PARABOLIC EVOLUTION EQUATIONS WITH ASYMPTOTICALLY AUTONOMOUS DELAY

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ABSTRACT. We study retarded parabolic non–autonomous evolution equations whose coefficients converge as $t \to \infty$ such that the autonomous problem in the limit has an exponential dichotomy. Then the non–autonomous problem inherits the exponential dichotomy and the solution of the inhomogeneous equation tends to the stationary solution at infinity. We use a generalized characteristic equation to deduce the exponential dichotomy and new representation formulas for the solution of the inhomogeneous equation.

1. INTRODUCTION

In the present paper we continue the investigation of the long–term behaviour of asymptotically autonomous evolution equations begun in [30]. There we studied the Cauchy problem

$$\dot{u}(t) = A(t)u(t) + f(t), \quad t > s \geq 0, \qquad u(s) = x, \tag{1.1}$$

on a Banach space X assuming that the linear operators $A(t)$, $t \geq 0$, are sectorial of the same type and satisfy the 'Acquistapace–Terreni' condition (see (P) below) and that there exists another sectorial operator A such that

$$\lim_{t\to\infty} R(w, A(t)) = R(w, A) \quad \text{in } \mathcal{L}(X, X_\alpha^A) \tag{1.2}$$

for some $\alpha \in (0, 1]$. Here X_α^B is the real interpolation space $(X, D(B))_{\alpha,\infty}$ for a sectorial operator B. Then Theorem 3.3 of [30] shows that the spectral condition

$$\sigma(A) \cap i\mathbb{R} = \emptyset \tag{1.3}$$

implies the exponential dichotomy of the evolution family $U(\cdot, \cdot)$ solving (1.1) with $f = 0$. (Recall that $\sigma(A(t)) \cap i\mathbb{R} = \emptyset$ for $t \in \mathbb{R}$ does not imply the exponential dichotomy of $U(\cdot, \cdot)$.) Moreover, if $f(t) \to f_\infty$, then the mild solution u of (1.1) converges to the stationary solution at infinity, i.e.,

$$\lim_{t\to\infty} u(t) = -A^{-1} f_\infty =: u_\infty$$

provided a compatibility condition for f and x holds which is necessary for the boundedness of u. If f is, e.g., Hölder continuous, then the mild solution is a classical one,

$$\lim_{t\to\infty} \dot{u}(t) = 0, \quad \text{and} \quad \lim_{t\to\infty} A(t)u(t) = Au_\infty \tag{1.4}$$

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due to [30, Thm.4.1]. These results extend a theorem by H. Tanabe, [32, Thm.5.6.1]; see also [15] for closely related facts and [30] for further references. Very recently, C.J.K. Batty and R. Chill showed in [5] that one can allow for $\alpha = 0$ in (1.2) (i.e., convergence in $\mathcal{L}(X)$). This paper extends [30] in several directions, e.g., the almost periodicity of $U(\cdot,\cdot)$ is studied.

We now complement (1.1) by a delay term and treat the retarded problem

$$\dot{u}(t) = A(t)u(t) + L(t)u_t + f(t), \quad t \geq s, \qquad u_s = \phi \in E. \tag{1.5}$$

Here we keep the assumptions on $A(\cdot)$ and f (taking $\alpha = 0$ in (1.2)), let $r \geq 0$, $E := C([-r,0],X)$, $L(\cdot) \in C_b(\mathbb{R}_+, \mathcal{L}_s(E,X))$, the space of uniformly bounded, strongly continuous operator functions, and set $u_t(\xi) := u(t+\xi)$ for $\xi \in [-r,0]$, $t \geq s$, and a function $u : [s-r,\infty) \to X$. We further assume that

$$\lim_{t\to\infty} L(t) = L \qquad \text{in } \mathcal{L}(C^\beta([-r,0],X),X) \tag{1.6}$$

for some $\beta \in [0,1)$. This covers in particular the case

$$L(t)\phi = B(t)\phi(-\tau(t)),$$

where $\tau \in C(\mathbb{R}_+,[-r,0])$ and $B(\cdot) \in C_b(\mathbb{R}_+,\mathcal{L}_s(X))$ converge (in $\mathbb{R}$ and $\mathcal{L}(X)$, respectively) as $t \to \infty$. Again we consider the limit problem

$$\dot{u}(t) = Au(t) + Lu_t, \quad t \geq 0, \qquad u_0 = \phi \in E. \tag{1.7}$$

Recall that $u(t) = [S(t)\phi](0)$ solves (1.7), where the semigroup $S(\cdot)$ on E is generated by

$$B := \tfrac{d}{d\xi}, \;\; D(B) := \{\phi \in C^1([-r,0],X) : \phi(0) \in D(A),\; \phi'(0) = A\phi(0) + L\phi\}. \tag{1.8}$$

It is known that $S(\cdot)$ has exponential dichotomy if and only if

$$\sigma(B) \cap i\mathbb{R} = \emptyset \iff \lambda \in \rho(A + L_\lambda) \quad \forall\, \lambda \in i\mathbb{R}, \tag{1.9}$$

where $L_\lambda \in \mathcal{L}(X)$ is given by $L_\lambda x := Le_\lambda x$ for $\lambda \in \mathbb{C}$ and $e_\lambda(\xi) := e^{\lambda\xi}$, $-r \leq \xi \leq 0$. The right hand side of (1.9) replaces the spectral condition (1.3) and can be considered as a 'generalized characteristic equation' for (1.7). The proofs of these facts are contained in [10, §VI.6], see also [33, §3.1, 3.2] and the references therein.

In this paper we extend our previous results from the undelayed case to the setting introduced above, see [7], [17, §9.5], [20], [22] for related investigations if $X = \mathbb{C}^n$. In Theorem 3.6 we establish that the homogeneous problem (1.5) with $f = 0$ has an exponential dichotomy if condition (1.9) holds. This theorem relies on Proposition 3.2 and 3.3 which show the robustness of exponential dichotomy of (parabolic) partial functional differential equations. In differing settings this subject is also treated in [4], [9], [14], [16], [18], [20], [28]. In Section 4 it is then proved that the mild solution u of the inhomogeneous problem (1.5) converges to the stationary solution at infinity, i.e.,

$$\lim_{t\to\infty} u(t) = -(A + L_0)^{-1} f_\infty.$$

(Here we do not assume the parabolicity of the problem.) Finally, if f and $L(\cdot)$ are, e.g., Hölder continuous in time, then the mild solution of (1.5) is a classical one and satisfies again (1.4) as we show in Section 5. There we also discuss a retarded parabolic partial differential equation.

To deduce the theorems sketched above, we will make extensive use of the regularity properties of the undelayed parabolic equation (1.1) established in [1], [2]. These facts were also used in [5] and [30] to derive convergence results for evolution families which are crucial to the present work. We further need the spectral theory of the associated 'evolution semigroup', cf. [6] or [10, §VI.9], to verify the exponential dichotomy of the homogeneous problem (1.5) with $f = 0$. Our approach relies in particular on a 'generalized characteristic equation' for non–autonomous delay problems derived in [14]. We also employ representation formulas for the mild solution of the inhomogeneous equation investigated in Section 4. Here our methods are inspired by the evolution semigroup approach and by the papers [12], [13], [14], [24]. Several auxiliary results are presented in the next section. Unexplained notation can be found in [10]. By c we denote a generic constant.

2. Prerequisites

Parabolic evolution equations. Let X be a Banach space and J be either $\mathbb{R}$ or $[a,\infty)$. A collection $\{U(t,s) : t \geq s,\ t,s \in J\} \subseteq \mathcal{L}(X)$ is said to be an *evolution family* if

(a) $U(t,s) = U(t,\tau)U(\tau,s), \quad U(s,s) = I,$ and
(b) $(t,s) \mapsto U(t,s)$ is strongly continuous

for $t \geq \tau \geq s$ and $t,\tau,s \in J$. A closed linear operator A on X is called *sectorial* of type (ϕ, K) if it is densely defined and

$$\|R(\lambda, A)\| \leq \frac{K}{1+|\lambda|} \quad \text{for} \quad \lambda \in \Sigma_\phi \subseteq \rho(A)$$

and constants $K \geq 0$, $\phi \in (\pi/2, \pi)$, where $\Sigma_\phi := \{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \phi\}$. Recall that then A generates an exponentially stable analytic semigroup $(e^{tA})_{t\geq 0}$. Given $\theta \in (0,1)$ and an operator A such that $A_w := A - w$ is sectorial for some $w \geq 0$, we set

$$\|x\|_\theta^A := \sup_{\lambda \in \Sigma_\phi} \|\lambda^\theta A_w R(\lambda, A_w)x\| \quad \text{and} \quad X_\theta^A := \{x \in X : \|x\|_\theta^A < \infty\}.$$

Note that $(X_\theta^A, \|\cdot\|_\theta^A)$ is a Banach space continuously embedded in X. Further, $X_1^A := D(A)$ is equipped with the graph norm of A, and $X_0^A := X$. We refer to [3, Chap.I,V], [10, §II.5], or [21, Chap.1,2] for the interpolation theory of (analytic) semigroups. The following hypothesis was introduced by P. Acquistapace and B. Terreni in [2].

(P) The operators $A_w(t) = A(t) - w$, $t \in J$, are sectorial of the same type (ϕ, K) and

$$\|A_w(t)^{-1} - A_w(s)^{-1}\|_{\mathcal{L}(X, X_\nu^t)} \leq L\,|t-s|^\mu$$

for $t,s \in J$ and constants $w, K, L \geq 0$, $\phi \in (\frac{\pi}{2}, \pi)$, and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$.

Here we let $X_\theta^t := X_\theta^{A(t)}$ and $\|x\|_\theta^t := \|x\|_\theta^{A(t)}$. Under these assumptions it is shown in [1, Thm.2.3] that there exists an evolution family $U(t,s)$, $t \geq s$, $t,s \in J$, such that $U(\cdot,s) \in C^1((s,\infty), \mathcal{L}(X))$ and $\partial_t U(t,s) = A(t)U(t,s)$ for $t > s$. Moreover, $u = U(\cdot,s)x \in C^1([s,\infty), X)$ is the unique solution of (1.1) for $x \in D(A(s))$ and $f = 0$, see also [2], [3], [34], [35]. We say that $A(\cdot)$ *generates* $U(\cdot,\cdot)$. This evolution family also possesses parabolic regularity in the sense that

$$\|U(t,s)\|_{\mathcal{L}(X^s_\theta,X^t_\theta)} + (t-s)^\theta \, \|U(t,s)\|_{\mathcal{L}(X,X^t_\theta)} \le C, \tag{2.1}$$

$$\|A(t)U(t,s)\|_{\mathcal{L}(X^s_\theta,X)} \le C\,(t-s)^{\theta-1} \tag{2.2}$$

$$\|U(t,s)-I\|_{\mathcal{L}(X^s_\theta,X)} \le C\,(t-s)^{\theta} \tag{2.3}$$

for $0 < t-s \le 1$, $0 \le \theta \le 1$, and a constant C depending only on the constants in (P). In [35, Thm.2.1, 2.3] similar estimates involving fractional powers are established. In the present context, (2.1) follows by interpolating the cases $\theta = 0, 1$ in [1, Thm.2.3], whereas (2.2) and (2.3) are consequences of the representation formula

$$U(t,s) = e^{(t-s)A(s)} + \int_s^t Z(\tau,s)\,d\tau \tag{2.4}$$

given in [1, (2.6)], the regularity of analytic semigroups, see [21, §2.2], and the estimates for $Z(t,s)$ stated in [1, Lemma 2.2]. (To verify (2.2), use that (2.4) yields

$$A(t)U(t,s) = \partial_t U(t,s) = A(s)e^{(t-s)A(s)} + Z(t,s)$$

for $t > s$.) Similarly one sees that

$$\|A(t)U(t,s) - A(t')U(t',s)\| \le C_\theta\, \varepsilon^{-1-\theta}\,(t'-t)^\theta \tag{2.5}$$

for $s+\varepsilon \le t \le t' \le s+1$, and $\theta \in [0, \mu+\nu-1)$. Note that (2.1) with $\theta = 0$ implies the *exponential boundedness* of $U(t,s)$, i.e., there are constants $M \ge 1$ and $\omega \in \mathbb{R}$ such that

$$\|U(t,s)\| \le Me^{\omega(t-s)} \qquad \text{for } \ t \ge s. \tag{2.6}$$

For $f \in L^1_{loc}(J,X)$ and $x \in X$ the *mild solution* of (1.1) is defined by

$$u(t) = U(t,s)x + \int_s^t U(t,\tau)f(\tau)\,d\tau. \tag{2.7}$$

Every classical, and even strong, solution of (1.1) is a mild one by [1, Prop.3.2]. Conversely, the mild solution is continuously differentiable, $u(t) \in D(A(t))$, and it satisfies (1.1) for $t \ge s$ if $x \in D(A(s))$ and f is Hölder continuous or bounded with respect to interpolation norms due to [2, Thm.6.1, 6.2].

Retarded evolution equations. We now turn our attention to the retarded problem (1.5). Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family on X with $J = [a,\infty), \mathbb{R}$ and let $L(\cdot) \in C_b(J, \mathcal{L}_s(E,X))$. Here and below we use notation introduced in the first section. Define the evolution family $\tilde{U}(\cdot,\cdot)$ on E by

$$(\tilde{U}(t,s)\phi)\,(\xi) := \begin{cases} \phi(t+\xi-s), & -r \le \xi \le s-t, \\ U(t+\xi,s)\phi(0), & s-t \le \xi \le 0, \end{cases} \tag{2.8}$$

for $\xi \in [-r,0]$, $\phi \in E$, and $t \ge s$, $s \in J$. Then there exists a unique exponentially bounded evolution family $V(\cdot,\cdot)$ on E satisfying

$$(V(t,s)\phi)\,(\xi) = (\tilde{U}(t,s)\phi)\,(\xi) + \int_s^{(t+\xi)\vee s} U(t+\xi,\tau)L(\tau)V(\tau,s)\phi\,d\tau \tag{2.9}$$

for $\xi \in [-r,0]$, $\phi \in E$, $t \ge s$, and $s \in J$, where $a \vee b := \max\{a,b\}$. This result is more or less known and proved in e.g. [28, Prop.3.2]. In the autonomous case, where $L(t) = L$ and $U(t,s) = e^{(t-s)A}$, we obtain $V(t,s) = S(t-s)$ for the semigroup $S(\cdot)$ generated by the operator B defined in (1.8), see Lemma VI.6.2 and VI.6.5 in [10] or [33, Thm.3.1.5]. Gronwall's inequality and the estimate (2.6) with $\omega \geq 0$ imply that

$$\|V(t,s)\| \leq M \exp[(\omega + M\|L(\cdot)\|_\infty)(t-s)]. \tag{2.10}$$

Given $\phi \in E$ and $s \in J$, we set

$$u(t) := \begin{cases} \phi(t-s), & t \in [s-r, s], \\ V(t,s)\phi(0), & t \geq s. \end{cases}$$

Formulas (2.9) and (2.8) yield $u_t = V(t,s)\phi$ for $t \geq s$, and u is thus the unique continuous solution of

$$u(t) = U(t,s)\phi(0) + \int_s^t U(t,\tau)L(\tau)u_\tau \, d\tau$$

for $t \geq s$. Further, let $f \in L^1_{loc}(J,X)$. In analogy to (2.7), we call a function $u \in C([s-r,\infty),X)$ satisfying

$$\begin{aligned} u(t) &= U(t,s)\phi(0) + \int_s^t U(t,\tau)\left(L(\tau)u_\tau + f(\tau)\right) d\tau, \quad t \geq s, \\ u(t) &= \phi(t-s), \quad s-r \leq t \leq s, \end{aligned} \tag{2.11}$$

the *mild solution* of the inhomogeneous problem (2.11) (or of (1.5) if $U(\cdot,\cdot)$ is generated by $A(\cdot)$). By a standard fix point argument one easily constructs a mild solution which is unique by Gronwall's inequality, see e.g. [11, Thm.3.2]. In Section 4 we derive representation formulas for the mild solutions, whereas in Proposition 5.2 its regularity is investigated in the parabolic case. For more results in this context we refer to [9], [12], [13], [14], [23], [24], [25], [31], [33, Chap.4].

Exponential dichotomy. An evolution family $U(\cdot,\cdot)$ on X with $J = [a,\infty)$ or $\mathbb{R}$ is said to have an *exponential dichotomy* (or to be *hyperbolic*) if there are projections $P(\cdot) \in C_b(J, \mathcal{L}_s(X))$ and constants $N, \delta > 0$ such that

(a) $U(t,s)P(s) = P(t)U(t,s)$,
(b) the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ of $U(t,s)$ is invertible (and we set $U_Q(s,t) := U_Q(t,s)^{-1}$),
(c) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|U_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$,

for $t \geq s$ and $s \in J$. Here and below we let $Q = I - P$ for a projection P. In the case that $P(t) = I$ for $t \in J$ we call $U(\cdot,\cdot)$ *exponentially stable.* We also need *Green's function*

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t \geq s, \ t,s \in J, \\ -U_Q(t,s)Q(s), & t < s, \ t,s \in J, \end{cases}$$

corresponding to a hyperbolic evolution family $U(\cdot,\cdot)$ with projections $P(\cdot)$.

The dichotomy projections are uniquely determined by (a)–(c) if $J = \mathbb{R}$, [27, Cor.3.3]. This uniqueness fails for $J = [a,\infty)$. If $U(t,s) = T(t-s)$ for $t \geq s$, $J = \mathbb{R}$, and a C_0–semigroup $T(\cdot)$, then $P(t) = P$ does not depend on t by [27, Cor.3.3] so that the standard spectral theory for semigroups applies (compare [10, §IV.3, V.1]).

Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family on X with $J = \mathbb{R}$. As can be seen by simple ODE examples, see e.g. [10, Ex.VI.9.9], we cannot hope to characterize the exponential dichotomy of an evolution family by the spectra of the operators generating it. Instead we use the spectra of the associated *evolution semigroup* given by

$$(T_U(t)f)(s) := U(s, s-t)f(s-t), \quad t \geq 0, \ s \in \mathbb{R}, \ f \in C_0(\mathbb{R}, X),$$

and its generator G_U, cf. [6] or [10, §VI.9]. It is not difficult to show that $T_U(\cdot)$ is in fact a C_0–semigroup on $C_0(\mathbb{R}, X)$ and that

$$(R(\lambda, G_U)f)(t) = \int_{-\infty}^{t} U_\lambda(t, s)f(s)\, ds \tag{2.12}$$

for $t \in \mathbb{R}$, $f \in C_0(\mathbb{R}, X)$, and $\operatorname{Re}\lambda > \omega$, where ω is given as in (2.6) and $U_\lambda(t,s) := e^{-\lambda(t-s)}U(t,s)$. A result by R. Rau, Y. Latushkin, and S. Montgomery–Smith says that

$$U(\cdot,\cdot) \text{ is hyperbolic} \iff 1 \in \rho(T_U(t_0)) \text{ for some } t_0 > 0 \iff 0 \in \rho(G_U) \tag{2.13}$$

and that then the dichotomy projections are given by

$$P(\cdot) = \frac{1}{2\pi i} \int_{\mathbb{T}} R(\lambda, T_U(t_0))\, d\lambda, \tag{2.14}$$

for the unit circle $\mathbb{T}$, see [6, Thm.3.17] or [10, Thm.VI.9.18]. Assume further that $L(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(E, X))$ and let $V(\cdot,\cdot)$ solve (2.9). Evaluating the equivalence (2.13) for the evolution semigroup on $C_0(\mathbb{R}, E)$ induced by $V(t,s)$, it was proved in [14, Cor.3.6] that

$$V(\cdot,\cdot) \text{ is hyperbolic on } E \iff G_U + \widehat{L(\cdot)} \text{ is invertible on } C_0(\mathbb{R}, X), \tag{2.15}$$

where $(\widehat{L(\cdot)}f)(t) := L(t)f_t$ for $f \in C_0(\mathbb{R}, X)$ and $D(G_U + \widehat{L(\cdot)}) := D(G_U)$.

Finally we state a result needed in Section 4 and 5.

Proposition 2.1. *Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family on a Banach space X with $J = \mathbb{R}$ and let A generate the C_0–semigroup $S(\cdot)$. Assume that $U(\cdot,\cdot)$ and $S(\cdot)$ have exponential dichotomy with projections $P(t)$ and P, respectively, and that $U(s+t, s) \to S(t)$ strongly for $t \geq 0$ as $s \to \infty$. Then $P(s) \to P$ strongly as $s \to \infty$.*

Proof. Let $\tilde{F} := \{f \in C(\mathbb{R}, X) : f(t) \to 0 \text{ as } t \to -\infty, \ f(t) \to f_\infty \text{ as } t \to \infty\}$ be endowed with the sup–norm. Clearly,

$$(\tilde{T}_U(t)f)(s) := U(s, s-t)f(s-t), \quad t \geq 0, \ s \in \mathbb{R}, \ f \in \tilde{F}, \tag{2.16}$$

defines an exponentially bounded semigroup $\tilde{T}_U(\cdot)$ on $\tilde{F}$. Its restriction to $F := C_0(\mathbb{R}, X)$ is the evolution semigroup $T_U(\cdot)$. The spaces $\tilde{F}$ and $F \times X =: \hat{F}$ are isomorphic via

$$\Phi : \tilde{F} \to \hat{F}, \ f \mapsto (f - Mf_\infty, f_\infty),$$

where $Mx := \varphi(\cdot)x$ for a fixed function $\varphi \in C(\mathbb{R})$ with support in $\mathbb{R}_+$ and $\lim_{t\to\infty} \varphi(t) = 1$. We consider the induced semigroup $\hat{T}_U(\cdot)$ on $\hat{F}$ given by

$$\hat{T}_U(t) := \Phi \tilde{T}_U(t) \Phi^{-1} = \begin{pmatrix} T_U(t) & \tilde{T}_U(t)M - MS(t) \\ 0 & S(t) \end{pmatrix}, \quad t \geq 0.$$

By the assumption and (2.13), $\lambda - T_U(1)$ and $\lambda - S(1)$ are invertible for $|\lambda| = 1$; hence $\lambda \in \rho(\hat{T}_U(1))$. This means that $\hat{T}_U(\cdot)$ and $\tilde{T}_U(\cdot)$ are hyperbolic with spectral projection

$\hat{\mathcal{P}}$ and $\tilde{\mathcal{P}}$, respectively. On the other hand, formula (2.16) defines a semigroup $T_U^b(\cdot)$ on $C_b(\mathbb{R}, X)$ which is hyperbolic with projection

$$P(\cdot) = \mathcal{P}^b = \frac{1}{2\pi i} \int_{\mathbb{T}} R(\lambda, T_U^b(1))\, d\lambda$$

by [26, Thm.5.6] and the exponential dichotomy of $U(\cdot,\cdot)$. Since $R(\lambda, T_U^b(1))f = R(\lambda, \tilde{T}_U(1))f$ for $f \in \tilde{F}$, we obtain $\tilde{\mathcal{P}} = \mathcal{P}^b = P(\cdot)$. Therefore $P(t)$ converges strongly to a projection P' as $t \to \infty$. Finally,

$$\begin{aligned} \hat{\mathcal{P}} &= \frac{1}{2\pi i} \int_{\mathbb{T}} R(\lambda, \hat{T}_U(1))\, d\lambda = \begin{pmatrix} P(\cdot) & * \\ 0 & P \end{pmatrix} \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \Phi R(\lambda, \tilde{T}_U(1)) \Phi^{-1}\, d\lambda = \Phi P(\cdot) \Phi^{-1} = \begin{pmatrix} P(\cdot) & * \\ 0 & P' \end{pmatrix} \end{aligned}$$

so that $P = P'$. □

3. Exponential dichotomy of retarded evolution equations

We start with two robustness results for exponential dichotomy, where first $L(\cdot)$ and then $U(\cdot,\cdot)$ is varied employing different techniques. Proposition 3.2 shows in particular that dichotomy is not sensitive to small delays in the parabolic case, see [4, Thm.4.2] and [18] for related autonomous results. We point out that in general exponential stability can be destroyed by arbitrary small delays, see [4], [8], [18]. The next preliminary lemma allows to prove the uniformity of the dichotomy constants needed in Proposition 3.8.

Lemma 3.1. *Let $U(\cdot,\cdot)$ be an evolution family with $J = \mathbb{R}$ satisfying (2.6) with $\omega \geq 0$ and $L(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(E, X))$. Assume that the evolution family $V(\cdot,\cdot)$ solving (2.9) on E has an exponential dichotomy with constants $N, \delta > 0$. Then $N \leq N_0$ and $\delta \geq \delta_0 > 0$ where N_0 and δ_0 only depend on $M, \omega, \|L(\cdot)\|_\infty, \|(G_U + \widehat{L(\cdot)})^{-1}\|$. Conversely, $\|(G_U + \widehat{L(\cdot)})^{-1}\| \leq 4N/\delta$.*

Proof. Due to [29, Lemma 3.1], N and δ only depend on the exponential estimate of $V(\cdot,\cdot)$ and the norm of G_V^{-1}. The first assertion thus follows from (2.10) and the formula

$$\begin{aligned} (G_V^{-1} f)(t) &= [(G_U + \widehat{L(\cdot)})^{-1} (f(\cdot, 0) + L(\cdot) Rf)]_t - (Rf)(t), \\ (Rf)(t, \xi) &= \int_\xi^0 f(\xi - \tau + t, \tau)\, d\tau, \qquad f \in C_0(\mathbb{R}, E),\ t \in \mathbb{R},\ \xi \in [-r, 0], \end{aligned}$$

shown in [14, Thm.3.5]. The proof of this theorem also yields

$$(G_U + \widehat{L(\cdot)})^{-1} \varphi = (G_V^{-1} \Phi_L \varphi)(\cdot, 0)$$

for $\varphi \in C_0(\mathbb{R}, X)$ and a map $\Phi_L : C_0(\mathbb{R}, X) \to C_0(\mathbb{R}, E)$ with norm less than 2, cf. [14, (3.12)]. This implies the second assertion since $\|G_V^{-1}\| \leq 2N/\delta$ by [10, Thm.VI.9.18]. □

Proposition 3.2. *Let $A(t)$, $t \in \mathbb{R}$, satisfy (P) and generate $U(\cdot,\cdot)$ on X. Let $V_k(\cdot,\cdot)$ solve (2.9) on E for $U(\cdot,\cdot)$ and $L_k(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(E, X))$, $k = 1, 2$. Assume that*

$$q := \sup_{t \in \mathbb{R}} \|L_1(t) - L_2(t)\|_{\mathcal{L}(C^\beta([-r,0],X),X)} < \infty \tag{3.1}$$

for some $\beta \in [0,1)$ and that $V_1(\cdot,\cdot)$ has an exponential dichotomy. Then there is a number $q_0 > 0$ such that $V_2(\cdot,\cdot)$ also has an exponential dichotomy with constants $N, \delta > 0$ provided that $q \le q_0$. Moreover, $N \le N_0$ and $\delta \ge \delta_0 > 0$ where N_0 and δ_0 only depend on $U(\cdot,\cdot)$, $L_1(\cdot)$, $\|L_2(\cdot)\|_\infty$, and q_0.

Proof. In view of (2.15) we have to show that $G_U + \widehat{L_2(\cdot)}$ with domain $D(G_U)$ is invertible on $C_0(\mathbb{R}, X)$ knowing that $G_U + \widehat{L_1(\cdot)}$ is invertible. Fix $\lambda > \omega$, where $\omega \ge 0$ is given by (2.6). Then $\lambda \in \rho(G_U)$, $(\lambda - G_U)(G_U + \widehat{L_1(\cdot)})^{-1}$ is bounded, and

$$G_U + \widehat{L_2(\cdot)} = \left[I - (\widehat{L_1(\cdot)} - \widehat{L_2(\cdot)})R(\lambda, G_U)(\lambda - G_U)(G_U + \widehat{L_1(\cdot)})^{-1}\right](G_U + \widehat{L_1(\cdot)}). \quad (3.2)$$

We deduce from (3.1), (2.12), (2.3), and (2.1) that

$$\begin{aligned}
&\|(\widehat{L_1(\cdot)} - \widehat{L_2(\cdot)})R(\lambda, G_U)f\|_\infty \\
&\quad\le q\Big[\|R(\lambda, G_U)f\|_\infty + \sup_{\substack{-r\le\eta\le\xi\le 0\\ t\in\mathbb{R}}} |\xi-\eta|^{-\beta}\,\|R(\lambda, G_U)f(t+\xi) - R(\lambda, G_U)f(t+\eta)\|\Big] \\
&\quad\le q\Big[c\,\|f\|_\infty + \sup_{\substack{-r\le\eta\le\xi\le 0\\ t\in\mathbb{R}}} |\xi-\eta|^{-\beta}\Big(\int_{t+\eta}^{t+\xi}\|U_\lambda(t+\xi,\tau)f(\tau)\|\,d\tau \\
&\qquad\quad + \int_{-\infty}^{t+\eta}\|(U_\lambda(t+\xi,t+\eta) - I)U_\lambda(t+\eta,\tau)f(\tau)\|\,d\tau\Big)\Big] \\
&\quad\le c\,q\,\|f\|_\infty
\end{aligned}$$

for $f \in C_0(\mathbb{R}, X)$ and constants $c > 0$ independent of q and f. Thus (3.2) implies the invertibility of $G_U + \widehat{L_2(\cdot)}$ for sufficiently small q. The final assertion is an easy consequence of the previous lemma and (3.2). ☐

Proposition 3.3. *Let $U(\cdot,\cdot)$ and $U_n(\cdot,\cdot)$, $n \in \mathbb{N}$, be evolution families on X with $J = \mathbb{R}$ satisfying (2.6) uniformly in n and let $L_n(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(E,X))$ be uniformly bounded in n. Let $V_n(\cdot,\cdot)$, respectively $\tilde{V}_n(\cdot,\cdot)$, solve (2.9) on E with $J = \mathbb{R}$ for $L_n(\cdot)$ and $U_n(\cdot,\cdot)$, respectively $U(\cdot,\cdot)$. Assume that*

$$\lim_{n\to\infty}\sup_{s\in\mathbb{R}}\|U_n(s+t,s) - U(s+t,s)\|_{\mathcal{L}(X)} = 0 \quad (3.3)$$

for all $t \ge 0$ and that $\tilde{V}_n(\cdot,\cdot)$ is hyperbolic with uniform constants. Then $V_n(\cdot,\cdot)$ is hyperbolic for large n and the dichotomy constants can be chosen independent of n.

Proof. Let $G_{U_n} = G_n$ be the generator of the evolution semigroup $T_n(\cdot)$ on $C_0(\mathbb{R}, X)$ corresponding to $U_n(\cdot,\cdot)$. Due to (2.15) we know that $\tilde{H}_n := G_U + \widehat{L_n(\cdot)}$ with domain $D(G_U)$ is invertible and we have to show the invertibility of $H_n := G_n + \widehat{L_n(\cdot)}$ with domain $D(G_n)$ for large n. Let $R_n(\cdot)$ and $\tilde{R}_n(\cdot)$ be the semigroups on $C_0(\mathbb{R}, X)$ generated by H_n and $\tilde{H}_n$, respectively. By standard perturbation theory, both semigroups are exponentially bounded uniformly in n (with a common exponential bound $\tilde{\omega}$) and we have

$$\begin{aligned}
R_n(t) - \tilde{R}_n(t) = T_n(t) - T_U(t) &+ \int_0^t (T_n(t-s) - T_U(t-s))\widehat{L_n(\cdot)}R_n(s)\,ds \\
&+ \int_0^t T_U(t-s)\widehat{L_n(\cdot)}(R_n(s) - \tilde{R}_n(s))\,ds \quad (3.4)
\end{aligned}$$

for $t \geq 0$. Observe that $t \mapsto \|R_n(t) - \tilde{R}_n(t)\|$ is lower semi–continuous and thus measurable. Therefore the function

$$\varphi : \mathbb{R}_+ \to \mathbb{R}_+; \quad \varphi(t) = \overline{\lim_{n\to\infty}} \|R_n(t) - \tilde{R}_n(t)\|$$

is measurable and also locally bounded. Assumption (3.3) implies that $T_n(t) \to T_U(t)$ in operator norm for $t \geq 0$ as $n \to \infty$. We thus deduce from (3.4)

$$\varphi(t) \leq c \int_0^t \varphi(s)\, ds \qquad \text{for} \quad t \geq 0$$

using Fatou's lemma. Therefore $R_n(t) - \tilde{R}_n(t)$ tends to 0 in operator norm for $t \geq 0$ due to Gronwall's inequality. This yields $R(\lambda, H_n) - R(\lambda, \tilde{H}_n) \to 0$ in $\mathcal{L}(C_0(\mathbb{R}, X))$ as $n \to \infty$ for a fixed $\lambda > \tilde{\omega}$. Observe that

$$I - \lambda R(\lambda, H_n) = \big\{I - \lambda\big(R(\lambda, H_n) - R(\lambda, \tilde{H}_n)\big)[I - \lambda R(\lambda, \tilde{H}_n)]^{-1}\big\}[I - \lambda R(\lambda, \tilde{H}_n)],$$
$$[I - \lambda R(\lambda, \tilde{H}_n)]^{-1} = I - \lambda \tilde{H}_n^{-1}.$$

As a result, $I - \lambda R(\lambda, H_n)$ has uniformly bounded inverse for large n due to Lemma 3.1 and the assumptions. Hence

$$H_n^{-1} = R(\lambda, H_n)[\lambda R(\lambda, H_n) - I]^{-1}$$

exists and is uniformly bounded for large n. The assertion now follows from (2.15) and Lemma 3.1. ☐

We return to the asymptotically autonomous case making the following assumptions.

(H1) The operators $A(t)$, $t \geq 0$, on X satisfy (P) and there is an operator A on X such that $A - w$ is sectorial and

$$o_A(t) := \sup_{s \geq t} \|R(w, A(s)) - R(w, A)\|_{\mathcal{L}(X)} \longrightarrow 0 \qquad \text{as } t \to \infty.$$

(H2) There are operators $L(\cdot) \in C_b(\mathbb{R}_+, \mathcal{L}_s(E, X))$ and $L \in \mathcal{L}(E, X)$ such that

$$o_L(t) := \sup_{s \geq t} \|L(s) - L\|_{\mathcal{L}(C^\beta([-r,0],X),X)} \longrightarrow 0$$

as $t \to \infty$ for some $\beta \in [0, 1)$.

In the remainder of this section $U(\cdot,\cdot)$ is the evolution family on X generated by $A(\cdot)$, $T(\cdot)$ is the semigroup on X generated by A, $V(\cdot,\cdot)$ is the evolution family on E solving (2.9), and $S(\cdot)$ is the semigroup on E generated by B (see (1.8)).

In order to establish our results on exponential dichotomy, we need some convergence and regularity properties of $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$, respectively. For $a \geq 0$ we introduce the evolution family

$$U_a(t,s) := \begin{cases} U(t,s), & t \geq s \geq a, \\ U(t,a)T(a-s), & t > a > s, \\ T(t-s), & a \geq t \geq s, \end{cases} \tag{3.5}$$

on X. We further define

$$F^t_{\theta,h} := \{\phi \in C^h([-r,0], X) : \phi(0) \in X^t_\theta\} \quad \text{and} \quad F^A_{\theta,h} := \{\phi \in C^h([-r,0], X) : \phi(0) \in X^A_\theta\}$$

for $t \geq 0$ and $h, \theta \in [0,1]$ and endow these spaces with their natural norms.

The first assertion in the next lemma follows from [5, Thm.4.7] (see also [30, Lemma 3.2] where (1.2) was supposed). The second assertion is a consequence of the first one, compare the proofs of [5, Prop.6.1] or [30, Thm.3.3].

Lemma 3.4. *Let (H1) hold. Then $U(s+t,s) \to T(t)$ in $\mathcal{L}(X)$ as $s \to \infty$ uniformly for $t \in [t_0, t_1] \subseteq (0,\infty)$. Moreover, $U_a(s+t,s) \to T(t)$ in $\mathcal{L}(X)$ as $a \to \infty$ uniformly for $s \in \mathbb{R}$ and $t \in [t_0,t_1] \subseteq (0,\infty)$.*

Lemma 3.5. *Let $A(t)$, $t \geq 0$, satisfy (P), $L(\cdot) \in C_b(\mathbb{R}_+, \mathcal{L}_s(E,X))$, $0 \leq h \leq \theta < 1$, and $\delta > 0$. Then $V(t,s) : F^s_{\theta,h} \to F^t_{\theta,h}$, $t \geq s \geq 0$, is exponentially bounded and $V(t,s) : E \to F^t_{\theta,h}$ is uniformly bounded for $r + \delta \leq t - s \leq t_0$. Moreover, $V(t,s) : F^s_{\theta,h} \to F^t_{0,h}$, $t \geq s \geq 0$, is exponentially bounded for $0 \leq h \leq \theta \leq 1$.*

Proof. Let $s, t_0 \geq 0$, $0 \leq h \leq \theta < 1$, and $\delta > 0$. From (2.1) and (2.9) we easily deduce

$$\begin{aligned} \|[V(t,s)\phi](0)\|^t_\theta &\leq \begin{cases} c\,\|\phi(0)\|^s_\theta + c\int_s^t (t-\tau)^{-\theta}\|\phi\|_E\,d\tau, & s \leq t \leq s+t_0, \\ c\,\|\phi(0)\| + c\int_s^t (t-\tau)^{-\theta}\|\phi\|_E\,d\tau, & s+\delta \leq t \leq s+t_0, \end{cases} \\ &\leq \begin{cases} c\,\|\phi\|_{F^s_{\theta,0}}, & s \leq t \leq s+t_0, \\ c\,\|\phi\|_E, & s+\delta \leq t \leq s+t_0, \end{cases} \end{aligned} \tag{3.6}$$

where c does not depend on ϕ, s, and t as indicated above (but possibly on t_0 and δ). For $0 \geq \xi \geq \eta \geq -r$ and $t + \eta \geq s$, we can write

$$\begin{aligned} &[V(t,s)\phi](\xi) - [V(t,s)\phi](\eta) \\ &\qquad = (U(t+\xi, t+\eta) - I)\,[V(t+\eta,s)\phi](0) + \int_{t+\eta}^{t+\xi} U(t+\xi,\tau)L(\tau)V(\tau,s)\phi\,d\tau. \end{aligned}$$

Then (2.3) and (3.6) imply

$$\|[V(t,s)\phi](\xi) - [V(t,s)\phi](\eta)\| \leq \begin{cases} c\,|\xi-\eta|^h\,\|\phi\|_{F^s_{h,0}}, & s \leq t \leq s+t_0, \\ c\,|\xi-\eta|^h\,\|\phi\|_E, & s+r+\delta \leq t \leq s+t_0. \end{cases}$$

If $t + \xi \geq s \geq t + \eta$, we obtain analogously

$$\begin{aligned} [V(t,s)\phi](\xi) - [V(t,s)\phi](\eta) &= U(t+\xi,s)\phi(0) - \phi(0) + \phi(0) - \phi(t+\eta-s) \\ &\qquad + \int_s^{t+\xi} U(t+\xi,\tau)L(\tau)V(\tau,s)\phi\,d\tau, \\ \|[V(t,s)\phi](\xi) - [V(t,s)\phi](\eta)\| &\leq c\,|\xi-\eta|^h\,\|\phi\|_{F^s_{h,h}} \end{aligned}$$

for $s \leq t \leq s + t_0$ since $0 \leq s - t - \eta, t + \xi - s \leq \xi - \eta$ in this case. The lemma is an immediate consequence of the above estimates and $X^s_\theta \hookrightarrow X^s_h$, see e.g. [21, Prop.1.2.3]. $\square$

We can now prove that $V(\cdot,\cdot)$ inherits the exponential dichotomy of $S(\cdot)$ which has been characterized in (1.9). For the undelayed problem this result was shown in [5, Thm.6.2, 6.3] (in a more general situation) and in [30, Thm.3.3] (assuming (1.2)).

Theorem 3.6. *Assume that (H1), (H2), and $\sigma(B) \cap i\mathbb{R} = \emptyset$ hold. Then the evolution family $V(\cdot,\cdot)$ solving (2.9) is hyperbolic on an interval $[a,\infty) \subseteq \mathbb{R}_+$ with projections $P_V(s)$. Moreover, $V(s+t,s) \to S(t)$ and $P_V(s) \to P_S$ strongly on E and uniformly for $t \in [0,t_0]$ as $s \to \infty$, where P_S is the dichotomy projection of $S(\cdot)$.*

Proof. (1) Given $a \geq 0$, we define $U_a(\cdot,\cdot)$ as in (3.5) and

$$L_a(t) := \begin{cases} L(t), & t \geq a, \\ (t-a+1)L(t) + (a-t)L, & a-1 < t < a, \\ L, & t \leq a-1. \end{cases}$$

Note that $L_a(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(E,X))$ and

$$\|L_a(t) - L\|_{\mathcal{L}(C^\beta([-r,0],X),X)} \leq \begin{cases} o_L(a-1), & t > a-1, \\ 0, & t \leq a-1, \end{cases}$$

by (H2). We apply Proposition 3.2 with $U(t,s)$ replaced by $T(t-s)$, $L_1(t)$ by L, and $L_2(t)$ by $L_a(t)$. Consequently there exists an $a_0 \geq 0$ such that the evolution family solving (2.9) for $T(t-s)$ and $L_a(t)$ has an exponential dichotomy for $a \geq a_0$. Lemma 3.4 and Proposition 3.3 (applied to $T(t-s)$, $U_a(t,s)$, and $L_a(t)$) show that there is a number $a_1 \geq a_0$ such that the evolution family $V_a(t,s)$ solving (2.9) for $U_a(t,s)$ and $L_a(t)$ has an exponential dichotomy if $a \geq a_1$. This implies the first assertion since $V(t,s) = V_a(t,s)$ for $t \geq s \geq a$ by the uniqueness of (2.9).

(2) Let $\phi \in F^s_{1,\beta}$, $s \geq a$, $t \in [0,t_0]$, and $\xi \in [-r,0]$ with $t+\xi \geq 0$. Then (2.9) yields

$$\begin{aligned} V(s+t,s)\phi(\xi) - S(t)\phi(\xi) &= U(s+t+\xi,s)\phi(0) - T(t+\xi)\phi(0) \\ &\quad + \int_s^{s+t+\xi} \left(U(s+t+\xi,\tau) - T(s+t+\xi-\tau)\right) L(\tau)V(\tau,s)\phi\, d\tau \\ &\quad + \int_s^{s+t+\xi} T(s+t+\xi-\tau)\left(L(\tau)-L\right)V(\tau,s)\phi\, d\tau \\ &\quad + \int_s^{s+t+\xi} T(s+t+\xi-\tau)L\left(V(\tau,s)\phi - S(\tau-s)\phi\right) d\tau. \\ &=: S_1 + S_2 + S_3 + S_4. \end{aligned} \tag{3.7}$$

Observe that

$$\begin{aligned} &U(s+h,s) - T(h) \\ &\quad = U(s+h,s) - e^{hA(s)} + \frac{1}{2\pi i}\int_\Gamma e^{\lambda h} A_w R(\lambda,A)(A_w^{-1} - A_w(s)^{-1})\, A_w(s)R(\lambda,A(s))\, d\lambda \end{aligned}$$

for a suitable path Γ in $\mathbb{C}$. Formula (2.6) and Lemma 2.2 of [1] thus imply

$$\|U(s+t+\xi,s)\phi(0) - T(t+\xi)\phi(0)\| \leq c'\left((t+\xi)^{\mu+\nu-1} + o_A(s)\right)\|\phi(0)\|_1^s$$

for a constant independent of s, ξ, $t \in [0,t_0]$, and ϕ. Let $\delta > 0$ and $\tau_0 := (\frac{\delta}{2c'})^{\frac{1}{\mu+\nu-1}}$. Using the above estimate for $0 \leq t+\xi \leq \tau_0$ and Lemma 3.4 for $\tau_0 \leq t+\xi \leq t_0$, we obtain a number $s_1(\delta) \geq a$ such that

$$\|S_1\| \leq \delta\, \|\phi(0)\|_1^s \tag{3.8}$$

for $s \geq s_1(\delta)$. We can further write

$$S_2 = \int_s^{s\vee(s+t+\xi-\delta)} \cdots\, d\tau + \int_{s\vee(s+t+\xi-\delta)}^{s+t+\xi} \cdots\, d\tau. \tag{3.9}$$

Combining (3.7)–(3.9) with (H2), Lemma 3.4 and 3.5, one finds $s_2(\delta) \geq s_1(\delta)$ such that

$$
\begin{aligned}
&\|V(s+t,s)\phi - S(t)\phi\|_E \\
&\qquad \leq c\,\delta \|\phi\|_{F^s_{1,0}} + c\,o_L(s) \sup_{0\leq\tau\leq t} \|V(\tau+s,s)\phi\|_{C^\beta} + c\int_s^{s+t} \|V(s+\tau,s)\phi - S(\tau)\phi\|_E\,d\tau \\
&\qquad \leq c\,\delta\|\phi\|_{F^s_{1,\beta}} + c\int_s^{s+t} \|V(s+\tau,s)\phi - S(\tau)\phi\|_E\,d\tau
\end{aligned}
$$

for $s \geq s_2(\delta)$, where the constants c do not depend on δ, ϕ, s, and $t \in [0,t_0]$. So Gronwall's equality shows that

$$\|V(s+t,s)\phi - S(t)\phi\|_E \leq c\,\delta\|\phi\|_{F^s_{1,\beta}} \tag{3.10}$$

for $s \geq s_2(\delta)$. Given an $\varepsilon > 0$ and a $\phi \in E$, fix $\mu > w$ and $\phi_1 \in C^\beta([-r,0],X)$ with

$$\sup_{-r\leq\xi\leq 0} \|\mu R(\mu,A)\phi(\xi) - \phi(\xi)\| \leq \varepsilon \quad \text{and} \quad \|\phi - \phi_1\|_E \leq \varepsilon.$$

Set $g(s) := \mu R(\mu, A(s))\phi_1 \in F^s_{1,\beta}$ for $s \geq 0$. Note that

$$\sup_{s\geq 0} \|g(s)\|_{F^s_{1,\beta}} < \infty \quad \text{and} \quad \overline{\lim_{s\to\infty}}\, \|g(s) - \phi\|_E \leq (K+1)\varepsilon \tag{3.11}$$

since $R(\mu, A(s))$ converges strongly to $R(\mu,A)$ as $s \to \infty$ due to (H1) and the Trotter–Kato theorem. Finally (3.11) and (3.10) lead to

$$
\begin{aligned}
&\overline{\lim_{s\to\infty}}\, \|V(s+t,s)\phi - S(t)\phi\|_E \\
&\leq \overline{\lim_{s\to\infty}}\, \left(\|V(s+t,s)(\phi - g(s))\|_E + \|V(s+t,s)g(s) - S(t)g(s)\|_E + \|S(t)(g(s)-\phi)\|_E\right) \\
&\leq c\varepsilon + \overline{\lim_{s\to\infty}}\, \|V(s+t,s)g(s) - S(t)g(s)\|_E \leq c\,\varepsilon
\end{aligned}
$$

for a constant c independent of ε, s, and $t \in [0,t_0]$. Hence, $V(s+t,s) \to S(t)$ and, by Proposition 2.1, $P_V(s) \to P_S$ strongly on E and uniformly in $t \in [0,t_0]$ as $s \to \infty$. □

Remark 3.7. If $V(\cdot,\cdot)$ is exponentially stable, then we can take $a = 0$ in the above theorem. More generally, this is true if $\dim Q_S E$ is finite, (3.12) holds, and the adjoint $U(a,0)^*$ is injective on $Q^*(a)E^*$. This follows from Lemma 2.4, 2.5, and Remark 2.6 of [20] and Proposition 3.8 below.

Under stronger assumptions we can also show that $V(t,s)$ inherits the dimension of the unstable subspace of $S(t)$. In view of [3, Ex.IV.2.6.3] and [34, §4], condition (3.12) is true for second order elliptic operators on L^p–spaces with Neumann boundary conditions and sufficiently regular data (provided that $\beta \leq 1/2$).

Proposition 3.8. *Let (H1), (H2), and $\sigma(B) \cap i\mathbb{R} = \emptyset$ hold. We further assume that*

$$\text{(1.2) holds and } X^t_\theta \cong X^A_\theta\,,\ t \geq 0, \text{ with uniformly equivalent norms} \tag{3.12}$$

for some $\alpha \in (0,1]$ and $\theta = \beta \vee (1-\alpha)$. Then $\dim Q_V(t)E = \dim Q_S E$ for $t \geq a$, where $Q_V(t)$, Q_S, and a are given as in Theorem 3.6.

Proof. (1) The proof of Theorem 3.6 and Lemma 3.1 show that there is an a_1 such that $V_a(\cdot,\cdot)$ is hyperbolic with uniform constants $N, \delta > 0$ for $a \geq a_1$, where $V_a(\cdot,\cdot)$ solves

(2.9) for $L_a(\cdot)$ and $U_a(\cdot,\cdot)$. Assumption (3.12) yields that $F^t_{\theta,\beta} \cong F^A_{\theta,\beta}$ for $t \geq 0$ and $\theta = \beta \vee (1-\alpha)$. Thus,

$$c(t) := \sup_{a \geq a_1, s \in \mathbb{R}} \|V_a(s+t,s)\|_{\mathcal{L}(E,F^A_{\theta,\beta})} < \infty$$

by Lemma 3.5 if $t \geq 2r+1 =: t_0$ so that, setting $Q_a(t) := Q_{V_a}(t)$,

$$\begin{aligned}\sup_{a \geq a_1, s \in \mathbb{R}} \|Q_a(s)\|_{\mathcal{L}(E,F^A_{\theta,\beta})} &= \sup_{a \geq a_1, s \in \mathbb{R}} \|V_a(s,s-t_0)V_{a,Q}(s-t_0,s)Q_a(s)\|_{\mathcal{L}(E,F^A_{\theta,\beta})} \\ &\leq c(t_0) N e^{-\delta t_0}.\end{aligned}$$

In the same way we see that $Q_S \in \mathcal{L}(E, F^A_{\theta,\beta})$. Similarly as (3.10) one derives

$$\lim_{a\to\infty} \sup_{s \in \mathbb{R}} \|V_a(s+t,s) - S(t)\|_{\mathcal{L}(F^A_{\theta,\beta},E)} = 0.$$

We now use (2.14) to deduce

$$(Q_S - Q_a(\cdot))Q_a(\cdot) = \frac{1}{2\pi i}\int_{\mathbb{T}} R(\lambda, T_S(1))(T_{V_a}(1) - T_S(1))Q_a(\cdot)R(\lambda, T_{V_a}(1))\, d\lambda$$

$$(Q_S - Q_a(\cdot))Q_S = \frac{1}{2\pi i}\int_{\mathbb{T}} R(\lambda, T_{V_a}(1))(T_{V_a}(1) - T_S(1))Q_S R(\lambda, T_S(1))\, d\lambda,$$

Due to the uniformity of N and δ, the norms of $R(\lambda, T_{V_a}(1))$ are independent of a (see the proof of [29, Prop.3.3]). Combining all these facts, we arrive at

$$\lim_{a\to\infty} \sup_{t \in \mathbb{R}} \|(Q_S - Q_a(t))Q_S\| = \lim_{a\to\infty} \sup_{t \in \mathbb{R}} \|(Q_S - Q_a(t))Q_a(t)\| = 0. \tag{3.13}$$

(2) Recall that a projection P has the resolvent $R(\lambda, P) = \lambda^{-1}(1 + (\lambda-1)^{-1}P)$ for $\lambda \neq 0, 1$. Therefore $R(\lambda, Q_a(t))$ is uniformly bounded for $a \geq a_1$ and $|\lambda - 1| = 1/2$ since $\|Q_a(t)\| \leq N$. Due to (3.13) and the (proof of) [19, Thm.IV.3.16], there is a number $a_2 \geq a_1$ such that $\sigma(Q_S Q_a(t))$, $a \geq a_2$, has an open and closed subset in the disc $|\lambda - 1| < 1/2$ and the corresponding spectral subspace E_1 has the dimension $\dim Q_a(t)E$. Since $(Q_S Q_a(t))|E_1$ is invertible, this identity leads to

$$\dim Q_a(t)E = \dim Q_S Q_a(t)E_1 \leq \dim Q_S Q_a(t)E \leq \dim Q_S E.$$

Analogously one shows the converse inequality so that

$$\dim Q_a(t)E = \dim Q_S E$$

for $a \geq a_3 \geq a_2$ and $t \in \mathbb{R}$. Because of property (b) in the definition of exponential dichotomy it remains to verify that

$$\dim Q_a(t)E = \dim Q_{a_1}(t)E \tag{3.14}$$

for some $t \in \mathbb{R}$ where we have fixed $a \geq a_3 \geq a_1$. Observe that the stable subspaces are given by

$$P_a(s)E = \{\phi \in E : \lim_{t\to\infty} V(t,s)\phi = 0\}, \quad s \geq a,$$

see e.g. [27, p.551], and thus coincide with $P_{a_1}(s)E$. As a consequence,

$$\begin{aligned}&P_a(t) = P_a(t)\,(P_{a_1}(t) + Q_{a_1}(t)) = P_{a_1}(t) + P_a(t)V(t,a)V_{a_1 Q}(a,t)Q_{a_1}(t), \\ &\|Q_a(t) - Q_{a_1}(t)\| \leq N^2 e^{-2\delta(t-a)} < 1\end{aligned}$$

for large $t \geq a$. The assertion (3.14) now follows from (the arguments in) [19, §I.4.6]. ☐

4. Convergence of mild solutions to the inhomogenous problem

We now turn our attention to the inhomogeneous problem (2.11) assuming that $f(t) \to f_\infty$ as $t \to \infty$. We want to show that

$$\lim_{t\to\infty} u(t) = -(A + L_0)^{-1} f_\infty =: u_\infty$$

for the mild solution u of (2.11). To that purpose we do not need to restrict ourselves to the parabolic case (H1). So we only assume that $L(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(X,E))$ and $U(\cdot,\cdot)$ is an evolution family on X with $J = \mathbb{R}$ satisfying (2.6) with $\omega \geq 0$ (in order to simplify several estimates). Let $V(\cdot,\cdot)$ be the evolution family on E solving (2.9).

We derive in Corollary 4.2 and Proposition 4.3 new 'variation of parameters formulas' expressing the mild solution in terms of $V(\cdot,\cdot)$ and f. Here the main difficulty comes from the fact that f maps into X whereas $V(t,s)$ acts on E. To deal with this problem, we set

$$(E_\lambda f)(t,\xi) := e^{\lambda\xi} \int_{-\infty}^{t+\xi} U_\lambda(t+\xi,\tau) f(\tau)\, d\tau, \quad t \in \mathbb{R},\ \xi \in [-r,0],$$

for $\lambda > \omega$ and $f \in L^\infty(\mathbb{R},X)$, cf. [14, (3.4)]. Clearly, $E_\lambda f \in C_b(\mathbb{R},E)$ and

$$\|E_\lambda f\|_\infty \leq \frac{M}{\lambda - \omega} \|f\|_\infty. \tag{4.1}$$

We first show the convergence as $\lambda \to \infty$ of the function

$$w_\lambda(t) = w_\lambda(t;\, s, f) := \int_s^t V(t,\tau)\lambda(E_\lambda f)(\tau)\, d\tau, \quad t \geq s. \tag{4.2}$$

Proposition 4.1. *Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family on X with $J = \mathbb{R}$ and $L(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(E,X))$. Define w_λ as in (4.2) for $f \in L^\infty(\mathbb{R},X)$, $t \geq s$, and $\lambda > \omega$. Then $w_\lambda(t)$ converges in E to $w(t) = w(t;\, s,f)$ as $\lambda \to \infty$ uniformly for $0 \leq t - s \leq t_0$ and $\|f\|_\infty \leq c$. Moreover,*

$$w(t;\, s,f) = \int_s^{(t+\cdot)\vee s} U(t+\cdot,\tau)(L(\tau)w(\tau;\, s,f) + f(\tau))\, d\tau \qquad \textit{for } t \geq s. \tag{4.3}$$

Proof. Let $t \geq s$, $f \in L^\infty(\mathbb{R},X)$, and $\lambda,\mu > \omega$. The identities (4.2) and (2.9) yield

$$\begin{aligned} w_\lambda(t) &= \int_s^t \tilde{U}(t,\tau)\lambda(E_\lambda f)(\tau)\, d\tau + \int_s^t \int_\tau^{(t+\cdot)\vee\tau} U(t+\cdot,\sigma)L(\sigma)V(\sigma,\tau)\lambda(E_\lambda f)(\tau)\, d\sigma\, d\tau \\ &=: I_1 + I_2. \end{aligned} \tag{4.4}$$

Notice that

$$[\tilde{U}(t,\tau)\lambda(E_\lambda f)(\tau)](\xi) = \begin{cases} \lambda e^{\lambda(t+\xi-\tau)} \int_{-\infty}^{t+\xi} U_\lambda(t+\xi,\sigma) f(\sigma)\, d\sigma, & t+\xi \leq \tau, \\ \lambda e^{\lambda(t+\xi-\tau)} \int_{-\infty}^{\tau} U_\lambda(t+\xi,\sigma) f(\sigma)\, d\sigma, & t+\xi > \tau, \end{cases}$$

by (2.8). First, let $t+\xi \geq s$. Integrating by parts, we can transform I_1 into

$$\begin{aligned}
I_1(\xi) = &\int_s^{t+\xi} \lambda e^{\lambda(t+\xi-\tau)} \int_{-\infty}^{\tau} U_\lambda(t+\xi,\sigma)f(\sigma)\,d\sigma\,d\tau \\
&+ \int_{t+\xi}^{t} \lambda e^{\lambda(t+\xi-\tau)}\,d\tau \int_{-\infty}^{t+\xi} U_\lambda(t+\xi,\sigma)f(\sigma)\,d\sigma \\
= &\int_s^{t+\xi} e^{\lambda(t+\xi-\tau)}U_\lambda(t+\xi,\tau)f(\tau)\,d\tau - \int_{-\infty}^{t+\xi} U_\lambda(t+\xi,\sigma)f(\sigma)\,d\sigma \\
&+ e^{\lambda(t+\xi-s)} \int_{-\infty}^{s} U_\lambda(t+\xi,\sigma)f(\sigma)\,d\sigma + (1-e^{\lambda\xi}) \int_{-\infty}^{t+\xi} U_\lambda(t+\xi,\sigma)f(\sigma)\,d\sigma \\
= &\int_s^{t+\xi} U(t+\xi,\tau)f(\tau)\,d\tau + U(t+\xi,s)\int_{-\infty}^{s} U_\lambda(s,\tau)f(\tau)\,d\tau \\
&- \int_{-\infty}^{t+\xi} e^{-\lambda(t-\sigma)}U(t+\xi,\sigma)f(\sigma)\,d\sigma.
\end{aligned} \tag{4.5}$$

Fubini's theorem further implies

$$\begin{aligned}
I_2(\xi) &= \int_s^{t+\xi}\int_s^{\sigma} U(t+\xi,\sigma)L(\sigma)V(\sigma,\tau)\lambda(E_\lambda f)(\tau)\,d\tau\,d\sigma \\
&= \int_s^{t+\xi} U(t+\xi,\sigma)L(\sigma)w_\lambda(\sigma)\,d\sigma.
\end{aligned} \tag{4.6}$$

Similarly, we obtain

$$[w_\lambda(t)](\xi) = (1-e^{\lambda\xi}) \int_{-\infty}^{t+\xi} U_\lambda(t+\xi,\sigma)f(\sigma)\,d\sigma \tag{4.7}$$

for $t+\xi \leq s$. Combining (4.4)–(4.7), we estimate

$$\begin{aligned}
\|w_\lambda(t)-w_\mu(t)\|_E &\leq 2Me^{\omega(t-s)}\Big(\frac{1}{\lambda-\omega}+\frac{1}{\mu-\omega}\Big)\|f\|_\infty \\
&\qquad + M\|L(\cdot)\|_\infty \int_s^t e^{\omega(t-\sigma)}\|w_\lambda(\sigma)-w_\mu(\sigma)\|_E\,d\sigma, \\
\|w_\lambda(t)-w_\mu(t)\|_E &\leq c\,e^{(\omega+d)(t-s)}\Big(\frac{1}{\lambda-\omega}+\frac{1}{\mu-\omega}\Big)\|f\|_\infty
\end{aligned}$$

using Gronwall's inequality, where c and d do not depend on μ,λ,t,s. Hence w_λ tends to a function w as asserted. The identity (4.3) follows from (4.4)–(4.7) by letting $\lambda\to\infty$. $\square$

We now define, for $t\geq s$, $\phi\in E$, and $f\in L^\infty(\mathbb{R},X)$,

$$v(t;\,s,f) = v(t) := V(t,s)\phi + \lim_{\lambda\to\infty}\int_s^t V(t,\tau)\lambda(E_\lambda f)(\tau)\,d\tau = V(t,s)\phi + w(t;\,s,f), \tag{4.8}$$

$$u(t) := \begin{cases} [v(t)](0), & t\geq s, \\ \phi(t-s), & s-r\leq t\leq s. \end{cases} \tag{4.9}$$

The formulas (2.9), (4.3), and (4.8) yield

$$v(t) = \tilde{U}(t,s)\phi + \int_s^{(t+\cdot)\vee s} U(t+\cdot,\tau)(L(\tau)v(\tau)+f(\tau))\,d\tau. \tag{4.10}$$

This identity shows that $v(t) = u_t$ and, hence, that u solves (2.11).

Corollary 4.2. *Under the above assumptions,* (4.8) *and* (4.9) *yield the unique mild solution u of* (2.11)*, and $u_t = v(t)$.*

We note that there are other formulas for mild solutions using 'fundamental solutions' or 'resolvent operators', see [9], [23], [31], [33, Chap.4], and the references therein. These authors consider different situations partially obtaining stronger results. We have chosen our approach in view of the techniques employed in the proof of Theorem 4.4.

In a second step we want to replace in the above formulas the integrals over $[s,t]$ by integrals over $\mathbb{R}$. To that purpose we assume that $V(\cdot,\cdot)$ has an exponential dichotomy with constants $N, \delta > 0$, projections $P_V(t)$, and Green's function $\Gamma_V(\cdot,\cdot)$. We then introduce

$$\tilde{w}_\lambda(t) = \tilde{w}_\lambda(t; f) := \int_{\mathbb{R}} \Gamma_V(t,\tau)\lambda(E_\lambda f)(\tau)\, d\tau \tag{4.11}$$

for $t \in \mathbb{R}$, $\lambda > \omega$, and $f \in L^\infty(\mathbb{R}, X)$. Observe that $\tilde{w}_\lambda$ is continuous and

$$\|\tilde{w}_\lambda(t; f)\| \le 2NM(1+\omega)\delta^{-1}\, \|f\|_\infty\,, \qquad \lambda \ge \omega + 1,\ t \in \mathbb{R}, \tag{4.12}$$

by (4.1). Using a standard argument (see e.g. [6, p.108]), we deduce from the definitions (4.11) and (4.2) that

$$\begin{aligned} \tilde{w}_\lambda(t; f) &= V(t,s)\tilde{w}_\lambda(s; f) + \int_s^t V(t,\tau)\lambda(E_\lambda f)(\tau)\, d\tau \\ &= V(t,s)\tilde{w}_\lambda(s; f) + w_\lambda(t;\, s, f) \end{aligned} \tag{4.13}$$

for $t \ge s$. This expression is the key to the next result.

Proposition 4.3. *Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family on X with $J = \mathbb{R}$ and $L(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(E, X))$. Assume that the evolution family $V(\cdot,\cdot)$ solving* (2.9) *on E has an exponential dichotomy. Define $\tilde{w}_\lambda$ as in* (4.11) *for $f \in L^\infty(\mathbb{R}, X)$ and $\lambda > \omega$. Then $\tilde{w}_\lambda$ converges in $C_b(\mathbb{R}, E)$ to a function $\tilde{w} = \tilde{w}(\cdot; f)$ as $\lambda \to \infty$ uniformly for $\|f\|_\infty \le c$. Moreover,*

$$\tilde{w}(t;\, f) = V(t,s)\tilde{w}(s; f) + w(t;\, s, f), \quad t \ge s, \tag{4.14}$$

and the mild solution of (2.11) *is given by $u(t) = [v(t)](0)$ and*

$$v(t) = V(t,s)(\phi - \tilde{w}(s;\, f)) + \tilde{w}(t;\, f) \tag{4.15}$$

for $t \ge s$, $\phi \in E$, and $f \in L^\infty(\mathbb{R}, X)$.

Proof. Formula (4.13) with s replaced by $t - s$, (4.12), and the exponential dichotomy of $V(\cdot,\cdot)$ imply

$$\|P_V(t)(\tilde{w}_\lambda(t) - \tilde{w}_\mu(t))\|_E \le c\, e^{-\delta s}\, \|f\|_\infty + N\, \|w_\lambda(t;\, t-s, f) - w_\mu(t;\, t-s, f)\|_E$$

for $\lambda, \mu \ge \omega + 1$, $t \in \mathbb{R}$, $s \ge 0$, and a constant $c \ge 0$. Fix $\varepsilon > 0$ and choose $s \ge 0$ such that $c\, e^{-\delta s}\|f\|_\infty \le \varepsilon$. Proposition 4.1 then shows that

$$\overline{\lim_{\lambda,\mu\to\infty}}\, \sup_{t\in\mathbb{R}} \|P_V(t)(\tilde{w}_\lambda(t) - \tilde{w}_\mu(t))\|_E \le \varepsilon.$$

As a result, $P_V(\cdot)\tilde{w}_\lambda$ converges in $C_b(\mathbb{R},E)$ as $\lambda\to\infty$. In a similar way we deduce from (4.13) with s replaced by t and t replaced by $s+t$ that

$$\begin{aligned}&V_Q(t,t+s)Q_V(t+s)[\tilde{w}_\lambda(t+s)-\tilde{w}_\mu(t+s)]\\&\qquad=Q_V(t)[\tilde{w}_\lambda(t)-\tilde{w}_\mu(t)]+V_Q(t,t+s)Q_V(t+s)[w_\lambda(t+s;\,t,f)-w_\mu(t+s;\,t,f)],\\&\|Q_V(t)(\tilde{w}_\lambda(t)-\tilde{w}_\mu(t))\|_E\le c\,e^{-\delta s}\,\|f\|_\infty+N\,e^{-\delta s}\,\|w_\lambda(t+s;\,t,f)-w_\mu(t+s;\,t,f)\|.\end{aligned}$$

As above we see that $Q_V(\cdot)\tilde{w}_\lambda$ converges in $C_b(\mathbb{R},E)$ as $\lambda\to\infty$. The identity (4.14) then follows from (4.13) and Proposition 4.1. The last assertion is a consequence of Corollary 4.2, (4.8), and (4.14). □

We are now in a position to establish the convergence of $v(t)$ as $t\to\infty$. Let $s\in\mathbb{R}$ and $f\in L^\infty(\mathbb{R},X)$ with support in $[s,\infty)$. Because of (4.15) and the boundedness of $\tilde{w}$, the function v is bounded if and only if $\phi-\tilde{w}(s;\,f)\in P_V(s)E$ if and only if

$$Q_V(s)\phi=\tilde{w}(s;\,f)=-\lim_{\lambda\to\infty}\int_s^\infty V_Q(s,\tau)\lambda(E_\lambda f)(\tau)\,d\tau. \tag{4.16}$$

If this is the case, then

$$v(t)=V(t,s)P_V(s)\phi+\tilde{w}(t;\,f),\qquad t\ge s. \tag{4.17}$$

Theorem 4.4. *Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family on X with $J=\mathbb{R}$ and $L(\cdot)\in C_b(\mathbb{R},\mathcal{L}_s(E,X))$. Take $s\in\mathbb{R}$, $\phi\in E$, and $f\in L^\infty(\mathbb{R},X)$ with support in $[s,\infty)$ such that* (4.16) *holds and $f(t)\to f_\infty$ in X as $t\to\infty$. Assume that the evolution family $V(\cdot,\cdot)$ solving* (2.9) *on E has an exponential dichotomy with projections $P_V(t)$. Let A generate a C_0–semigroup $T(\cdot)$ on X, $L\in\mathcal{L}(X,E)$, and $S(\cdot)$ be the C_0–semigroup on E generated by B (see* (1.8)*). Suppose further that $S(\cdot)$ has an exponential dichotomy with projection P_S. Finally, assume that $U(\tau+t,\tau)\to T(t)$ as $\tau\to\infty$ strongly on X for $t\ge0$ and that $V(\tau+t,\tau)\to S(t)$ and $P_V(\tau)\to P_S$ as $\tau\to\infty$ strongly on E for $t\ge0$. Then the function v given by* (4.8) *converges in E to $v_\infty(\xi)=u_\infty=-(A+L_0)^{-1}f_\infty$, $\xi\in[-r,0]$, and the mild solution $u(t)=[v(t)](0)$ of* (2.11) *tends to u_∞ in X as $t\to\infty$.*

Proof. Proposition 4.3 implies that

$$\lim_{t\to\infty}\tilde{w}(t)=\lim_{t\to\infty}\lim_{\lambda\to\infty}\tilde{w}_\lambda(t)=\lim_{\lambda\to\infty}\lim_{t\to\infty}\tilde{w}_\lambda(t) \tag{4.18}$$

if the double limit on the right hand side exists. Because of (4.17) and the exponential dichotomy of $V(\cdot,\cdot)$ we thus have to show that the limits

$$\lim_{t\to\infty}\tilde{w}_\lambda(t)=:w_{\lambda,\infty}\qquad\text{and}\qquad\lim_{\lambda\to\infty}w_{\lambda,\infty}\equiv-(A+L_0)^{-1}f_\infty \tag{4.19}$$

exist in E. We first write

$$\begin{aligned}\tilde{w}_\lambda(t)=&\int_0^\infty V(t,t-\tau)P_V(t-\tau)\lambda e^{\lambda\cdot}\int_{-\infty}^{t-\tau+\cdot}U_\lambda(t-\tau+\cdot,\sigma)f(\sigma)\,d\sigma\,d\tau \qquad (4.20)\\&-\int_0^\infty V_Q(t,t+\tau)Q_V(t+\tau)\lambda e^{\lambda\cdot}\int_{-\infty}^{t+\tau+\cdot}U_\lambda(t+\tau+\cdot,\sigma)f(\sigma)\,d\sigma\,d\tau\\=&\int_0^\infty V(t,t-\tau)P_V(t-\tau)\lambda e^{\lambda\cdot}\int_0^\infty U_\lambda(t-\tau+\cdot,t-\tau+\cdot-\sigma)f(t-\tau+\cdot-\sigma)\,d\sigma\,d\tau\\&-\int_0^\infty V_Q(t,t+\tau)Q_V(t+\tau)\lambda e^{\lambda\cdot}\int_0^\infty U_\lambda(t+\tau+\cdot,t+\tau+\cdot-\sigma)f(t+\tau+\cdot-\sigma)\,d\sigma\,d\tau.\end{aligned}$$

We may assume that $T(\cdot)$ also satisfies (2.6). Fix $\varepsilon \in (0,1)$, $\tau \in \mathbb{R}$, and $\xi \in [-r,0]$. For $t \in \mathbb{R}$, $\lambda \geq \omega+1$, and $t_0 > 0$, we further compute

$$\begin{aligned}
D_{\mp}(t,\tau,\xi) &:= \int_0^\infty [U_\lambda(t\mp\tau+\xi, t\mp\tau+\xi-\sigma)f(t\mp\tau+\xi-\sigma) - e^{-\lambda\sigma}T(\sigma)f_\infty]\,d\sigma \\
&= \int_0^\infty U_\lambda(t\mp\tau+\xi, t\mp\tau+\xi-\sigma)\,(f(t\mp\tau+\xi-\sigma)-f_\infty)\,d\sigma \\
&\quad + \int_0^\infty e^{-\lambda\sigma}[U(t\mp\tau+\xi, t\mp\tau+\xi-\sigma)-T(\sigma)]\,f_\infty\,d\sigma, \\
\|D_{\mp}(t,\tau,\xi)\| &\leq \int_0^{t_0} Me^{(\omega-\lambda)\sigma}\,\|f(t\mp\tau+\xi-\sigma)-f_\infty\|\,d\sigma + \int_{t_0}^\infty 4Me^{(\omega-\lambda)\sigma}\|f\|_\infty\,d\sigma \\
&\quad + \int_0^{t_0} e^{-\lambda\sigma}\|[U(t\mp\tau+\xi, t\mp\tau+\xi-\sigma)-T(\sigma)]f_\infty\|\,d\sigma \\
&\leq M \sup_{0\leq\sigma\leq t_0} \|f(t\mp\tau+\cdot-\sigma)-f_\infty\|_E + 4Me^{-t_0}\,\|f\|_\infty \\
&\quad + \int_0^{t_0} \|[U(t\mp\tau+\cdot, t\mp\tau-\sigma+\cdot)-T(\sigma)]f_\infty\|_E\,d\sigma.
\end{aligned}$$

Fixing a large t_0 and letting $t\to\infty$, this estimate leads to

$$\overline{\lim_{t\to\infty}} \sup_{\xi\in[-r,0]} \|D_{\mp}(t,\tau,\xi)\| \leq \varepsilon$$

for each $\tau\in\mathbb{R}$ due to the assumptions and the theorem of dominated convergence. Therefore the inner integrals in (4.20) converge in E to

$$\int_0^\infty e^{-\lambda\sigma}T(\sigma)f_\infty\,d\sigma = R(\lambda,A)f_\infty$$

as $t\to\infty$ for each $\tau\in\mathbb{R}$. By assumption, $V(t,t-\tau)P_V(t-\tau)$ tends strongly to $S(\tau)P_S$ as $t\to\infty$ for each $\tau\in\mathbb{R}$. One easily sees that $V_Q(t,t+\tau)Q_V(t+\tau)\to S_Q(-\tau)Q_S$ strongly (compare part (1) of the proof of [30, Thm.4.1]). Moreover, the outer integrands in (4.20) are bounded by $ce^{-\delta\tau}$ for $\lambda\geq\omega+1$. As a result of Lebesgue's convergence theorem and [10, Exer.V.1.19], we thus obtain

$$\begin{aligned}
\lim_{t\to\infty}\tilde{w}_\lambda(t) &= \int_0^\infty S(\tau)P_S\lambda e^{\lambda\cdot}R(\lambda,A)f_\infty\,d\tau - \int_0^\infty S_Q(-\tau)Q_S\lambda e^{\lambda\cdot}R(\lambda,A)f_\infty\,d\tau \\
&= -B^{-1}\lambda e^{\lambda\cdot}R(\lambda,A)f_\infty =: w_{\lambda,\infty}.
\end{aligned}$$

Hence, $w_{\lambda,\infty}\in D(B)$ and $w'_{\lambda,\infty}(\xi) = Bw_{\lambda,\infty}(\xi) = -\lambda e^{\lambda\xi}R(\lambda,A)f_\infty$ which yields

$$w_{\lambda,\infty}(\xi) = x_\lambda + (1-e^{\lambda\xi})R(\lambda,A)f_\infty \tag{4.21}$$

for some $x_\lambda = w_{\lambda,\infty}(0)\in D(A)$. Moreover, $w'_{\lambda,\infty}(0) = Ax_\lambda + Lw_{\lambda,\infty}$ so that

$$-\lambda R(\lambda,A)f_\infty = Ax_\lambda + L_0x_\lambda + L(1-e^{\lambda\cdot})R(\lambda,A)f_\infty.$$

Since $S(\cdot)$ is hyperbolic, $A+L_0$ is invertible by [10, Thm IV.3.6, Prop.VI.6.7]. Thus,

$$x_\lambda = -(A+L_0)^{-1}\left[\lambda R(\lambda,A)f_\infty + L(1-e^{\lambda\cdot})R(\lambda,A)f_\infty\right] \longrightarrow -(A+L_0)^{-1}f_\infty$$

as $\lambda\to\infty$. This fact combined with (4.21) leads to the second identity in (4.19). $\square$

5. The inhomogeneous problem in the parabolic case

We assume again that (H1), (H2), and $\sigma(B) \cap i\mathbb{R} = \emptyset$ hold. Fix a as obtained in Theorem 3.6. We study the problem

$$\begin{aligned} u(t) &= U(t,a)\phi(0) + \int_a^t U(t,\tau)\,(L(\tau)u_\tau + f(\tau))\,d\tau, \quad t \geq a, \\ u(t) &= \phi(t-a), \quad a - r \leq t \leq a, \end{aligned} \tag{5.1}$$

for $\phi \in E$ and $f \in C_b([a,\infty), X)$ converging to f_∞ in X as $t \to \infty$. Because of Lemma 3.4 and Theorem 3.6, we can apply Theorem 4.4 to conclude that

$$\exists\, u_\infty := \lim_{t\to\infty} u(t) = -(A + L_0)^{-1} f_\infty \tag{5.2}$$

if and only if (4.16) with $s = a$ holds. (Here we extend f by 0 to a function on $\mathbb{R}$ and use the hyperbolic evolution family solving (2.9) for $U_a(\cdot,\cdot)$ and $L_a(\cdot)$ as in the proof of Theorem 3.6.) To obtain classical solutions of (1.5), we have to require more regularity for $L(\cdot)$ and f. Here we use the space

$$E_\theta^t := \{\phi \in E : \|\phi\|_{E_\theta^t} := \sup_{\xi\in[-r,0]} \|\phi(\xi)\|_\theta^{t+\xi} < \infty\}$$

endowed with the norm $\|\cdot\|_{E_\theta^t}$ and the assumptions

(H3) $f \in C_b^\gamma(\mathbb{R}_+, X)$ and $L(\cdot) \in C_b^\gamma(\mathbb{R}_+, \mathcal{L}(E,X))$ for some $\gamma \in (0,1)$;

(H3') $f \in C_b(\mathbb{R}_+, X)$, $\|f(t)\|_\gamma^t$ and $L(t) : E_\gamma^t \to X_\gamma^t$ are uniformly bounded for $t \geq 0$ and some $\gamma \in (0,1)$.

We now come to the second main result of this paper extending [30, Thm.4.1] where the undelayed case was treated.

Theorem 5.1. *Assume that (H1), (H2), $\sigma(B) \cap i\mathbb{R} = \emptyset$, and that either (H3) or (H3') hold. Fix $a \geq 0$ as obtained in Theorem 3.6. Let $\phi \in E$ satisfy* (4.16) *with $s = a$ and $f(t) \to f_\infty$ in X as $t \to \infty$. Then the mild solution u of* (5.1) *is continuously differentiable on $[a + 2r + 2, \infty)$, $u(t) \in D(A(t))$, and*

$$\dot{u}(t) = A(t)u(t) + L(t)u_t + f(t) \quad \textit{for} \;\; t \geq a + 2r + 2. \tag{5.3}$$

Moreover, u fulfills (5.2)*,*

$$\lim_{t\to\infty} \dot{u}(t) = 0, \quad \textit{and} \quad \lim_{t\to\infty} A(t)u(t) = Au_\infty. \tag{5.4}$$

Proof. We have already proved (5.2). Assume for a moment that u satisfies (5.3) and that $\dot{u}(t)$ tends to 0 as $t \to \infty$. Then

$$\lim_{t\to\infty} A(t)u(t) = -\lim_{t\to\infty}(L(t)u_t + f(t)) = -L_0 u_\infty - f_\infty = Au_\infty.$$

Thus the second part of (5.4) follows from the first one and (5.3). Extend f by 0 to $\mathbb{R}$ and recall that $u(t) = [v(t)](0)$ and

$$v(t) = V(t,a)P_V(a)\phi + \tilde{w}(t;\, f) \tag{5.5}$$

$$\begin{aligned} &= V(t,a)P_V(a)\phi + V(t,s)\tilde{w}(s;\, f) + \int_s^{t+\cdot} U(t+\cdot,\tau)\,(L(\tau)w(\tau;\, s, f) + f(\tau))\,d\tau \\ &=: S_1(t) + S_2(t,s) + S_3(t,s) \end{aligned} \tag{5.6}$$

for $t \geq s+r$ and $s \geq a$ by (4.9), (4.17), (4.14), and (4.3). We first assume that (H3) holds and establish the differentiability of $V(\cdot,s)\psi$ in step (1). The derivatives of $S_k(t)$ are estimated in step (2). The required regularity of u and (5.3) also follow from (1) and (2). Then we perform (more briefly) the analogous steps (1') and (2') supposing (H3').

(1) We verify that $V(\cdot,s)\psi$ is Hölder continuous in E for $\psi \in E$. Let $s \geq a$ and $s+r+1 \leq t' \leq t \leq t'+t_0$. Using (2.9) we calculate

$$\begin{aligned}
&(V(t,s)\psi)(\xi) - (V(t',s)\psi)(\xi) \\
&\quad = [U(t+\xi,t'+\xi)-1]\Big(U(t'+\xi,s)\psi(0) + \int_s^{t'+\xi} U(t'+\xi,\tau)L(\tau)V(\tau,s)\psi\,d\tau\Big) \\
&\qquad + \int_{t'+\xi}^{t+\xi} U(t+\xi,\tau)L(\tau)V(\tau,s)\psi\,d\tau \\
&\quad = [U(t+\xi,t'+\xi)-1]\Big(U(t'+\xi,t'+\xi-1)(V(t'-1,s)\psi)(\xi) \\
&\qquad + \int_{t'+\xi-1}^{t'+\xi} U(t'+\xi,\tau)L(\tau)V(\tau,s)\psi\,d\tau\Big) + \int_{t'+\xi}^{t+\xi} U(t+\xi,\tau)L(\tau)V(\tau,s)\psi\,d\tau
\end{aligned}$$

for $\xi \in [-r,0]$. Therefore (2.3) and (2.1) allow to estimate

$$\begin{aligned}
&\|V(t,s)\psi - V(t',s)\psi\|_E \\
&\quad \leq c\,(t-t')^\theta\Big(\|V(t'-1,s)\psi\|_E + \sup_{-r\leq\xi\leq 0}\int_{t'+\xi-1}^{t'+\xi}(t'+\xi-\tau)^{-\theta}\,\|V(\tau,s)\psi\|_E\,d\tau\Big) \\
&\qquad + c\,(t-t')\sup_{t'-r\leq\tau\leq t}\|V(\tau,s)\psi\|_E \\
&\quad \leq c\,(t-t')^\theta \sup_{t'-r-1\leq\tau\leq t}\|V(\tau,s)\psi\|_E
\end{aligned} \tag{5.7}$$

for $\theta \in (0,1)$ and constants c independent of ψ, t, t', s as chosen above. Hence, $L(\cdot)V(\cdot,s)\psi \in C^\gamma([s+r+1,\infty),X)$ by (H3). We further have

$$\begin{aligned}
[V(t,s)\psi]\,(\xi) &= [V(t,t_1)V(t_1,s)\psi]\,(\xi) \\
&= U(t+\xi,t_1)[V(t_1,s)\psi](0) + \int_{t_1}^{t+\xi} U(t+\xi,\tau)L(\tau)V(\tau,s)\psi\,d\tau
\end{aligned} \tag{5.8}$$

for $t > t_1 + r$ and $t_1 \geq s$. Thus $t \mapsto V(t,s)\psi$ is continuously differentiable in E and

$$\begin{aligned}
\partial_t V(t,s)\psi\,(\xi) &= A(t+\xi)U(t+\xi,t_1)[V(t_1,s)\psi](0) + L(t+\xi)V(t+\xi,t_1)V(t_1,s)\psi \\
&\quad + A(t+\xi)\int_{t_1}^{t+\xi} U(t+\xi,\tau)L(\tau)V(\tau,s)\psi\,d\tau \\
&=: S_4(t,s)\,(\xi) + S_5(t,s)\,(\xi) + S_6(t,s)\,(\xi)
\end{aligned} \tag{5.9}$$

for $t > t_1 + r \geq s + 2r + 1$ and $\xi \in [-r,0]$ due to [2, Thm.6.1(i)].

(2) We can now estimate the derivative of the first summand in (5.6) using (5.9) with $\psi = P_V(a)\phi$, $s = a$, $t \geq a + 2r + 2$, and $t_1 = t - 1 - r$. By means of (2.1), [2, Thm.6.1(ii)], (H3), (5.7), and the exponential dichotomy of $V(\cdot,\cdot)$, we so obtain

$$\begin{aligned}
\|\tfrac{d}{dt}\,S_1(t)\|_E &\leq c\,\|V(t+r-1,a)P_V(a)\phi\|_E + c\,\|L(\cdot)V(\cdot,a)P_V(a)\phi\|_{C^\gamma([t-1-r,t],X)} \\
&\leq c\,e^{-\delta(t-a-2r-2)}\|\phi\|_E
\end{aligned} \tag{5.10}$$

for $t \geq a + 2r + 2$ and constants c independent of t and ϕ.

For $\frac{d}{dt}\tilde{w}(t) = \partial_1 S_2(t,s) + \partial_1 S_3(t,s)$ we first establish a global Hölder estimate. (Here ∂_1 denotes the derivative with respect to the first variable.) The equality (4.3), [1, Thm.4.3], and (4.1) yield

$$\|w(\cdot;\, s, f)\|_{C^\theta([s,s+T],E)} \leq c_T \left[\|f\|_\infty + \|w(\cdot;\, s, f)\|_{C([s,s+T],E)}\right] \leq c'_T \tag{5.11}$$

for $\theta \in (0,1)$ and constants independent of s. Employing (H3) and [2, Thm.6.1(i)], we then derive

$$[\partial_1 S_3(t,s)](0) = A(t)[S_3(t,s)](0) + L(t)w(t;\, s, f) + f(t) \tag{5.12}$$

for $t \geq s + r$. Together with (5.6), (5.8), and (5.9), this gives us the asserted regularity of $u(t)$ and

$$\begin{aligned} \dot{u}(t) &= A(t)u(t) + L(t)[V(t,a)P_V(a)\phi + V(t,s)\tilde{w}(s;\, f) + w(t;\, s, f)] + f(t) \\ &= A(t)u(t) + L(t)u_t + f(t) \end{aligned}$$

for $t > a + 2r + 1$, where we have used $v(t) = u_t$ in the second equality.

We further deduce from (5.6), [2, Thm.6.1(ii)], (H3), and (5.11) that

$$\|\partial_1 S_3(t,s) - \partial_1 S_3(t',s)\|_E \leq c\,|t - t'|^\gamma \tag{5.13}$$

for $s + 2r + 3 \geq t, t' \geq s + r + 1/2$ and a constant independent of s. Moreover, (5.6) combined with (5.11) and [2, Thm.6.1(ii)] implies that

$$\sup_{t\in[s+r,s+2r+2]} \|\partial_1 S_3(t,s)\|_E \leq c \tag{5.14}$$

where c does not depend on s.

In order to obtain analogous estimates for $\partial_1 S_2(t,s)$, we involve (5.9) with $\psi = \tilde{w}(s;\, f)$. Let $s + 2r + 3/2 \leq t_1 + r + 1/2 \leq t', t \leq t_1 + r + 2 =: t_2$. We estimate

$$\begin{aligned} \|S_4(t,s) - S_4(t',s)\|_E &\leq c\,|t-t'|^\gamma\, \|V(t_1,s)\tilde{w}(s;\, f)\|_E \\ \|S_5(t,s) - S_5(t',s)\|_E &\leq c\,|t-t'|^\gamma\, \|V(\cdot,s)\tilde{w}(s;\, f)\|_{C^\gamma([t_1+1/2,t_2],E)} \\ &\leq c\,|t-t'|^\gamma \sup_{\tau\in[t_1-r-1/2,t_2]} \|V(\tau,s)\tilde{w}(s;\, f)\|_E \\ \|S_6(t,s) - S_6(t',s)\|_E &\leq c\,|t-t'|^\gamma\, \|L(\cdot)V(\cdot,s)\tilde{w}(s;\, f)\|_{C^\gamma([t_1,t_2],X)} \\ &\leq c\,|t-t'|^\gamma \sup_{\tau\in[t_1-r-1,t_2]} \|V(\tau,s)\tilde{w}(s;\, f)\|_E \end{aligned}$$

using (2.5), (H3), (5.7), and [2, Thm.6.1(ii)]. Take $s = t-2-2r$ and $t_1 = s+r+1 = t-r-1$ in the above estimates and recall that $\tilde{w}$ is uniformly bounded. Then we arrive at

$$\|\partial_1 S_2(t, t-2-2r) - \partial_1 S_2(t', t-2-2r)\|_E \leq c\,|t-t'|^\gamma \tag{5.15}$$

for a constant not depending on the above chosen t, t'. Similarly one sees that

$$\sup_{t\geq s+2r+2} \|\partial_1 S_2(t, t-2-2r)\|_E < \infty. \tag{5.16}$$

Choosing $s = t - 2r - 2$ in (5.6), the inequalities (5.13)–(5.16) yield

$$\tfrac{d}{dt}\tilde{w} \in C_b^\gamma([a+2r+2,\infty), E) \tag{5.17}$$

as required. On the other hand, $\tilde{w}(t;\, f)$ tends in E to $v_\infty \equiv u_\infty$ as $t \to \infty$ by (4.18) and (4.19). Let $\hat{w}(t) := \tilde{w}(t) - v_\infty$. The interpolation result [21, Cor.1.2.19], [21, Cor.0.2.2], and (5.17) then imply

$$\begin{aligned}\sup_{n\le t\le n+1} \|\tfrac{d}{dt}\tilde{w}(t)\|_E &\le \|\tfrac{d}{dt}\hat{w}\|_{C^h([n,n+1],E)} \\ &\le c\,\|\hat{w}\|^{1-\theta}_{C([n,n+1],E)}\,\|\hat{w}\|^{\theta}_{C^{1+\gamma}([n,n+1],E)} \\ &\le c \sup_{n\le t\le n+1} \|\tilde{w}(t) - v_\infty\|_E^{1-\theta} \quad\longrightarrow\; 0 \qquad \text{as} \;\; n\to\infty,\end{aligned}$$

where $h \in (0,\gamma)$, $\theta = \frac{1+h}{1+\gamma}$, and the constants do not depend on $n \ge a + 2r + 2$. Because of (5.5) and (5.10), the assertion is verified in the case that (H3) holds.

(1') Employing (5.8) for $t_1 = t - r - 1 \ge s \ge a$ and (2.1), we derive

$$\|V(t,s)\psi\,(\xi)\|_\gamma^{t+\xi} \le c \sup_{\tau\in[t-r-1,t]} \|V(\tau,s)\psi\|_E \tag{5.18}$$

for $t \ge s + r + 1 \ge a + r + 1$. The equality (5.9) then follows from (5.8), (5.18), (H3'), and [2, Thm.6.2(i)].

(2') Take $s = a$, $t \ge a + 2 + 2r$, $t_1 = t - r - 1$, and $\psi = P_V(a)\phi$ in (5.9). Then (2.1), (5.18), (H3'), [2, Thm.6.2(iii)], and the exponential dichotomy of $V(\cdot,\cdot)$ imply that

$$\|\partial_t\, V(t,a)P_V(a)\phi\|_E \le c\, e^{-\delta(t-a-2r-2)}\|\phi\|_E\,. \tag{5.19}$$

From (4.3), [1, Thm.4.3], and (4.1) we further deduce

$$\|w(t;\, s,f)\|_{E_\theta^t} \le c_T \tag{5.20}$$

for $\theta \in (0,1)$, $t \in [s, s+T]$, and a constant independent of s. The identity (5.3) for $t \ge a + 2r + 2$ is now a consequence of (5.6), (5.9), (5.20), (H3'), and [2, Thm.6.2(i)].

We fix $s = t - 2 - 2r \ge a$ in (5.6). Then [2, Thm.6.2(iii)], (H3'), and (5.20) lead to

$$\sup_{t\ge a+2r+2} \|\partial_1 S_3(t, t-2-2r)\|_{E_\gamma^t} < \infty. \tag{5.21}$$

To deal with S_2, we involve (5.9) for $\psi = \tilde{w}(s;\, f)$, $t_1 = t - r - 1$, $s = t - 2r - 2$, and $t \ge a + 2r + 2$. This gives

$$\begin{aligned}\|\partial_1\, V(t,t-2r-2)\tilde{w}(t-2r-2)\|_{E_\gamma^t} &\le c \sup_{t-r-1\le\tau\le t} \|V(\tau,t-2r-2)\tilde{w}(t-2r-2)\|_{E_\gamma^\tau} \\ &\le c \sup_{t-2r-2\le\tau\le t} \|V(\tau,t-2r-2)\tilde{w}(t-2r-2)\|_E \\ &\le c \end{aligned}\tag{5.22}$$

thanks to [30, Prop.2.4], (5.18), (H3'), and [2, Thm.6.2(iii)]. Putting (5.21) and (5.22) together, we conclude

$$\sup_{t\ge a+2r+2} \|\tfrac{d}{dt}\tilde{w}(t)\|_{E_\gamma^t} = \sup_{t\ge a+2r+2} \|\partial_1 S_2(t,t-2r-2) + \partial_1 S_3(t,t-2r-2)\|_{E_\gamma^t} < \infty.$$

We set $y(t) := [\tilde{w}(t)](0) = u(t) - [V(t,a)P_V(a)\phi](0)$ and remark that

$$\sup_{t\ge a+2r+2} \|\dot{y}(t)\|_\gamma^t < \infty. \tag{5.23}$$

Because of $\dot{y}(t) = A(t)y(t) + L(t)y_t + f(t)$, the assumptions, (5.2), and (5.19) imply

$$\begin{aligned}R(w,A(t))\dot{y}(t) &= -y(t) + wR(w,A(t))y(t) + R(w,A(t))L(t)y_t + R(w,A(t))f(t)\\ &\to -u_\infty + wR(w,A)u_\infty + R(w,A)L_0u_\infty + R(w,A)f_\infty\\ &= R(w,A)\left[(A+L_0)u_\infty + f_\infty\right] = 0 \qquad (5.24)\end{aligned}$$

as $t \to \infty$. On the other hand, the moment inequality (see e.g. [10, Thm.II.5.34]), the embedding [10, Prop.II.5.33], and (5.23) allow to estimate

$$\begin{aligned}\|\dot{y}(t)\| &\le c\,\|(w-A(t))^{-1}\dot{y}(t)\|^{1-\theta}\,\|(w-A(t))^h\dot{y}(t)\|^\theta\\ &\le c\,\|(w-A(t))^{-1}\dot{y}(t)\|^{1-\theta}\,(\|\dot{y}(t)\|_\gamma^t)^\theta\\ &\le c\,\|R(w,A(t))\dot{y}(t)\|^{1-\theta}\end{aligned}$$

for $h \in (0,\gamma)$, $\theta := \frac{1}{1+h}$, and constants c independent of $t \ge a+2r+2$. As a consequence of (5.24) we thus obtain that $\dot{y}(t)$ tends to 0 as $t \to \infty$. Together with (5.19) this establishes the theorem also under the hypothesis (H3'). □

The above proof leads to the following regularity result for mild solutions.

Proposition 5.2. *Assume that $A(\cdot)$ satisfies (P) and that $L(\cdot) \in C_b(\mathbb{R}_+, \mathcal{L}_s(E,X))$ satisfies either (H3) or (H3'). Then the mild solution u of* (2.11) *is continuously differentiable on $(s+r,\infty)$, $u(t) \in D(A(t))$ for $t > s+r$, and u fulfills* (1.5) *for $t > s+r$. If in addition $\phi(0) \in D(A(s))$ and either $\phi \in C^\gamma([-r,0],X)$ (in case (H3)) or $\phi \in E_\gamma^s$ (in case (H3')), then the assertion holds with '$t > s+r$' replaced by '$t \ge s$'.*

Proof. The first assertion follows from (4.8), (4.3), (5.8), and the arguments used in part (1) and (1') of the proof of Theorem 5.1. Similarly one sees that the function $\tau \mapsto L(\tau)u_\tau + f(\tau)$, $\tau \ge s$, is either Hölder continuous or bounded in X_γ^τ under the additional assumption on ϕ. Thus (2.11) and [2, Thm.6.1, 6.2] imply the second assertion. □

Remark 5.3. Using Lemma 3.5 and modifying the proofs in a straightforward way, one can replace (H3) in Theorem 5.1 and Proposition 5.2 by the condition

(H3a) $f \in C_b^\gamma(\mathbb{R}_+, X)$ and $L(\cdot) \in C_b^\gamma(\mathbb{R}_+, \mathcal{L}(C^h([-r,0],X),X))$ for some $\gamma, h \in (0,1)$.

Example 5.4. We investigate the retarded heat equation

$$\begin{aligned}&\partial_t\, u(t,x) = a(t,x)\,\partial_{xx}u(t,x) + c(t,x)u(t,x) - l(t,x)u(t-\tau(t),x) + f(t,x),\ t \ge 0,\ x \in [0,\pi],\\ &\quad u(t,0) = u(t,\pi) = 0, \quad t \ge 0,\\ &\quad u(t,x) = \phi(t,x), \quad -2 \le t \le 0,\ x \in [0,\pi],\end{aligned}$$

where $a, c, l \in C_b^\mu(\mathbb{R}_+, C[0,\pi])$, $\tau \in C^\mu(\mathbb{R}_+,[0,2])$, $\mu \in (0,1)$, $a(t,x) \ge a_0 > 0$, and

$$a(t,x) \to 1, \quad c(t,x) \to 1+\varepsilon, \quad l(t,x) \to \tfrac{\pi}{2}\, e^\varepsilon, \quad \tau(t) \to 1$$

uniformly for $x \in [0,\pi]$ as $t \to \infty$ for some $\varepsilon \ge 0$. Let $X = L^2(0,\pi)$ and $D = W^{2,2}(0,\pi) \cap W_0^{1,2}(0,\pi)$ be endowed with the usual norms. Set

$$A(t)\varphi = a(t,\cdot)\,\varphi'' + c(t,\cdot)\,\varphi \quad \text{and} \quad Af = \varphi'' + (1+\varepsilon)\varphi$$

for $\varphi \in D$ and $t \ge 0$. One easily verifies (H1) with $\nu = 1$ and (3.12) with $\alpha = \theta = 1$ using elliptic regularity. On $E = C([-2,0],X)$ we further define

$$L(t)\phi = -l(t,\cdot)\,\phi(-\tau(t)) \quad \text{and} \quad L\phi = -\tfrac{\pi}{2}\, e^\varepsilon\, \phi(-1).$$

Observe that (H2) is satisfied for each $\beta \in (0,1)$ and that (H3a) holds with $h = \gamma = \mu$ if $f \in C_b^\mu(\mathbb{R}_+, X)$. (One could also work with space regularity of l and f using (H3').)

Define B as in (1.8). In [33, pp.78-80] it is shown that $\sigma(B) \cap i\mathbb{R} = \emptyset$ for small $\varepsilon > 0$. For $\varepsilon = 0$, there are two simple eigenvalues $\pm i\frac{\pi}{2}$ with corresponding eigenfunctions $\phi_\pm$ and spectral projection Q; the rest of the spectrum has real part smaller than $d < 0$.

Thus, for small $\varepsilon > 0$, $V(t,s)$ has an exponential dichotomy on an interval $[a, \infty)$ with two dimensional unstable subspaces. We obtain the convergence of the mild solution if $f(t) \to f_\infty$ and (4.16) holds with $s = a$. If $\varepsilon = 0$, one sees by a rescaling argument that there is a splitting such that $V(t,s)P(s)$ is exponentially stable. For $\phi \in E$ there are numbers $c_\pm$ such that $Q\phi = c_+\phi_+ + c_-\phi_-$. Theorem 3.6 also implies that

$$V(s+t,s)Q(s)\phi \longrightarrow S(t)Q\phi = c_+\, e^{it\frac{\pi}{2}}\phi_+ + c_-\, e^{-it\frac{\pi}{2}}\phi_-$$

as $s \to \infty$ for $t \geq 0$ and $\phi \in E$.

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