A Simple Proof of a Theorem of Arnol'd about Updown Sequences

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In [1], Arnol'd has calculated the number of maximal boundary morsifications $K(B_n)$ of the singularity B_n, which is a boundary singularity of the function x^n on the line $\{x\}$, equipped with the "boundary" $x = 0$.

Surprisingly, the number $K(B_n)$ was shown to be equal to the number U_{n+1} of $(n+1)$-updown sequences. By the way, a n-updown sequence is a sequence of integers $i_1 < i_2 > i_3 < \ldots i_n$, such that $\{i_1, \ldots, i_n\} = \{1, 2, \ldots, n\}$. Define $U_0 = 1$. The numbers U_n form the sequence

n	0	1	2	3	4	5	6	7	8	...
U_n	1	1	1	2	5	16	61	272	1385	...

The exponential generating function of these numbers is given by

$$\sum_{n=0}^{\infty} U_n \, \frac{t^n}{n!} = \sum_{n=0}^{\infty} U_{2n} \, \frac{t^{2n}}{(2n)!} + \sum_{n=0}^{\infty} U_{2n+1} \, \frac{t^{2n+1}}{(2n+1)!} = sec(t) + tan(t)$$

We have that $U_{2n} = E_{2n}$ (Euler number or secant number), and $U_{2n-1} =$

T_{2n-1} (tangent number), and $T_{2n-1} = \frac{(-1)^{n-1}4^n(4^n-1)}{2n}|B_{2n}|$, where $B_n = n-$th Bernoulli number.

To show that equality, Arnol'd had to prove an interesting (astonishing in his own words) result involving updown sequences (Theorem 2', p. 544). However, there is a flaw in his proof: the proof fails for $k = 0$. (See theorem below).

In this note, we present a simpler and maybe more elegant proof of that result.

Definition. *Let U_n^k = the number of $(n+1)$-updown sequences such that $i_2 - i_1 = k + 1$.*

Remark. Hence, $U_{n+1} = U_n^0 + \ldots + U_n^{n-1}$.

Then the result to be proved is the following:

Theorem. *Let $s + k \leq n$. Then the number of $(n+2)$-updown sequences such that $i_2 - i_1 = s + 1$ and $i_2 + k = n + 2$ is equal to U_n^k (and hence does not depend on s).*

Proof. Since there is a bijection between the $(n+2)$-updown sequences such that $i_2 - i_1 = s+1$, $i_2 + k = n+2$ and the $(n+2)$-down-up sequences $j_1 > j_2 < j_3 > \ldots j_{n+2}$, $\{j_1, \ldots, j_{n+2}\} = \{1, 2, \ldots, n+2\}$, such that $j_1 - j_2 = s+1$ and $j_2 - k = 1$, we shall construct a bijection

$\psi :\ \{(n+1)\text{-updown seq. } i_1 < i_2 > i_3 < \ldots i_{n+1};\ i_2 - i_1 = k+1\} \longrightarrow$

$\{(n+2)\text{-down-up seq. } j_1 > j_2 < j_3 > \ldots j_{n+2};\ j_1 - j_2 = s+1,\ j_2 - k = 1\}$

For this, consider an arbitrary $(n+1)$-updown sequence $i_1 < i_2 > i_3 < \ldots i_{n+1}$ such that $i_2 - i_1 = k + 1$, and let $C_1 = \{i_2, i_3, \ldots, i_{n+1}\}$ and $C_2 = (C_1 \cup \{i_1, n+2\}) \setminus \{k+1, k+s+2\}$. Now, order C_1 and C_2, writing

$C_1 = \{r_1, r_2, \ldots, r_n\},\ r_v < r_{v+1}, C_2 = \{s_1, s_2, \ldots, s_n\},\ s_v < s_{v+1}$ and define the increasing bijective function

$$\varphi : \begin{cases} C_1 & \to & C_2 \\ r_v & \mapsto & s_v \end{cases}$$

Now, define ψ by putting $j_{r+1} = \varphi(i_r),\ r \geq 2$ and consider the sequence $k+s+2, k+1, j_3, \ldots, j_{n+2}$. We have that it is a $(n+2)$-down-up sequence, since $j_3 = \varphi(i_2) > k+1$. In fact, it is easy to see that

$$\varphi(i_2) = \begin{cases} i_2 & \text{if } (k+s+2) > i_2 \\ i_2 + 1 & \text{otherwise} \end{cases}$$

Finally, it is routine to check that ψ is bijective, using the above computation for $\varphi(i_2)$.

REFERENCE

1. V.I. Arnol'd, *Bernoulli-Euler updown numbers associated with function singularities, their combinatorics and arithmetics*, Duke Math. J. 63 (1991), 537-555

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