Characterizations of Efficient Points via Nonlinear Scalarization

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We present characterizations of (weakly) efficient points for optimization in the space $\mathcal{C}(T)$ of continuous functions. Our main tool is a class of nonlinear scalarization functionals introduced by GERTH and WEIDNER. We discuss how the scalarized problem can be used to derive optimality criteria in terms of generalized differentials and that this scalarization generalizes other scalarization methods such as linear scalarization. Further, we will give some continuity properties with respect to the scalarization parameters.

1 Motivation

In this paper, we consider the vector optimization problem

$$(P) \qquad F(x) \to \min, \qquad x \in X,$$

where $F : \mathfrak{X} \to \mathcal{C}(T)$ is a function which maps $\mathfrak{X}$ in the space of continuous functions, $X \subseteq \operatorname{dom} F$ is the feasible set and T a compact subset of a topological space. For a unified notation we will denote $f(x,t) = F(x)(t)$ whenever we evaluate elements of $\mathfrak{Y} = \mathcal{C}(T)$ pointwise.

The space $\mathcal{C}(T)$ of continuous functions (over a compact set T) endowed with the maximum norm

$$\|c\| = \sup_{t\in T} |c(t)| = \max_{t\in T} |c(t)|$$

and the natural order cone

$$\mathcal{C}^+(T) = \{c \in \mathcal{C}(T) : c(t) \geq 0 \ \ \forall\, t \in T\}$$

is a BANACH lattice but in general not order complete. Furthermore we have

$$\operatorname{int} \mathcal{C}^+(T) = \{c \in \mathcal{C}(T) : c(t) > 0 \ \ \forall\, t \in T\} \neq \emptyset$$

and $0 \in \operatorname{bd} \mathcal{C}^+(T)$. For an survey on properties of $(\mathcal{C}(T), \mathcal{C}^+(T), \|.\|)$, see for instance [2] or [7].

Definition. Let $\mathfrak{X}$ a real BANACH space and $\mathfrak{Y}$ an ordered vector space, let $K \subset \mathfrak{Y}$ a convex cone, $K \cap -K = \{0\}$, $\operatorname{int} K \neq \emptyset$ and $F : \mathfrak{X} \to \mathfrak{Y}$ a given function and $X \subseteq \operatorname{dom} F$. An element $x_0 \in X$ is called

- *efficient* with respect to X and K iff $F(X) \cap (F(x_0) - K\backslash\{0\}) = \emptyset$,
- *weakly efficient* with respect to X and K iff $F(X) \cap (F(x_0) - \operatorname{int} K) = \emptyset$.

The set of all (weakly) efficient elements is denoted by $\operatorname{Eff}(F(X), K)$ resp. $\operatorname{Eff}_w(F(X), K)$.

Obviously, $\operatorname{Eff}(F(X), K) \subseteq \operatorname{Eff}_w(F(X), K)$ holds.

If we chose $\mathfrak{Y} = \mathcal{C}(T)$ and $K = \mathcal{C}^+(T)$, the definition above becomes an infinite-dimensional generalization of the classical PARETO optimality:

$$\begin{array}{lcll} x_0 \in \mathrm{Eff}(F(X), \mathcal{C}^+(T)) & \iff & \nexists\, x \in X: & f(x,t) \le f(x_0,t) \ \ \forall\, t \in T, \\ & & & f(x,t_0) < f(x_0,t_0) \text{ for some } t_0 \in T, \\ x_0 \in \mathrm{Eff}_w(F(X), \mathcal{C}^+(T)) & \iff & \nexists\, x \in X: & f(x,t) < f(x_0,t) \ \ \forall\, t \in T. \end{array}$$

So we will often add the word "PARETO" when we consider explicitly $K = \mathcal{C}^+(T)$.

In the following, we will investigate how known results from finite-dimensional objective spaces (namely $\mathbb{R}^n$) rule in the space $\mathcal{C}(T)$ and, on the other hand, how general results from arbitrary ordered vector spaces can be specified for our setting. We have built up our studies on a nonlinear scalarization method which was introduced by GERTH and WEIDNER. They used a MINKOWSKI-like functional for separating the set $F(X)$ and the ordering cone.

In [11], ZUBIRI gave conditions for weak efficiency of functions $F : A \to \mathcal{C}(T)$ via linear scalarization. We will show how a special choice of parameters in our scalarization leads to results similar to those of ZUBIRI. In [1], CARRIZOSA and PLASTRIA investigated weak efficiency of convex $\mathcal{C}(T)$-valued functions over $\mathbb{R}^n$ and received a geometrical optimality criterion which will turn out to be a special case of our Theorem 2.3.

Let us finish this introduction with some remarks on the role of order completeness. An ordered space $\mathfrak{Y}$ is said to be *(sequential) order complete* if for any (countable) set which is bounded from below, the infimum exists. The space $\mathcal{C}(T)$ is order complete if and only if the closure of each open subset of T is open in T, i. e. only if T is finite and in some other rare cases without practical importance. But without order completeness, any calculus based on sequences (or more general on nets) and their limits becomes rather difficult, often we have to invoke space-specific conditions to assure existence of limits at all. In example 2.1 we consider a function F which is differentiable in each component $t \in T$, the function of these partial derivatives $f_1'(x,.)$ is even continuous, but this mapping is not attained as a limit of differential quotients in the space $\mathcal{C}(T)$. Note that, up to now, in literature there exist no (generalized) subdifferential definition and calculus on abstract spaces which lack order completeness.

2 Scalarization

2.1 Scalarization by monotone functionals

Our main tool for the characterization of efficient elements will be a class of scalarizing functionals $z : \mathcal{C}(T) \to \mathbb{R}$ which fulfill certain monotony properties. So, let us recall the notions of monotony:

Definition. Let B be a nonempty subset of $\mathfrak{Y}$. A functional $z : \mathfrak{Y} \to \mathbb{R}$ is called

- *B-monotone*, if $y_1 \in y_2 - B$ implies $z(y_1) \le z(y_2)$;
- *strongly B-monotone*, if $y_1 \in y_2 - B\backslash\{0\}$ implies $z(y_1) < z(y_2)$.

We recall that a strongly monotone functional is also monotone, but in general strong $(\mathrm{int}\, B)$-monotony does not imply B-monotony. If $B \subseteq \mathrm{cl}\, \mathrm{int}\, B$ (which is fulfilled for instance if B is convex and $\mathrm{int}\, B \ne \emptyset$), then a continuous, $(\mathrm{int}\, B)$-monotone functional is B-monotone.

We build up our work on the following equivalences between optimal elements and solutions of the scalarized problem. They are contained in [4, Theorem 2.2] and [9, Theorem 4.3.2 and Satz 4.3.5].

Lemma 2.1 *Let $K \subsetneq \mathfrak{Y}$ be a closed convex cone with nonempty interior and $K\backslash\{0\} \subset \operatorname{int} C$ for some convex cone $C \subsetneq \mathfrak{Y}$, let $y_0 = F(x_0)$. The following assertions are equivalent:*

(a) $x_0 \in \operatorname{Eff}(F(X), K)$.

(b) There exists a strongly K-monotone functional $z : \mathfrak{Y} \to \mathbb{R}$ such that $z(y_0) \leq z(y)\ \forall\ y \in F(X)$.

(c) There exists a K-monotone functional $z : \mathfrak{Y} \to \mathbb{R}$ such that $z(y_0) \leq z(y)\ \forall\ y \in F(X)$ and y_0 is the only solution of this inequality.

Further, $x_0 \in \operatorname{Eff}_w(F(X), K)$ is equivalent to the existence of a strongly $(\operatorname{int} K)$-monotone functional $z : \mathfrak{Y} \to \mathbb{R}$ such that $z(y_0) \leq z(y)\ \forall\ y \in F(X)$.

All of these functionals can be constructed based on the separating functionals z introduced below. We remark that, in general, we can not choose the functional mentioned in (b) to be continuous: In the proof of [9, Satz 4.3.5] such a non-continuous but strongly K-monotone separating functional z is given.

A functional z as mentioned in the Lemma above is called *scalarization*. It shifts the origin problem into real-valued optimization for which optimality criteria and methods are much easier to rule.

2.2 Scalarization via nonlinear separating functionals

Let $\mathfrak{X}$ be a real Banach space. We fix (as parameters) an element $k \in \mathcal{C}(T)$ and a proper subset C of $\mathcal{C}(T)$ which fulfills the following condition:

(V1) The set C has nonempty interior, $\operatorname{cl} C + \operatorname{int} K \subseteq \operatorname{int} C$ for some convex cone $K \subset \mathcal{C}(T)$ with $\operatorname{int} K \neq \emptyset$; $k \in \operatorname{int} K$.

Condition (V1) is satisfied, for instance, if C itself is a convex cone and $k \in \operatorname{int} C$ or if C can be written as $C = \tilde{C} - a$ with $a \in \mathcal{C}(T)$ and some convex cone $\tilde{C}$ satisfying (V1), $k \in \operatorname{int} \tilde{C}$. We define a mapping $z^{C,k} : \mathcal{C}(T) \to \mathbb{R}$ by

$$z^{C,k}(y) := \inf\{t \in \mathbb{R} :\ y \in tk - \operatorname{cl} C\}. \tag{1}$$

A functional of this type, defined for an arbitrary ordered linear topological space $\mathfrak{Y}$ instead of $\mathcal{C}(T)$, was used by Gerth (Tammer) and Weidner in [3] in order to scalarize vector optimization problems. They proved the following properties for $z^{C,k}$:

Lemma 2.2 *Let C and k such that condition (V1) is fulfilled, let B a non-empty subset of $\mathcal{C}(T)$. Then for the functional $z = z^{C,k}$ holds:*

(i) z is a well-defined continuous mapping from $\mathcal{C}(T)$ onto $(-\infty, +\infty)$;

(ii) z is convex iff $\operatorname{cl} C$ is convex; z is sublinear iff $\operatorname{cl} C$ is a convex cone;

(iii) $z(y) > 0 \Leftrightarrow y \in \operatorname{int} C$ and $z(y) = 0 \Leftrightarrow y \in \operatorname{bd} C$; in particular $z(0) = 0 \Leftrightarrow 0 \in \operatorname{bd} C$.

(iv) z is B-monotone iff $\operatorname{bd} C + B \subseteq \operatorname{cl} C$, and strongly B-monotone iff $\operatorname{bd} C + B\backslash\{0\} \subseteq \operatorname{int} C$;

(v) Let $A \subset \mathcal{C}(T)$ be a nonempty subset of $\mathcal{C}(T)$ and C a convex cone. Then we have:

$$A \cap (-\operatorname{int} C) = \emptyset \iff z(A) \geq 0, \quad z(\operatorname{int} A) > 0. \tag{2}$$

The main assertions are contained in the proof of [3, Theorem 2.1] with $-C$ instead of C. For a complete survey of the properties of z look at [9, Theorem 3.2.1 ff].

In the following, we will often refer to [9] although most of the results were firstly published at [3]. In her habilitation script, [9], WEIDNER has drawn together the properties of the scalarization functional z related to the properties of the cone C.

Now we will use the separating property (2) in Lemma 2.2 in order to characterize efficient points.

Theorem 2.1 *Let $C \subset \mathcal{C}(T)$ a convex cone with nonempty interior and $k \in \operatorname{int} C$. Consider the scalar optimization problem*

$$(P_{C,k,F(x_0)}) \qquad \begin{array}{l} h(\tau, x) := \tau \to \min \\ \text{where } F(x) - \tau k \in F(x_0) - \operatorname{cl} C, \quad (\tau, x) \in \mathbb{R} \times X. \end{array}$$

Then holds: $x_0 \in \operatorname{Eff}_w(F(X), C) \iff (0, x_0)$ solves the problem $(P_{C,k,F(x_0)})$. If in addition each solution (τ, x) of $(P_{C,k,F(x_0)})$ fulfills $F(x) = F(x_0)$, x_0 is even efficient.

Note that the goal function $h : \mathbb{R} \times \mathfrak{X} \to \mathbb{R}$ in problem $(P_{C,k,F(x_0)})$ is linear, so there exists at most one optimal value τ for each C, k.

Proof of Theorem 2.1. We set $A = F(X) - F(x_0)$ and $K = B = C$. Then $z = z^{C,k}$ defined by (1) is a convex, sublinear, continuous and strongly ($\operatorname{int} C$)-monotone functional (Lemma 2.2). Further, we derive: $x_0 \in \operatorname{Eff}_w(F(X), C) \iff$

$$z^{C,k}(F(X) - F(x_0)) \geq 0, \quad z^{C,k}(\operatorname{int} F(X) - F(x_0)) > 0 \tag{3}$$

Regarding $z(0) = 0$, the first inequality is equivalent to

$$\begin{aligned} \inf \{t \in \mathbb{R} : F(x) \in F(x_0) + tk - \operatorname{cl} C\} &\geq 0 \\ &= \inf \{t \in \mathbb{R} : F(x_0) \in F(x_0) + tk - \operatorname{cl} C\} \\ &= \min \{t \in \mathbb{R} : F(x_0) \in F(x_0) + tk - \operatorname{cl} C\} \end{aligned}$$

for all $x \in X$, i. e. $(0, x_0)$ solves $(P_{C,k,F(x_0)})$.

From Lemma 2.1 c) we derive finally that x_0 is efficient if in addition each solution (τ, x) of $(P_{C,k,F(x_0)})$ fulfills $F(x) = F(x_0)$. ■

Corollary 2.1 *If $(0, x_1)$ is a solution of $(P_{C,k,F(x_0)})$ different from $(0, x_0)$, then $F(x_1) - F(x_0) \in \operatorname{bd} C$ holds.*

Proof. Obviously, $z^{C,k}(F(x_1) - F(x_0)) = 0$, so Lemma 2.2 (iii) implies the assertion. ■

2.3 (Weak) PARETO efficiency

From Theorem 2.1 we derive the following result for the case of PARETO efficiency:

Theorem 2.2 $x_0 \in \operatorname{Eff}_w(F(X), \mathcal{C}^+(T)) \iff x_0$ *solves the problem*

$$(P1) \qquad \max_{t \in T} [f(x, t) - f(x_0, t)] \to \min, \quad x \in X,$$

where $f(x, t) \equiv F(x)(t)$ denotes the pointwise evaluation of $F(x)$. In addition, if each other solution $\bar{x}$ of $(P1)$ satisfies

$$f(\bar{x}, t) = f(x_0, t) \qquad \forall\, t \in T, \tag{4}$$

then $x_0 \in \operatorname{Eff}(F(X), \mathcal{C}^+(T))$.

Remind that in general the functional $z = z^{\mathcal{C}^+(T),k}$ will not be strongly $\mathcal{C}^+(T)$-monotone, i. e. optimal points of problem $(P1)$ have not to be efficient.

Clearly, equation (4) in Theorem 2.2 can be replaced by $\min_{t\in T}\,[f(\bar{x},t) - f(x_0,t)] = 0$ or $\|F(\bar{x}) - F(x_0)\| = 0$.

Proof. Pointwise evaluated, we can write problem $(P_{\mathcal{C}^+(T),k,F(x_0)})$ as

$$\begin{aligned} &\tau \to \min \\ &\text{where } \frac{f(x,t) - f(x_0,t)}{k(t)} \le \tau \quad \forall\, t \in T, \end{aligned}$$

or equivalently,

$$\max_{t\in T} \frac{f(x,t) - f(x_0,t)}{k(t)} \to \min, \quad x \in X. \tag{5}$$

Since $k \in \operatorname{int} \mathcal{C}^+(T)$, we have $k(t) > 0$ for all $t \in T$ and therefore $1/k(t) \in \mathbb{R}$ and $1/k(t) > 0$ for all $t \in T$. Further, for each $x \in X$, $f(x,.)$ is a continuous function over the compact set T, so the maximum in (5) is attained. We complete the proof concluding that we can omit the parameter $k \in \operatorname{int} \mathcal{C}^+(T)$ in the problem above since it can be chosen arbitrary. ■

We continue by applying a formular for the subdifferential $\partial_x f(x,t)$ (in the sense of Clarke) of pointwise maximum of functions recently given by Luu and Oettli in [5]. In the following, $f_1'(x_0,t)$ denotes the Fréchet derivative of $f(.,t)$ in $x \in X$, $\partial_1 f(x,t)$ the subdifferential of $f(.,t)$ in the sense of Clarke, and $N_X(x)$ the normal cone at X in x.

Lemma 2.3 (Luu/Oettli, [5]) *Let $\mathfrak{X}$ a Banach space and T a compact subset of a topological space, let functions $g : \mathfrak{X} \times T \to \mathbb{R}$ be given such that $g(x,.)$ is continuous for each $x \in \mathfrak{X}$ and $G(x) := \sup_{t\in T} g(x,t)$. Let x_0 be a local minimum of G over X.*

(i) If $g(.,t)$ is Fréchet differentiable in x_0 for all $t \in T$ and if the limits

$$\langle g_1'(x_0,t), d\rangle = \lim_{\lambda\to 0} \frac{g(x_0 + \lambda d, t) - f(x_0,t)}{\lambda} \tag{6}$$

are attained uniformly over T, then

$$0 \in \operatorname{cl}_{\sigma(\mathfrak{X}',\mathfrak{X})} \operatorname{conv}\left(\bigcup_{t\in T} g_1'(x_0,t)\right) + N_X(x_0)$$

holds.

(ii) If $g(.,t)$ is Lipschitz in a neighborhood of x_0 with the same constant L for all $t \in T$ and if the limits

$$\langle g_1^\circ(x_0,t), d\rangle = \limsup_{x\to x_0,\ \lambda\to 0} \frac{g(x + \lambda d, t) - f(x,t)}{\lambda} \tag{7}$$

are attained uniformly over T, then

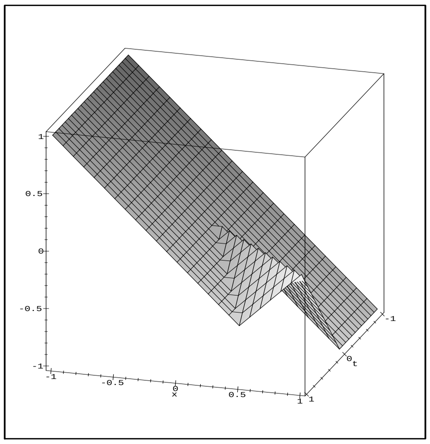
$$0 \in \operatorname{cl}_{\sigma(\mathfrak{X}',\mathfrak{X})} \operatorname{conv}\left(\bigcup_{t\in T} \partial_1 f(x_0,t)\right) + N_X(x_0)$$

holds.

Uniformly convergence as assumed in the lemma induces continuity of the mappings $t \mapsto f_1'(x,t)$ resp. $t \mapsto \partial_1 f(x,t)$. Conversely, if one of these mappings is continuous, uniformly convergence has not to be fulfilled and therefore our lemma may be not applicable. Example 2.1 illustrates such a case.

Example 2.1 We consider functions $g : \mathbb{R}^2 \to \mathbb{R}$,

$$\begin{aligned} g(x,t) &:= \max\{-x, -|t-x|\} \\ &= \left\{ \begin{array}{ll} -|x-t| & \text{for } 0 \le x,\ 0 \le t \le 2x \\ -x & \text{else} \end{array} \right\} \\ G(x) &:= \sup_{t\in T} g(x,t) \end{aligned}$$



with $X = T = [-1,1]$. Obviously, $x_0 = 0$ is a minimizer of G. The functions $g(.,t)$ are FRÉCHET-differentiable in x_0 with $g_1'(x_0,t) = -1$ for all $t \in T$, so

$$0 \notin \mathrm{cl}_{\sigma(\mathcal{X}',\mathcal{X})} \mathrm{conv} \left(\bigcup_{t\in T} g_1'(x_0,t) \right).$$

What's wrong? The mapping $t \mapsto g_1'(x_0,t)$ is continuous, but the limits

$$\langle g_1'(x,t), d\rangle = \lim_{\lambda\to 0} \frac{g(x_0+\lambda d,t) - g(x_0,t)}{\lambda}$$

are not attained uniformly in t. Indeed, let for instance $d = 1$ and $0 < \varepsilon < 1$ arbitrary. Then for $\lambda = t = \varepsilon/2 \ge 0$ we have $\lambda^{-1}(g(\lambda,t) - g(0,t)) = 0$.

Theorem 2.3 *Let $\mathcal{X}$ a BANACH space, T a compact subset of a topological space, and $x_0 \in \mathrm{Eff}_w(F(X), \mathcal{C}^+(T))$.*

(i) If $f(.,t)$ is FRÉCHET differentiable in x_0 for all $t \in T$ and if the limits in (6) are attained uniformly over T, then

$$0 \in \mathrm{cl}_{\sigma(\mathcal{X}',\mathcal{X})} \mathrm{conv} \left(\bigcup_{t\in T} f_1'(x_0,t) \right) + N_X(x_0)$$

holds.

(ii) If $f(.,t)$ is LIPSCHITZ in a neighborhood of x_0 with the same constant L for all $t \in T$ and if the limits in (7) are attained uniformly in T, then

$$0 \in \mathrm{cl}_{\sigma(\mathcal{X}',\mathcal{X})} \mathrm{conv} \left(\bigcup_{t\in T} \partial_1 f(x_0,t) \right) + N_X(x_0)$$

holds.

Here $f(x,t) \equiv F(x)(t)$ denotes the pointwise evaluation of $F(x)$.

Proof. Both assertions follow from Theorem 2.2 and Lemma 2.3. ■

In [10], we have given a similar characterization for weakly efficient points if $f(.,t)$ are convex functions: $x_0 \in \mathrm{Eff}_w(F(X), \mathcal{C}^+(T)) \iff$

$$0 \in \mathrm{cl}_{\sigma(\mathcal{X}',\mathcal{X})} \mathrm{conv} \left(\bigcup_{t\in T} \partial_1^c f(x_0,t) \right) + N_X(x_0), \tag{8}$$

where $\partial_1^c f(x_0,t)$ denotes the subdifferential of convex analysis for the function $f(.,t)$ in x_0. There are two fundamental differences to Theorem 2.3:

1. Without convexity, one can guarantee only necessity of (8). This is a fact already known from real-valued optimization and therefore it is not surprising.

2. Without convexity, one has to demand uniform convergence of the generalized differential quotients.

The second observation quite remarkable. On one hand it is a concession to the lack of order completeness of $\mathcal{C}(T)$, since if we replace for instance $\mathcal{C}(T)$ by $\mathbb{R}^n$, uniform convergence follows directly from pointwise convergence. On the other hand, if we assume convexity, the differential quotient is monotonical decreasing for $\lambda \to 0$, and so DINI's theorem of monotonical convergence yields uniform convergence.

2.4 Linear scalarization and proper efficiency

The most common scalarization method for vector valued optimization problems is the so-called *linear scalarization* which leads to proper efficiency.

Definition. Let $\mathfrak{X}$ a real BANACH space and $\mathfrak{Y}$ an ordered vector space, let $K \subset \mathfrak{Y}$ a convex cone, $K \cap -K = \{0\}$, $\operatorname{int} K \neq \emptyset$ and $F : X \to \mathfrak{Y}$ a given function, $X \subseteq \operatorname{dom} F$. Set

$$K^* := \{u \in \mathfrak{Y}' : u(y) \geq 0 \quad \forall\, y \in K\},$$

where $\mathfrak{Y}'$ the topological dual space to $\mathfrak{Y}$. An element $x_0 \in X$ is called *properly efficient* with respect to X and K iff there exists some $z \in \operatorname{int} K^*$ such that

$$z(F(x_0)) \leq z(F(x)), \qquad x \in X.$$

The set of all properly efficient point will be denoted by $\operatorname{Eff}_p(F(X), K)$.

Remind that $\operatorname{Eff}_p(F(X), K) \subseteq \operatorname{Eff}(F(X), K)$.

In our case we have $\mathfrak{Y} = \mathcal{C}(T)$. Due to the RIESZ representation theorem we can identify $\mathcal{C}(T)'$ with the space $\mathcal{BM}(T)$ of regular BOREL measures μ: For each $u \in \mathcal{C}(T)'$ there exists a measure $\mu \in \mathcal{BM}(T)$ such that

$$\langle u, x\rangle = \int_T x(t)\,\mu(\mathrm{d}t), \qquad x \in \mathcal{C}(T),$$

holds and $\|u\| = |\mu|$. $|\mu|$ denotes the total variation of μ. The dual cone to $\mathcal{C}^+(T)$ is given by

$$\mathcal{BM}^+(T) := \{\mu \in \mathcal{BM}(T) : \mu(S) \geq 0 \quad \forall\, S \subseteq T\}$$

and $\mu \in \operatorname{int} \mathcal{BM}^+(T)$ iff $\mu(S) > 0$ for all $S \subseteq T$.

In the following, we consider cones $C = D_\mu$,

$$D_\mu := \{c \in \mathcal{C}(T) : \int_T c(t)\,\mu(\mathrm{d}t) \geq 0\},$$

where $\mu \in \mathcal{BM}^+(T)$. Obviously, each D_μ is a convex cone and

$$\operatorname{int} D_\mu := \{c \in \mathcal{C}(T) : \int_T c(t)\,\mu(\mathrm{d}t) > 0\}.$$

Further, $\mathcal{C}^+(T) \subset D_\mu$ and $\operatorname{bd} D_\mu + \mathcal{C}^+(T)\backslash\{0\} \subseteq \operatorname{int} D_\mu$ for all $\mu \in \operatorname{int} \mathcal{BM}^+(T)$; therefore we have $x_0 \in \operatorname{Eff}(F(M), \mathcal{C}^+(T))$ whenever we can find some $\mu \in \operatorname{int} \mathcal{BM}^+(T)$ such that $x_0 \in \operatorname{Eff}(F(M), D_\mu)$. We can even state

Theorem 2.4 *If x_0 solves the problem*

$$(P2) \qquad \int_T f(x,t)\,\mu(\mathrm{d}t) \to \min, \quad x \in X,$$

for some measure $\mu \in \operatorname{int} \mathcal{BM}^+(T)$, then $x_0 \in \operatorname{Eff}(F(X), \mathcal{C}^+(T))$.

Proof. By $\operatorname{bd} D_\mu + \mathcal{C}^+(T)\setminus\{0\} \subseteq \operatorname{int} D_\mu$ and Lemma 2.2, the functional $z^{C,k}$ with $C = D_\mu$ and $k \in \operatorname{int} D_\mu$ is strongly $\mathcal{C}^+(T)$-monotone. Therefore x_0 is efficient with respect to the cone D_μ iff it minimizes $z^{C,k}(F(x))$ over X or equivalently $(0, x_0)$ solves

$$\begin{aligned} &\tau \to \min \\ &\text{where } F(x) - \tau k - F(x_0) \in -\operatorname{cl} D_\mu, \quad x \in X. \end{aligned}$$

The side condition writes

$$\int_T [f(x,t) - \tau k(t) - f(x_0,t)]\,\mu(\mathrm{d}t) \le 0.$$

Since $\int_T k(t)\,\mu(\mathrm{d}t) \neq 0$, we can proceed as in the proof of Theorem 2.2: problem $(P_{D_\mu,k,F(x_0)})$ is eqivalent to (P2). ■

In [11], Zubiri recently gave a similar crition for weak efficiency under additional convexity and boundedness assumptions. His supplementary assertions on upper bounds for $|\mu|$ result from appropriate choice of μ. A version of his optimality criterion involving a lower bound a of $F(X)$ is given in section 3.3.

3 A class of substitute problems

3.1 Substitute problems versus scalarization

We consider the scalar problem

$$(P_{C,k,a}) \qquad \begin{aligned} &\tau \to \min \\ &\text{where } F(x) \in a + \tau k - \operatorname{cl} C, \quad (\tau, x) \in \mathbb{R} \times X, \end{aligned}$$

with $C \subset \mathcal{C}(T)$, $k \in \mathcal{C}(T)$ and $a \in \mathcal{C}(T)$ as parameters.

Problems of this type as a method for vector optimization have been studied by Pascoletti/Serafini, [6], and more detailed by Weidner, [9] (see also the references mentioned therein). Weidner devotes a whole section of her habilitation to a survey how most of the known scalarization methods (in $\mathbb{R}^n$) fit into this general concept.

Let $\mathcal{D}_{C,k,a} := \bigcup_{\tau\in\mathbb{R}}(a + \tau k - \operatorname{cl} C)$.

Lemma 3.1 *The problem $(P_{C,k,a})$ has a optimal solution iff $z^{C-a,k}$ due to (1),*

$$z^{C-a,k}(y) = \inf\{\tau \in \mathbb{R} : \ y \in a + \tau k - \operatorname{cl} C\},$$

is well-defined on $F(X) \cap \mathcal{D}_{C,k,a}$ and $\min\{z^{C-a,k}(y) \ : \ y \in F(X) \cap \mathcal{D}_{C,k,a}\}$ exists. In this case, the set of optimal values τ_0 correspond to that of the problem $\min\{z^{C-a,k}(y) \ : \ y \in F(X) \cap \mathcal{D}_{C,k,a}\}$.

The proof is contained in [9, Satz 6.1.1].

As we stated in section 2.2, we have $\mathcal{D}_{C,k,a} = \mathcal{C}(T)$ if $C - a$ and k fulfill condition (V1). Since in most applications this can be satisfied, we will restrict ourselves to this case.

Theorem 3.1 *Let $C-a$ and k fulfill condition (V1), let $F(X)+C$ be closed and bounded from below. Then problem $(P_{C,k,a})$ has a solution.*

Proof. Since (V1) is fulfilled, $z^{C-a,k}$ is well-defined and continuous (cf. Lemma 2.2). Since $F(X)+C$ is closed and bounded from below, the set $\{t \in \mathbb{R} : \; a+tk \in F(X)+\operatorname{cl} C\}$ is closed and bounded from below, i. e. the problem

$$(\tilde{P}_{C,k,a}) \qquad \begin{array}{l} \tau \to \min \\ \text{where } a+\tau k \in F(X)+\operatorname{cl} C, \quad \tau \in \mathbb{R}, \end{array}$$

has a solution τ_0. So, there exists some $x_0 \in X$ such that $a+\tau_0 k \in F(x_0)+\operatorname{cl} C$ and (τ_0, x_0) solves problem $(P_{C,k,a})$. ∎

The proof is due to Sterna-Karwat, [8]. Closedness of $F(X)+C$ can be guaranteed if for instance $F(X)$ is compact and C closed.

Note that the goal function $(\tau, x) \mapsto \tau$ of $(P_{C,k,a})$ is linear and therefore convex, so there exists at most one optimal value τ_0 for each C, k, a.

In Theorem 2.1 we stated that x_0 is weakly efficient with respect to a convex cone C and X iff $(0, x_0)$ solves $(P_{C,k,a})$ with $a = F(x_0)$ and arbitrary $k \in \operatorname{int} C$. If $(0, x_0)$ solves $(P_{C,k,F(x_0)})$ uniquely with respect to $y = F(x)$, x_0 is even efficient. In this paragraph, we will show how we can reach each (weakly) efficient point choosing different $a \in \mathcal{C}(T)$.

Theorem 3.2 *Let C and k fulfill condition (V1). Then we have*

$$\begin{array}{rcl} x_0 \in \operatorname{Eff}_w(F(X), C) & \iff & (0, x_0) \text{ solves } (P_{C,k,F(x_0)}), \\ & \iff & \exists\; a \in \mathcal{C}(T) : \; (0, x_0) \text{ solves } (P_{C,k,F(x_0)}). \\ x_0 \in \operatorname{Eff}(F(X), C) & \iff & (0, x_0) \text{ solves } (P_{C,k,a}) \text{ uniquely in } y = F(x), \\ & \iff & \exists\; a \in \mathcal{C}(T) : \; (0, x_0) \text{ solves } (P_{C,k,a}) \text{ uniquely in } y = F(x). \end{array}$$

Let $C \subsetneq \mathcal{C}(T)$ be a convex cone with nonempty interior and $F(X) \subseteq a + \operatorname{int} C$; then

$$\begin{array}{rcl} x_0 \in \operatorname{Eff}_w(F(X), C) & \iff & (1, x_0) \text{ solves } (P_{C,F(x_0)-a,a}). \\ x_0 \in \operatorname{Eff}(F(X), C) & \iff & (1, x_0) \text{ solves } (P_{C,F(x_0)-a,a}) \text{ uniquely in } y = F(x). \end{array}$$

On the contrary, if $F(X) \subseteq a - \operatorname{int} C$; then

$$\begin{array}{rcl} x_0 \in \operatorname{Eff}_w(F(X), C) & \iff & (-1, x_0) \text{ solves } (P_{C,a-F(x_0),a}). \\ x_0 \in \operatorname{Eff}(F(X), C) & \iff & (-1, x_0) \text{ solves } (P_{C,a-F(x_0),a}) \text{ uniquely in } y = F(x). \end{array}$$

The theorem corresponds to [9, Satz 6.2.3 and Satz 8.1].

3.2 On the optimal value function

We associate to problem $(P_{C,k,a})$ the set-valued map

$$\mathcal{T}(a,k) = \mathcal{T}_C(a,k) := \{\tau \in \mathbb{R} : a \in -\tau k + F(X) + \operatorname{cl} C\}$$

and its marginal function $m : \mathcal{C}(T) \to \mathbb{R} \cup \{-\infty, +\infty\}$ given by

$$\begin{array}{rcl} m(a,k) = m_C(a,k) & := & \inf\{\tau \in \mathbb{R} : x \in X, \; a \in -\tau k + F(x) + \operatorname{cl} C\} \\ & = & \inf\{\tau \in \mathbb{R} : a \in -\tau k + F(X) + \operatorname{cl} C\}. \end{array}$$

Further, we define a function $g : \operatorname{dom} m \to \mathcal{C}(T)$ by

$$g(a,k) = g_C(a,k) := m(a)\,k + a.$$

All over this section, we assume $C \subsetneq \mathcal{C}(T)$ to be a convex cone with nonempty interior, $k \in \operatorname{int} C$ and $a \in \mathcal{C}(T)$ arbitrary.

The following assertions are obvious.

Lemma 3.2 *The values $\mathcal{T}(a,k)$ of $\mathcal{T}$ are nonempty convex sets; they are closed and bounded from below if $F(X) + \operatorname{cl} C$ is closed and $F(X)$ is bounded from below, respective.*

In consequence, $m(a,k)$ is finite whenever $F(X)$ is bounded from below; and $g(a,k) \in F(X) + \operatorname{cl} C$ holds if in addition $F(X)+\operatorname{cl} C$ is closed. In this later case, we have $g(a,k) \in \operatorname{Eff}_w(F(X), C)$.

Theorem 3.3 *For each $k \in \operatorname{int} C$ resp. each $a \in \mathcal{C}(T)$, the functions $m(.,k)$ and $m(a,.)$ are upper semicontinuous on $\mathcal{C}(T)$ resp. on $\operatorname{int} C$. If in addition $F(X) + \operatorname{cl} C$ is closed, then these functions are even continuous.*

Proof. Fix $k \in \operatorname{int} C$. $m(.,k)$ is upper-semicontinuous, if the sets $\{a : m(a,k) \geq r\}$ are closed for each $r \in \mathbb{R}$, or equivalently, if the sets $\{a : m(a,k) < r\}$ are open. So, fix some $r \in \mathbb{R}$ and choose $a_0 \in \operatorname{dom} m$ such that $m(a_0,k) < r$. Then there exists $\tau \in \mathbb{R}$, $\tau < r$ with $a_0 + \tau k \in F(X) + \operatorname{cl} C$. Therefore,

$$a_0 \in -rk + (r-\tau)k + F(X) + \operatorname{cl} C \subseteq -rk + F(X) + \operatorname{int} C,$$

i. e. $U := -rk + F(X) + \operatorname{int} C$ is an open neighborhood of a_0. For each $a \in U$ there exists some $\varepsilon > 0$ such that

$$a \in -rk + \varepsilon k + F(X) + \operatorname{int} C - \varepsilon k \subseteq -(r-\varepsilon)k + F(X) + \operatorname{cl} C,$$

which implies $m(a,k) \leq r - \varepsilon < r$.

To show lower semicontinuity, choose some $r \in \mathbb{R}$. If $m(a,k) > r$, then $a \notin -rk + F(X) + \operatorname{cl} C$. Closedness of $F(X) + \operatorname{cl} C$ implies openess of $\mathcal{C}(T)\backslash\{-rk + F(X) + \operatorname{cl} C\}$ and therefore openess of $\{a \in \mathcal{C}(T) : m(a,k) > r\}$.

The proof of the semicontinuity of the function $m(a,.)$ for fixed $a \in \mathcal{C}(T)$ works in the same way. ■

Corollary 3.1 *For each $k \in \operatorname{int} C$ resp. each $a \in \mathcal{C}(T)$, the functions $g(.,k)$ and $g(a,.)$ are upper semicontinuous on $\mathcal{C}(T)$ resp. on $\operatorname{int} C$. If in addition $F(X) + \operatorname{cl} C$ is closed, then these functions are even continuous.*

In Theorem 3.2 we have shown that there is another paramter variation which describes the set of efficient points: a and k vary both but depend linearly on each other. So we have investigated continuity in this case, too:

Theorem 3.4 *The function $a \mapsto m(a, F(x_0) - a)$ is upper semicontinuous on $F(X) - \operatorname{int} C$ for each $x_0 \in X$. If in addition $F(X) + \operatorname{cl} C$ is closed, this function is even continuous.*

Proof. We can adapt the proof of Theorem 3.3: We fix some $r \in \mathbb{R}$, $x_0 \in X$ and choose $a_0 \in F(X) - \operatorname{int} C$ such that $m(a_0, F(x_0) - a_0) < r$. Then there exists $\tau \in \mathbb{R}$, $\tau < r$ with $a_0 + \tau(F(x_0) - a_0) \in F(X) + \operatorname{cl} C$. Choose some $\lambda \in (0,1)$ such that $r_0 := \tau + \lambda(r - \tau) \neq 1$. Consequently, we have

$$\begin{aligned} a_0 &\in -r_0(F(x_0) - a_0) + (r - r_0)(F(x_0) - a_0) + F(X) + \operatorname{cl} C \\ &\subseteq -r_0(F(x_0) - a_0) + F(X) + \operatorname{int} C, \end{aligned}$$

i. e. $U := (1-r_0)^{-1}(-r_0(F(x_0)-a_0)+F(X)+\operatorname{int} C)\cap(F(X)-\operatorname{int} C)$ is an open neighbourhood of a_0. For each $a \in U$ we have

$$a \in -r_0(F)x_0) - a) + F(X) + \operatorname{int} C \subseteq -r_0(F)x_0) - a) + F(X) + \operatorname{cl} C$$

which implies $m(a, F(x_0) - a) \leq r_0 < r$.

To show lower semicontinuity, fix some $r \in \mathbb{R}$ and $x_0 \in X$. If $m(a, F(x_0) - a) > r$, then $a \notin -r(F(x_0) - a) + F(X) + \operatorname{cl} C$. Closedness of $F(X) + \operatorname{cl} C$ implies openess of $\mathcal{C}(T)\backslash\{-rk + F(X) + \operatorname{cl} C\}$ and therefore openess of $\{a \in F(X) - \operatorname{cl} C : m(a, k) > r\}$. ■

Corollary 3.2 *For each $x_0 \in X$, the function $a \mapsto g(a, F(x_0) - a)$ is upper semicontinuous on $F(X) - \operatorname{cl} C$. If in addition $F(X) + \operatorname{cl} C$ is closed, then this function is even continuous.*

Sterna-Karwat recently studied the marginal function m of problem $(P_{C,k,a})$, see for instance [8]. She proved semicontinuity of m and g with respect to a and k simultaneously under less general conditions than ours. Nevertheless, Theorem 3.2 shows that we can reach all (weakly) efficient elements varying either k or a.

We remark that in classical Parametric Optimization, continuity properties of the marginal function can be proved only under some compactness assumptions. In our forthcomming studies, we will investigate how our problem can be embedded in Parametric Optimization and why our assumptions for continuity can be chosen much weaker.

3.3 Special substitute problems

In this section, we will briefly discuss some corollaries of Theorem 3.2 for $C = \mathcal{C}^+(T)$ and $C = D_\mu$.

Theorem 3.5 *Suppose $F(X) \subseteq a + \operatorname{int} \mathcal{C}^+(T)$. Then $x_0 \in \operatorname{Eff}_w(F(X), \mathcal{C}^+(T)) \iff x_0$ solves problem*

$$(P3) \qquad \max_{t\in T} \frac{f(x,t) - a(t)}{f(x_0,t) - a(t)} \to \min, \quad x \in X,$$

where $f(x,t) \equiv F(x)(t)$ denotes the pointwise evaluation of $F(x)$. If for each other solution x of $(P3)$ $F(x) = F(x_0)$ holds, x_0 is even efficient.

Proof. $(1, x_0)$ solves problem $(P_{C,F(x_0)-a,a})$ iff

$$\begin{aligned} 1 &= \inf\{\tau \in \mathbb{R} : F(x_0) \in a + \tau(F(x_0) - a) - \mathcal{C}^+(T)\} \\ &\leq \inf\{\tau \in \mathbb{R} : F(x) \in a + \tau(F(x_0) - a) - \mathcal{C}^+(T)\} \end{aligned}$$

The problem can be transformed into

$$\begin{aligned} &\tau \to \min \\ &\text{where } f(x,t) \leq a(t) + \tau\, \underbrace{(f(x_0,t) - a(t))}_{>0} \quad \forall\, t \in T, \end{aligned}$$

and straightforward to

$$\max_{t\in T} \frac{f(x,t) - a(t)}{f(x_0,t) - a(t)} \to \min, \quad x \in X.$$

Now from Theorem 3.2 the assertions follow. ■

If $F(X) \subseteq a - \operatorname{int} \mathcal{C}^+(T)$, we have to replace $(P3)$ in Theorem 3.5 by

$$(\tilde{P}2) \qquad \max_{t\in T} \frac{f(x,t) - a(t)}{a(t) - f(x_0,t)} \to \min, \quad x \in X.$$

Problem $(P3)$ corresponds to the scalarization by weightened TCHEBYCHEV norms with weights

$$w(t) := \frac{1}{f(x_0,t) - a(t)}.$$

So, Theorem 3.5 is a infinite-dimensional generalization of [9, Satz 7.7.1].

Applying the results of LUU and OETTLI, we can derive again the assertions of Theorem 2.3: For x_0 beeing a solution of (P3) it is necessary that

$$0 \in \operatorname{cl}_{\sigma(\mathfrak{X}',\mathfrak{X})} \operatorname{conv} \left(\bigcup_{t \in T} \frac{1}{f(x_0,t) - a(t)} f_1'(x_0,t) \right) + N_X(x_0)$$

holds if $f(.,t)$ is FRÉCHET differentiable in x_0 for all $t \in T$ and if the limits in (6) are attained uniformly over T, and

$$0 \in \operatorname{cl}_{\sigma(\mathfrak{X}',\mathfrak{X})} \operatorname{conv} \left(\bigcup_{t \in T} \frac{1}{f(x_0,t) - a(t)} \partial_1 f(x_0,t) \right) + N_X(x_0)$$

if $f(.,t)$ is LIPSCHITZ in a neighborhood of x_0 with the same constant L for all $t \in T$ and if the limits in (7) are attained uniformly over T. Since $0 \in \operatorname{conv}\{e_1, e_2\}$ is equivalent to $0 \in \operatorname{conv}\{\alpha_1 e_1, \alpha_2 e_2\}$ for positive factors $\alpha_i \in \mathbb{R}$, we can omit the weightening factors in the formulars above and receive again the results of Theorem 2.3.

Theorem 3.6 *Suppose $F(X) \subseteq a + \operatorname{int} \mathcal{C}^+(T)$. If x_0 solves for some $\mu \in \mathbb{BM}^+(T)$ the problem*

$$(P3) \qquad \int_T [f(x,t) - a(t)]\, \mu(\mathrm{d}t) \to \min, \quad x \in X,$$

where $f(x,t) \equiv F(x)(t)$ denotes the pointwise evaluation of $F(x)$, then $x_0 \in \operatorname{Eff}(F(X), \mathcal{C}^+(T))$.

The proof is similar to that of Theorem 2.4 and Theorem 3.5.

Problems $(P3)$ and $(P3)$ remind of TCHEBYCHEV scalarization and proper efficiency in the sense of GEOFFRION. In a forthcomming paper, we will discuss different kinds of proper efficiency for multicriterial optimization in $\mathcal{C}(T)$ and how they can be embedded in the concept of scalarization via nonlinear functionals.

4 Conclusions

In this paper, we have derived optimality assertions for multicriteria optimization problems with infinitely many goal functions. These results generalize many known optimality criteria from $\mathbb{R}^n$ to the space $\mathcal{C}(T)$ of continuous functions. On the other hand, we have discussed how the lack of order completeness of $\mathcal{C}(T)$ disturbs any approach to optimality criteria via generalized derivatives.

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