K}$. Hence $H_\xi(z,t)$, $H_\eta(z,t)$ (where $t = \xi + i\eta$) are (ν,μ)-solutions in $\mathbb{K}$ uniformly bounded for every $t \in B$.

Due to the smoothness of ν, μ the derivatives $H_{\xi z}(z,t)$, $H_{\xi\bar z}(z,t)$ are continuous and uniformly bounded in $\mathbb{K}$ for every $t \in B$. Therefore

$$\int_B v^*(t)H_\xi(z,t)d\sigma_t \quad \text{with} \quad v^*(t) = \frac{-1}{4\pi I(\mathbb{S})}v(t)\frac{\partial |A(t)|^2}{\partial\eta}$$

may be differentiated under the integral sign and thus is also a (ν,μ)-solution in $\mathbb{K}$. The same holds for the other three parts of $i\int_{\mathbb{S}} v[H(z,\cdot)]d\sigma$.

Due to the arbitrariness of z_0 this proves that $F(z) := i \int_{\mathbb{S}} v[H(z,\cdot)]d\sigma$ is a (ν,μ)-solution in $\mathbb{C} \setminus \mathfrak{R}$ (if ν, μ are smooth).

Next, we show that $F(\infty) = 0$. First we observe that

$$H(z,t) = \log(z-t) + a(z,t)$$

where $a(z,t)$ is an analytic function of z in $\{|z| > R^*\}$ for each $t \in \mathbb{C}$, and

$$\lim_{z\to\infty} a(z,t) = 0 \quad \forall t \in \mathbb{C}.$$

(cf. [6], definition IV.3.3).

Let $U(\infty) = \{|z| > R_\infty\}$ where R_∞ is any constant greater than $\max\{R^*, \sup\{|z| : z \in \mathfrak{R}\}\}$. The difference quotient

$$\Delta(z,t,n) := \frac{H(z,t+h_n) - H(z,t)}{h_n} \ ,$$

h_n real and $\neq 0$, $h_n \to 0$ as $n \to \infty$, is an analytic function of z in the closure of $U(\infty)$ with a zero at ∞ for every n. By the maximum modulus principle, $|\Delta(z,t,n)|$ attains its maximum in the closure of $U(\infty)$ on $\{|z| = R_\infty\}$. On this bounded set the $\Delta(z,t,n)$ are uniformly bounded $\forall t \in \mathfrak{R}$, $\forall n$, according to the considerations at the beginning of the proof (under using a covering of $\{|z| = R_\infty\}$ by a finite number of sufficiently small disks). Thus the analytic functions $\Delta(\cdot,t,n)$ are uniformly bounded also in $U(\infty)$ for these t and n, $H_\xi(z,t)$ is in $U(\infty)$ also uniformly bounded $\forall t \in \mathfrak{R}$, and $H_\xi(\infty,t) = 0$. Therefore, $\int_B v^*(t)H_\xi(z,t)d\sigma_t$ has also a zero at ∞ (e.g. by the Lebesgue theorem on dominated convergence). In the same way the assertion follows for the remaining parts of $i \int_{\mathbb{S}} v[H(z,\cdot)]d\sigma$ as well as for $\int_{\mathbb{S}} u[H^-(z,\cdot)]d\sigma$.

The validity of theorem 4 for ν, μ, which only satisfy (9), (10), is now followed immediately by means of theorem 1 above and e.g. [6], theorem II.4.1, which concludes the proof of theorem 4.

By means of theorem 1 we obtain further in an obvious way

Theorem 5. *Let E be a bounded measurable set $\subset \mathbb{C}$, let ν, μ satisfy* (9), (10), *and v_1, v_2 be real-valued functions from $L_p(E)$, $p > 2$.*

Then (with $t = \xi + i\eta$)

$$\int_E (v_1(t)H_\xi(z,t) + v_2(t)H_\eta(z,t))d\sigma_t$$

is a (ν,μ)-solution in $\mathbb{C} \setminus$ (closure of E) with a zero at ∞.

For proof one merely has to approximate $v_j(t)\chi_E(t)$ (χ_E the characteristic function of E) by appropriate continuous functions, to apply theorem 4 and to carry out a limiting process using also theorem 1.

(A more detailed consideration, which however requires an extension of theorem 1(II), shows that theorem 5 also holds with $p = 1$.)

4. Decomposition of (ν,μ)-solutions

Now let G be a two-fold connected domain $\subset \mathbb{C}$, and let

$G_1 = G \cup D_1$, D_1 the bounded component of $\mathbb{C} \setminus G$,
$G_2 = G \cup D_2$, D_2 the unbounded component of $(\mathbb{C} \cup \{\infty\}) \setminus G$.

Until further notice, the coefficients ν, μ are to satisfy the conditions (9), (10). Then the following theorem holds.

Theorem 6. *Let f be a (ν,μ)-solution in G. Then there exist (ν,μ)-solutions f_1 in G_1 and f_2 in G_2 with $f_2(\infty) = 0$ such that*

$$f = f_1 + f_2 \quad \text{in} \quad G\ .$$

f_1 is called the regular part of f (with respect to G), f_2 is called the principle part of f (with respect to G). f_1, f_2 are uniquely determined by these properties.

Proof. Let E_n, $n = 1, 2, \cdots$, be a sequence of open sets such that $E_n \subset E_{n+1}$, $E_n \subset\subset G$ $\forall n$ and $\bigcup\limits_n E_n = G$. There are two sequences of strips $\mathbb{S}_n^1$, $\mathbb{S}_n^2$ in G such that

$$\mathbb{S}_{n+1}^2 \prec \mathbb{S}_n^2 \prec \mathbb{S}_n^1 \prec \mathbb{S}_{n+1}^1$$

and

$$\mathbb{S}_n^2 \prec E_n \prec \mathbb{S}_n^1$$

for every n. By theorem 3 and (14)

$$f(z) = \int\limits_{\mathbb{S}_m^1} (u[H^-(z,\cdot)] + iv[H(z,\cdot])d\sigma - \int\limits_{\mathbb{S}_l^2} (u[H^-(z,\cdot)] + iv[H(z,\cdot])d\sigma$$

$\forall z \in E_n$ if $m, l \geq n$.

Let

$$f_{j,m}(z) = (-1)^{j-1} \int\limits_{\mathbb{S}_m^j} (u[H^-(z,\cdot)] + iv[H(z,\cdot])d\sigma\ , \quad j = 1, 2\,.$$

Then

$$f(z) = f_{1,m}(z) + f_{2,l}(z) \quad \text{in} \quad E_n \quad \forall m, l \geq n\ .$$

If one keeps l fixed, this implies that $f_{1,m}(z)$ does not depend on m for $z \in E_n$. By theorem 4, $f_{1,m}(z)$ is a (ν,μ)-solution (in particular) in the interior domain of $\mathbb{S}_n^1$, and by the uniqueness theorem for (ν,μ)-solutions $f_{1,m}(z)$ does not depend on m there as well. Hence

$$f_1(z) := \lim_{m\to\infty} f_{1,m}(z)$$

is a (ν, μ)-solution in the interior domain of $\mathbb{S}_n^1$ for every n, i.e. $f_1(z)$ is a (ν, μ)-solution in G_1, and

$$f(z) = f_1(z) + f_{2,l}(z) \quad \text{in} \quad E_n \quad \forall l \geq n \ .$$

In the same way we now find an $f_2(z)$ with the properties asserted. The uniqueness of these f_1, f_2 is an immediate consequence of Liouville's theorem for (ν, μ)-solutions, which finishes the proof of theorem 6.

If the bounded component of $\mathbb{C} \setminus G$ consists only of a point a, then the principle part of f with respect to G is called principle part of f at a. If the unbounded component of $\mathbb{C} \setminus G$ is empty, then the regular part of f with respect to G is called principle part of f at ∞.

Of course, theorem 6 has an obvious analogue, if G is an n-fold connected subdomain of $\mathbb{C}$. We only state this in a special case.

Theorem 7. *Let f be a (ν, μ)-solution in $\mathbb{C} \setminus \{a_1, a_2, \cdots, a_N\}$, where $a_1, \cdots, a_N$ are finite points.* (i) *Then*

$$f(z) = f_1(z, \infty) + \sum_{j=1}^{N} f_2(z, a_j) \ ,$$

where $f_1(z, \infty)$, $f_2(z, a_j)$ mean the principle parts of f at ∞ and a_j, respectively.

(ii) *If a_j is a pole of f then $f_2(z, a_j)$ also has a pole at a_j, and if a_j is an essential singularity of f then also of $f_2(z, a_j)$. The corresponding statements hold for $f_1(z, \infty)$.*

(If ν, μ satisfy e.g. a Bojarski condition at a_j, cf. (19) below, then more can be said on account of theorem 8 below.)

Proof. Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be the principle part of the (ordinary) Laurent expansion of $f(z)$ at ∞. Then

$$UP(\cdot)(z) =: f_1(z, \infty)$$

(U the transformation defined in [6], theorem IV.1.9) is a (ν, μ)-solution in $\mathbb{C}$, and its principle part of the ordinary Laurent expansion at ∞ is the same as that of f. Hence $f^*(z) := f(z) - f_1(z, \infty)$ is also a (ν, μ)-solution in $\mathbb{C} \setminus \{a_1, a_2, \cdots, a_N\}$, and $f^*(\infty) = 0$.

Let $f_2^*(z, a_j)$ be the principle part of f^* at a_j, i.e. that unique (ν, μ)-solution in $\mathbb{C} \setminus \{a_j\}$ with a zero at ∞, which satisfies

$$f^*(z) = f_1^*(z) + f_2^*(z, a_j)$$

where $f_1^*(z)$ is a (regular) (ν, μ)-solution in a full neighbourhood of a_j; the uniqueness of this decomposition is again an obvious consequence of Liouville's theorem. (Analogously with the proof of theorem 6, this $f_2^*(z, a_j)$ arises as the limit of a sequence

$$-Re \int_{\mathbb{S}_n} f^*(t)[M(t,z)]d\sigma_t - iIm \int_{\mathbb{S}_n} f^*(t)[N(t,z)]d\sigma_t$$

with e.g. $\mathbb{S}_n = (\mathfrak{R}_n, A)$, $\mathfrak{R}_n = \{t : r_{1n} \leq |t - a_j| \leq r_{2n}\}$, $0 < r_{1n} < r_{2n}$ $\forall n \in \mathbb{N}$, $r_{2n} \to 0$ as $n \to \infty$, and $A(t) = t - a_j$.)

With the principle part $f_2(z, a_j)$ of f at a_j defined in the same way, it must hold $f_2(z, a_j) = f_2^*(z, a_j)$ (because $f(z) - f^*(z)$ is a (ν, μ)-solution even in $\mathbb{C}$). Thus

$$h(z) := f^*(z) - \sum_{j=1}^{N} f_2(z, a_j)$$

is a (ν, μ)-solution in $\mathbb{C} \setminus \{a_1, a_2, \cdots, a_N\}$ with a zero at ∞ and removable singularities at the a_j. We obtain then from Liouville's theorem our assertion (i). Assertion (ii) is now an immediate consequence of e.g. the Casorati-Weierstraß theorem for (ν, μ)-solutions.

In the analytic case, $\nu(z) = \mu(z) \equiv 0$ in $\mathbb{C}$, the statement of theorem 6 is an immediate consequence of the Cauchy integral formula and simple properties of Cauchy integrals. If, in theorem 7, the points $a_1, \cdots, a_N$ and ∞ are at most poles of f and if again $\nu(z) = \mu(z) \equiv 0$ in $\mathbb{C}$, then theorem 7(i) simply means the representation of f by partial fractions.

5. Generalized Laurent expansions for (ν, μ)-solutions

Now let G be an annulus, and without loss of generality we assume that

$$G = \{z : \rho < |z| < \rho^*\} \quad \text{where} \quad 0 \leq \rho < \rho^* \leq \infty \ . \tag{17}$$

Until further notice, a strip $\mathbb{S} = (\mathfrak{R}, A)$ in G is now to be of the special shape

$$\mathfrak{R} = \{z : R_1 \leq |z| \leq R_2\} \quad \text{with} \quad \rho < R_1 < R_2 < \rho^* \tag{18}$$

and $A(z) \equiv z$ in $\mathfrak{R}$.

The coefficients ν, μ are now supposed to satisfy (9), (10) as well as a Bojarski condition, cf. [4], p. 499, at the center of G, i.e. here

$$\frac{\nu(z) - \nu(0)}{z}, \frac{\mu(z) - \mu(0)}{z} \in L_p(\mathbb{C}) \quad \text{with a} \quad p > 2 \ . \tag{19}$$

Then the existence and uniqueness of the generalized powers $[az^n]_{\nu,\mu}$ of the (ν, μ)-system (and also of the $(\nu, \overline{\mu})$- and the $(\nu, -\overline{\mu})$-systems) are assured and we may ask for expansions of a (ν,μ)-solution f in (at least subsets of) G of the form

$$f(z) = \sum_{n=-\infty}^{+\infty} [a_n z^n]_{\nu,\mu} \; . \tag{20}$$

As in [10] let

$$\Lambda(z,g) := (1 - \frac{\bar{z}}{z}\nu(z))g(z) - \mu(z)\overline{g(z)} \; . \tag{21}$$

Further let

$$\Omega_j(t,g) := Re\left\{\frac{\partial[t^{-j}]_{\nu,\overline{\mu}}}{\partial t} t\Lambda(t,g)\right\} + iIm\left\{\frac{\partial[t^{-j}]_{\nu,-\overline{\mu}}}{\partial t} t\Lambda(t,g)\right\} \tag{22}$$

for every $j \in \mathbb{Z}\setminus\{0\}$, and

$$\Omega_0(t,z,g) := Re\left\{M_t(t,z)t\Lambda(t,g)\right\} + iIm\left\{N_t(t,z)t\Lambda(t,g)\right\} \; ,$$

$$\Omega_0(t,g) := \Omega_0(t,0,g) \, . \tag{23}$$

Let $\mathbb{S} = (\mathfrak{R}, id)$ be a strip in G, and g be any (ν,μ)-solution in G. A simple calculation shows that

$$\begin{aligned} Re\int_{\mathbb{S}} g(t)\left[[t^{-j}]_{\nu,\overline{\mu}}\right] d\sigma_t + iIm\int_{\mathbb{S}} g(t)\left[[t^{-j}]_{\nu,-\overline{\mu}}\right] d\sigma_t \\ = \frac{1}{2\pi I(\mathbb{S})}\int_{\mathfrak{R}} \Omega_j(t,g)d\sigma_t \end{aligned} \tag{24}$$

for every $j \in \mathbb{Z}\setminus\{0\}$ and

$$\begin{aligned} Re\int_{\mathbb{S}} g[M(\cdot,z)]d\sigma + iIm\int_{\mathbb{S}} g[N(\cdot,z)]d\sigma \\ = \frac{1}{2\pi I(\mathbb{S})}\int_{\mathfrak{R}} \Omega_0(t,z,g)d\sigma_t \end{aligned} \tag{25}$$

for $z \notin \mathfrak{R}$. Thus, by theorem 2(i),

$$\frac{1}{|\mathfrak{R}|}\int_{\mathfrak{R}} \Omega_j(t,g)d\sigma_t \tag{26}$$

is for every $j \in \mathbb{Z}\setminus\{0\}$ independent of $\mathfrak{R}$ for every $\mathfrak{R}$ as in (18). Further, because of (14) and (25),

$$\frac{1}{|\mathfrak{R}|}\int_{\mathfrak{R}} \Omega_0(t,g)d\sigma_t = g_1(0) \tag{27}$$

for every $\mathfrak{R}$ as in (18), where g_1 is the regular part of g with respect to G, cf. theorem 6 and its proof. Hence, also the left-hand-side in (27) is independent of these $\mathfrak{R}$.

The next theorem clarifies the question after the above mentioned generalized Laurent expansions.

Theorem 8. *Let f be a (ν,μ)-solution in G, G as in* (17)*, and ν, μ are to satisfy* (9)*,* (10) *and* (19)*. With f_1, f_2 being the regular and principle parts, respectively, of f with respect to G, it holds*

$$(i) \quad f_1(z) = \sum_{j=0}^{\infty} [a_j z^j]_{\nu,\mu} \quad in \quad \{|z| < \rho^* \kappa^{-2}\}$$

(where $[a_0 z^0]_{\nu,\mu} := a_0$) and

$$(ii) \quad f_2(z) = \sum_{j=1}^{\infty} [a_{-j} z^{-j}]_{\nu,\mu} \quad in \quad \{|z| > \rho\kappa^2\} \ ,$$

κ a constant from $[1,\infty)$ depending only on ν, μ (cf. (32) *below). The series in (i), (ii) converge absolutely and uniformly in every compact subset of the respective domains. The constants a_j are uniquely determined for every $j \in \mathbb{Z}$ and are given by*

$$a_j = \tfrac{-1}{j(1-|b|^2)} \left\{ Re \int_{\mathbb{S}} f(t) \left[[t^{-j}]_{\nu,\overline{\mu}} \right] d\sigma_t + iIm \int_{\mathbb{S}} f(t) \left[[t^{-j}]_{\nu,-\overline{\mu}} \right] d\sigma_t \right\}$$

for every $j \in \mathbb{Z} \setminus \{0\}$ and

$$a_0 = Re \int_{\mathbb{S}} f(t)[M(t,0)] d\sigma_t + iIm \int_{\mathbb{S}} f(t)[N(t,0)] d\sigma_t$$

where $\mathbb{S}$ is any strip in G homotopic to a strip as with (18)*, and b is an algebraic expression of $\nu(0)$, $\mu(0)$ with $|b| \le k$ (cf.* [6]*, p. 52).*

Thus, if

$$\frac{\rho^*}{\rho} > \kappa^4$$

then the series (i), (ii) have a common domain of convergence and hence there exists an expansion (20) of f itself. This (of course only sufficient) condition is in any case satisfied if $\rho = 0$ or if $\rho^* = \infty$. In particular a (ν,μ)-solution f admits a generalized Laurent expansion (20) at each isolated singularity, and the type of the expansion then must correspond in the usual way with the type of the singularity, i.e. for instance if it is a pole of order N, then the expansion must read

$$f(z) = \sum_{n=-N}^{+\infty} [a_n z^n]_{\nu,\mu} \quad \text{with} \quad a_{-N} \neq 0 \ ;$$

the latter follows in an obvious way from general properties of (ν,μ)-solutions at poles where a condition (19) is satisfied, cf. [6], theorem III.3.2, from the asymptotic behaviour of generalized powers, cf. [7], and from the uniqueness statement concerning the a_j in theorem 8.

In the Beltrami case, $\mu \equiv 0$, the formulae for the a_j simplify to

$$a_j = \frac{-1}{j} \int_{\mathbb{S}} f(t) \left[[t^{-j}]_{\nu,0} \right] d\sigma_t \quad \forall j \in \mathbb{Z} \setminus \{0\}$$

(because $b = 0$ if $\mu(0) = 0$) and

$$a_0 = \int_{\mathbb{S}} f(t) [\log [t^1]_{\nu,0}] d\sigma_t \,;$$

moreover, $[t^k]_{\nu,0} = ([t^1]_{\nu,0})^k$ for every $k \in \mathbb{Z}$.

If also $\nu \equiv 0$ in $\mathbb{C}$, then $[t^1]_{\nu,0} \equiv t$, $\kappa = 1$, and

$$a_j = \frac{-1}{j} \int_{\mathbb{S}} f(t) [t^{-j}] d\sigma_t \quad \forall j \in \mathbb{Z} \setminus \{0\} \,,$$

$$a_0 = \int_{\mathbb{S}} f(t) [\log t\,] d\sigma_t$$

are the elementary Laurent coefficients of f in the annulus G, now in strip integral form (arising also directly by means of the averaging procedure of section 2 from the usual contour integral formulae for the a_j).

Proof of theorem 8. We start with proving (ii). By theorem 6, f_2 is a (ν, μ)-solution in $G_2 = \{|z| > \rho\}$ with a zero at ∞. According to the considerations near (26), (27), now applied to f_2 in G_2,

$$\frac{1}{|\mathfrak{R}|} \int_{\mathfrak{R}} \Omega_j(t, f_2) d\sigma_t$$

is, for each $j \in \mathbb{Z}$, independent of $\mathfrak{R} = \{z : r_1 \le |z| \le r_2\}$ with $\rho < r_1 < r_2 < \infty$.

To start with we assume that f_2 admits an expansion with the properties mentioned above. Then there exists an $\mathfrak{R} = \{z : r_1 \le |z| \le r_2\}$ contained in G_2 where the expansion for f_2 converges uniformly. Hence, for each $l \in \mathbb{Z}$,

$$\int_{\mathfrak{R}} \Omega_l(t, f_2) d\sigma_t = \sum_{j=1}^{\infty} \int_{\mathfrak{R}} \Omega_l(t, [a_{-j} t^{-j}]_{\nu,\mu}) d\sigma_t \,,$$

and that now again for every annulus $\mathfrak{R}$ with $\rho < r_1 < r_2 < \infty$ and center zero.

By [10], theorem 3,

$$\frac{1}{|\mathfrak{R}|} \int_{\mathfrak{R}} \Omega_l(t, [a t^n]_{\nu,\mu}) d\sigma_t = -na(1 - |b|^2) \delta_{l,n} \tag{28}$$

$\forall l, n \in \mathbb{Z}$, $l \neq 0$; here b is the constant mentioned in theorem 8, and $\delta_{l,n}$ means the Kronecker symbol.

Thus for every $j \in \mathbb{N}$

$$j(1-|b|^2)a_{-j} = \frac{1}{|\mathfrak{R}|}\int_{\mathfrak{R}} \Omega_{-j}(t,f_2)d\sigma_t = \frac{1}{|\mathfrak{R}^*|}\int_{\mathfrak{R}^*} \Omega_{-j}(t,f)d\sigma_t \tag{29}$$

(the latter by the Cauchy integral theorem, applied to the regular part f_1 of f, where $\mathfrak{R}^*$ is an annulus with center at zero contained in G).

Next we ask for the convergence of the series to the right of (ii) with the a_{-j} defined by (29). First we obtain the estimate

$$|a_{-j}|(1-|b|^2)j \leq \frac{1}{\pi(r_2^2-r_1^2)} \int\limits_{r_1<|t|<r_2} \left(\left|\frac{\partial [t^j]_{\nu,\overline{\mu}}}{\partial t}\right| + \left|\frac{\partial [t^j]_{\nu,-\overline{\mu}}}{\partial t}\right| \right) |t\Lambda(t,f_2)| d\sigma_t \,. \tag{30}$$

Further we have

$$[t^j]_{\nu,\overline{\mu}} = (\chi_1(t))^j\,, \quad [t^j]_{\nu,-\overline{\mu}} = (\chi_2(t))^j \tag{31}$$

where χ_1, χ_2 are $\frac{1+k}{1-k}$-quasiconformal mappings of $\mathbb{C}$ onto itself with $\chi_1(0)=\chi_2(0)=0$. By [12] there exists a constant $\kappa \geq 1$ depending only on k, on the L_p-norms of $\frac{\nu(z)-\nu(0)}{z}$ and $\frac{\mu(z)-\mu(0)}{z}$ and on R^*, such that for every $\chi=\chi_1$ or $\chi=\chi_2$ from (31) with any $j \in \mathbb{Z}\setminus\{0\}$ it holds

$$\kappa \geq \left|\frac{\chi(t)}{t}\right| \geq \frac{1}{\kappa} \quad \forall t \in \mathbb{C}\setminus\{0\} \tag{32}$$

(for concrete values of κ in special cases cf. [19], p. 107, [16]).

Putting $M(r,h)=\max\{|h(t)| : |t|=r\}$ we thus have

$$\begin{aligned} &\int\limits_{r_1<|t|<r_2} \left|\frac{\partial [t^j]_{\nu,\overline{\mu}}}{\partial t}\right| |t\Lambda(t,f_2)| d\sigma_t \leq \\ &\quad j \int\limits_{r_1<|t|<r_2} |\chi_{1t}(t)| \left|\frac{\chi_1(t)}{t}\right|^{j-1} |t|^j (1+k)|f_2(t)| d\sigma_t \leq \\ &\quad (1+k)j\kappa^{j-1}M(r_1,f_2) \left(\int\limits_{r_1<|t|<r_2} |\chi_{1t}(t)|^2 d\sigma_t \right)^{1/2} \cdot \sqrt{\pi(r_2^2-r_1^2)}\, r_2^j \,. \end{aligned}$$

Furthermore, because of (32) and $|\chi_{\bar{t}}| \leq k|\chi_t|$,

$$\int\limits_{r_1<|t|<r_2} |\chi_t(t)|^2 d\sigma_t \leq \frac{\pi\kappa^2 r_2^2}{1-k^2} \tag{33}$$

for $\chi=\chi_1$ and $\chi=\chi_2$.

Now let $r_2 = dr_1$, $d > 1$. Then

$$\frac{1}{\pi(d^2r_1^2 - r_1^2)} \int\limits_{r_1<|t|<dr_1} \left|\frac{\partial[t^j]_{\nu,\overline{\mu}}}{\partial t}\right| |t\Lambda(t,f_2)| d\sigma_t$$

$$\leq j\sqrt{\frac{1+k}{1-k}} M(r_1,f_2) \frac{d}{\sqrt{d^2-1}} (dr_1\kappa)^j \,.$$

The same estimate exists for the other part on the right-hand-side in (30). Thus for any fixed $d > 1$, for any fixed $r_1 > \rho$ and every $j \in \mathbb{N}$ it holds

$$|a_{-j}| \leq \frac{2M(r_1,f_2)}{\sqrt{(1-k)^3}} \frac{d}{\sqrt{d^2-1}} (dr_1\kappa)^j \,. \tag{34}$$

By [12]

$$|[az^l]_{\nu,\mu}| \ \leq |a|\kappa^{|l|}|z|^l \tag{35}$$

for every $l \in \mathbb{Z}$ and every $z \in \mathbb{C} \setminus \{0\}$, hence

$$|[a_{-j}z^{-j}]_{\nu,\mu}| \leq |a_{-j}|\kappa^j|z|^{-j} \quad \forall j \in \mathbb{N},$$

which gives in conjunction with (34) the absolute and uniform convergence of the series on the right-hand-side of (ii) in every compact subset of $G_2^* := \{|z| > \kappa^2\rho\}$.

Next we have to show that

$$h(z) := f_2(z) - \sum_{j=1}^{\infty} [a_{-j}z^{-j}]_{\nu,\mu}$$

vanishes identically in G_2^*. In any case, h is a (ν,μ)-solution in G_2^* and $h(\infty) = 0$. If $h \not\equiv 0$, then h were to posses a Laurent expansion

$$h(z) =: \frac{\alpha_m}{z^m} + \frac{\alpha_{m+1}}{z^{m+1}} + \cdots$$

at ∞ with $m \geq 1$ and $\alpha_m \neq 0$. There exists exactly one $[c_{-m}z^{-m}]_{\nu,\mu}$ with $c_{-m} \neq 0$ whose Laurent expansion at ∞ starts with $\frac{\alpha_m}{z^m}$. Hence

$$g(z) := h(z) - [c_{-m}z^{-m}]_{\nu,\mu}$$

is a (ν,μ)-solution in G_2^*, and

$$g(z) = \frac{\beta_{m+1}}{z^{m+1}} + \frac{\beta_{m+2}}{z^{m+2}} + \cdots \tag{36}$$

for $|z| > \max\{R^*, \kappa^2\rho\} =: R^{**}$.

Because of (28), (29),

$$\int_{\mathfrak{R}} \Omega_{-m}(t,h) d\sigma_t = 0 \tag{37}$$

for every $\mathfrak{R} = \{z : r_1 \le |z| \le r_2\} \subset G_2^*$. On the other hand, by (28),

$$\frac{1}{|\mathfrak{R}|}\int_{\mathfrak{R}} \Omega_{-m}(t,[c_{-m}t^{-m}]_{\nu,\mu} + g(t))d\sigma_t = \\ mc_{-m}(1-|b|^2) + \frac{1}{|\mathfrak{R}|}\int_{\mathfrak{R}} \Omega_{-m}(t,g(t))d\sigma_t\,. \tag{38}$$

Observing (31) (with $j = m$) and (32) we obtain

$$\left|\frac{1}{|\mathfrak{R}|}\int_{\mathfrak{R}} \Omega_{-m}(t,g(t))d\sigma_t\right| \le \\ \frac{m}{\pi(r_2^2-r_1^2)}\int_{\mathfrak{R}}\left(|\chi_{1t}\chi_1^{m-1}| + |\chi_{2t}\chi_2^{m-1}|\right)|t|(1+k)|g(t)|d\sigma_t \le \\ \frac{m(1+k)}{\pi(r_2^2-r_1^2)}\int_{\mathfrak{R}}(|\chi_{1t}|+|\chi_{2t}|)\,\kappa^{m-1}|t|^m|g(t)|d\sigma_t \le \\ \frac{m(1+k)\kappa^{m-1}}{\pi(r_2^2-r_1^2)}\left(\int_{\mathfrak{R}}(|\chi_{1t}|+|\chi_{2t}|)^2 d\sigma_t\right)^{1/2}\left(\int_{\mathfrak{R}}|t^m g(t)|^2 d\sigma_t\right)^{1/2}. \tag{39}$$

Let e.g. $r_2 = 2r_1$, $r_1 > R^{**}$. By (33) and (36),

$$\frac{1}{|\mathfrak{R}|}\int_{\mathfrak{R}} \Omega_{-m}(t,g(t))d\sigma_t = O\left(\frac{1}{r_1}\right). \tag{40}$$

From (37), (38) we then obtain a contradiction to $c_{-m} \ne 0$ by letting r_1 tend to ∞. Thus $h(z) \equiv 0$ in G_2^*.

Now we come to the proof of (i). By [11] an expansion (i) with the properties mentioned in theorem 8 exists if additionally

$$\lim_{z\to 0}\nu(z) = \nu(0)\,, \quad \lim_{z\to 0}\mu(z) = \mu(0)$$

is supposed. As indicated in [12] this condition can subsequently be eliminated. But it can be avoided, from the beginning, as we will now show.

f_1 as being the regular part of f with respect to G is a (ν,μ)-solution in $G_1 = \{|z| < \rho^*\}$. If f_1 admits an expansion (i) with the properties asserted, then there exists an $\mathfrak{R} = \{z : r_1 \le |z| \le r_2\} \subset G_1$ such that

$$\int_{\mathfrak{R}} \Omega_l(t,f_1)d\sigma_t = \sum_{j=0}^{\infty}\int_{\mathfrak{R}} \Omega_l(t,[a_j t^j]_{\nu,\mu})d\sigma_t$$

for every $l \in \mathbb{Z}$, and

$$\frac{1}{|\mathfrak{R}|}\int_{\mathfrak{R}} \Omega_l(t,f_1)d\sigma_t$$

is for each $l \in \mathbb{Z}$ independent of the above mentioned $\mathfrak{R} \subset G_1$. Due to (28) we obtain for these $\mathfrak{R}$ and for $\mathfrak{R}^*$ as in (29)

$$-j(1-|b|^2)a_j = \frac{1}{|\mathfrak{R}|}\int_{\mathfrak{R}} \Omega_j(t, f_1)d\sigma_t = \frac{1}{|\mathfrak{R}^*|}\int_{\mathfrak{R}^*} \Omega_j(t, f)d\sigma_t \tag{41}$$

for every $j \in \mathbb{N}$ (the latter on account of e.g. the expansion (ii) for f_2 already shown) and

$$\begin{aligned} a_0 = f_1(0) &= \frac{1}{|\mathfrak{R}|}\int_{\mathfrak{R}} \Omega_0(t, f_1)d\sigma_t \\ &= \frac{1}{|\mathfrak{R}^*|}\int_{\mathfrak{R}^*} \Omega_0(t, f)d\sigma_t \end{aligned} \tag{42}$$

because of (27) and theorem $3(ii)$.

Analogously with (30), (31) (with $-j$ in place of j) we obtain the estimate

$$\begin{aligned} &|a_j|(1-k^2) \le \\ &\quad \frac{1}{\pi(r_2^2-r_1^2)}\int_{\mathfrak{R}} \left(|\chi_{1t}\chi_1^{-j-1}| + |\chi_{2t}\chi_2^{-j-1}|\right) |t\Lambda(t, f_1(t)|d\sigma_t \end{aligned} \tag{43}$$

for $j \ge 1$, where χ_1, χ_2 are again $\frac{1+k}{1-k}$-quasiconformal mappings of $\mathbb{C}$ onto itself satisfying (32). This yields

$$|a_j|(1-k) \le \frac{1}{\pi(r_2^2-r_1^2)}\int_{\mathfrak{R}} (|\chi_{1t}| + |\chi_{2t}|)\, \kappa^{j+1}|t|^{-j}|f_1(t)|d\sigma_t$$

$$\le \frac{\kappa^{j+1}M(r_2, f_1)}{\pi(r_2^2-r_1^2)}\left[\left(\int_{\mathfrak{R}} |\chi_{1t}|^2 d\sigma_t\right)^{1/2} + \left(\int_{\mathfrak{R}} |\chi_{2t}|^2 d\sigma_t\right)^{1/2}\right]\left(\int_{\mathfrak{R}} |t|^{-2j}d\sigma_t\right)^{1/2}.$$

By (33) and (35) this obviously implies the absolute and uniform convergence of the series to the right of (i) in every compact subset of $G_1^* := \{z : |z| < \rho^*\kappa^{-2}\}$.

We still have to show that

$$h_1(z) = f_1(z) - \sum_{j=0}^{\infty}[a_j z^j]_{\nu,\mu} \equiv 0 \quad \text{in} \quad G_1^*.$$

In any case h_1 is a (ν, μ)-solution in G_1^* and $h_1(0) = 0$. If $h_1 \not\equiv 0$ then h_1 were to possess at 0 an asymptotic expansion

$$h_1(z) = c_m(z + \mathfrak{b}\bar{z})^m - b\overline{c_m}(\bar{z} + \overline{\mathfrak{b}}z)^m + O(|z|^{m+\alpha})$$

where $\mathfrak{b}$ is (like b) an algebraic expression of $\nu(0)$, $\mu(0)$ with $|\mathfrak{b}| \le k$, α a constant > 0, m an integer ≥ 1 and $c_m \ne 0$, cf. [6], p. 70. Hence

$$h_1(z) - [c_m z^m]_{\nu,\mu} =: g_1(z)$$

is a (ν,μ)-solution in G_1^*, $g_1(z) = O(|z|^{m+\alpha})$, and therefore even $g_1(z) = O(|z|^{m+1})$. Analogously with (37), (38) we have

$$0 = \frac{1}{|\mathfrak{R}|}\int_{\mathfrak{R}} \Omega_m(t,h_1)d\sigma_t =$$

$$- mc_m(1-|b|^2) + \frac{1}{|\mathfrak{R}|}\int_{\mathfrak{R}} \Omega_m(t,g_1(t))d\sigma_t\,. \tag{44}$$

Further, analogously with (39),

$$\left|\frac{1}{|\mathfrak{R}|}\int_{\mathfrak{R}} \Omega_m(t,g_1(t))d\sigma_t\right| \leq$$

$$\frac{m(1+k)\kappa^{m+1}}{\pi(r_2^2-r_1^2)}\left(\int_{\mathfrak{R}}(|\chi_{1t}|+|\chi_{2t}|)^2d\sigma_t\right)^{1/2}\left(\int_{\mathfrak{R}}|t^{-m}g_1(t)|^2d\sigma_t\right)^{1/2}.$$

If we put, e.g., $r_1 = \frac{1}{2}r_2$, then we have analogously with (40) that (as $r_2 \to 0$)

$$\frac{1}{|\mathfrak{R}|}\int_{\mathfrak{R}} \Omega_m(t,g_1(t))d\sigma_t = O(r_2)\,. \tag{45}$$

Thus the assumption $h_1 \not\equiv 0$ leads to a contradiction.

Concerning the remaining assertions on the a_j in theorem 8 only observe (29), (41), (42), (24), (25) and theorem 2 above as well as the fact that for the regular part F_1 and the principle part F_2 of $F \equiv 0$ with respect to G it must hold $F_1 \equiv 0$ and $F_2 \equiv 0$.

6. Proof of theorem 1

To prove (I) we first fix an open set B^* such that

$$B \subset\subset B^* \subset\subset B_0\,.$$

Further let
$\nu^*(z) = \nu(z)$, $\mu^*(z) = \mu(z)\,\forall z \in B^*$, $\nu^*(z) = \mu^*(z) = 0$ otherwise in $\mathbb{C}$.

By convolution of ν, μ and ν^*, μ^* with appropriate mollifiers $\omega_n(\cdot)$ we obtain sequences of ν_n, μ_n and ν_n^*, μ_n^* satisfying (a)-(d) each and, moreover,

$$\nu_n^*(z) = \nu_n(z), \quad \mu_n^*(z) = \mu_n(z) \quad \forall z \in B$$

and for all $n \geq n_0$ (where n_0 depends only on the choice of the ω_n and the choice of B^*).

Let U, U_n be the integral transformations belonging to ν^*, μ^* and ν_n^*, μ_n^*, respectively, according to [6], theorem IV.1.9. There exists a function $g(\cdot)$ uniquely defined in B_0 and analytic in B^* such that

$$Ug(z) = f(z) \quad \text{in} \quad B_0\,.$$

From the properties of U, U_n (cf. [6], [9]) it almost immediately follows that

$$f_n(z) := U_n g(z)$$

are (ν_n, μ_n)-solutions in B tending uniformly to $f(z)$ in every compact subset of B_0 as $n \to \infty$, which proves (I) of theorem 1.

As to the proof of (II), by [6], IV.1 and IV.3 it holds

$$H_n(t,z) = \log(t-z) + \frac{1}{\pi} \int \frac{F_n(\zeta, z)}{\zeta - t} d\sigma_\zeta$$

where $F_n(\cdot, z)$ is the unique solution of

$$F_n(t,z) = -\frac{\nu_n(t)}{t-z} - \frac{\mu_n(t)}{\overline{t-z}} + \nu_n(t) T F_n(\cdot, z)(t) + \mu_n(t) \overline{T F_n(\cdot, z)(t)} \tag{46}$$

in $L_s(\mathbb{C})$ with an s satisfying

$$kC(s) < 1\,, \quad 1 < s < 2\,;$$

here $C(s)$ means the norm of the complex Hilbert transformation $T : L_s(\mathbb{C}) \to L_s(\mathbb{C})$.

The L_s-norm of the inhomogeneous term in (46) is uniformly bounded for every $z \in \mathbb{C}$ and for every $n \in \mathbb{N}$; obviously the bound depends only on k, R^* and $s \in (1,2)$. Therefore, there exists a bound K_0 depending also only on k, R^* and s such that

$$\|F_n(\cdot, z)\|_{L_s(\mathbb{C})} \le K_0\,.$$

Due to (c) and (10),

$$F_n(t,z) = 0 \quad \forall |t| > R^* + 1\,.$$

Further,

$$P_G h(t) := -\frac{1}{\pi} \int\limits_G \frac{h(\zeta)}{\zeta - t} d\sigma_\zeta$$

is a bounded operator on $L_s(G)$ into $L_s(G)$ for each $s \in (1,2)$, if G is a bounded domain, cf. [20], p. 40, and its norm depends only on s and G.

Hence, if $G = G_t$ is a domain with the properties required in theorem 1(II), then

$$\|H_n(\cdot, z)\|_{L_s(G_t)} \le K_1 \quad \forall z \in D_z$$

with a constant K_1 depending only on k, R^*, G_t, D_z, s and the choice of the branch of $\arg(t-z)$ for $(t,z) \in G_t \times D_z$.

Since $H_n(\cdot, z)$ is (i.e. can be defined to be) a (ν_n, μ_n)-solution in G_t for every $z \in D_z$, lemma I.3.21 from [6] yields

$$|H_n(t,z)| \le K \quad \forall (t,z) \in D_t \times D_z$$

where K depends only on k, R^*, G_t, D_z, s, the choice of the branch of $\arg(t-z)$ for $(t,z) \in G_t \times D_z$, and the distance of ∂D_t from ∂G_t, which proves (II)(i).

The remaining assertions in theorem 1(II) follow from

$$H_{nt}(t,z) = \frac{1}{t-z} - TF_n(\cdot,z)(t)\,, \quad H_t(t,z) = \frac{1}{t-z} - TF(\cdot,z)(t)$$

(where $F(\cdot,z)$ is the solution of (46) with ν_n, μ_n replaced by ν, μ) by means of (46) and the corresponding equation for $F(\cdot,z)$ under using the properties of the norm of the Hilbert transformation T in L_s (namely $C(2) = 1$ and the continuity of $C(\cdot)$ in $(1,\infty)$) and the properties of the sequences ν_n, μ_n.

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