Young Tableaux and the Frobenius-pullback of Tensor Differential Forms

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1. Introduction

Symmetry properties of tensors play an important role in physics. This paper deals with projective manifolds and general differential tensor forms "which hopefully will lead to a new birational geometry of algebraic varieties" [12].

Let Ω^1 be the cotangential bundle on a smooth projective manifold Y. It is well known that the dimensions $h^{0,r} = \dim H^0(Y, \Omega^r)$, $\dim H^0(Y, S^r\Omega^1)$ and $\dim H^0(Y, (\Omega^1)^{\otimes r})$ are birational invariants of Y. The same holds true for tensor differential forms with any symmetry property corresponding to an irreducible representation of the symmetric group S_r , i.e. to a Young tableau T [3].

For instance, the plurigenus P_f is obtained for the rectangle T with $n = \dim Y$ rows and f columns.

The vector bundle $(\Omega^1)^{\otimes r}$ decomposes to the direct sum $(\Omega^1)^{\otimes r} = \bigoplus\limits_T \Omega^T$ where T runs through the so called standard Young tableaux and where Ω^T denotes the sheaf $e_T((\Omega^1)^{\otimes r})$ of germs of T-symmetric tensor differential forms with the Young symmetrizer e_T [8], [9], [14].

The sheaf Ω^T is locally free. In the case of $n <$ depth T one has $\Omega^T = 0$, otherwise rank $(\Omega^T) = \prod\limits_{1 \le i < j \le n} (\frac{l_i - l_j}{j-i} + 1)$ where l_i denotes the length of the i-th row of T with $l_i = 0$ if $i >$ depth T. Examples of complete intersections show that the birational invariants $\dim H^0(Y, \Omega^T)$ are independent of each other.

The ground field $\boldsymbol{k}$ is assumed to be algebraically closed. Assume $\operatorname{char} \boldsymbol{k} = 0$ or $\operatorname{char} \boldsymbol{k} = p >$ number of boxes of each Young tableau T appearing in this paper. In the following let $s \geq 0$ be a fixed integer. In the case $\operatorname{char} \boldsymbol{k} = p > 0$ we set $q := p^s$. If $s = 0$ then $\operatorname{char} \boldsymbol{k} = 0$ is permitted. In this case set $q = 1$.
If $\operatorname{char} \boldsymbol{k} = p > 0$ then for each local free sheaf $\mathcal{F}$ on Y we denote by $\mathcal{F}^{(s)}$ the s-times Frobenius-pullback of $\mathcal{F}$. It is easy to see
$(\mathcal{F} \oplus \mathcal{G})^{(s)} = \mathcal{F}^{(s)} \oplus \mathcal{G}^{(s)}$, $(\mathcal{F} \otimes \mathcal{G})^{(s)} = \mathcal{F}^{(s)} \otimes \mathcal{G}^{(s)}$, $(\bigwedge^r \mathcal{F})^{(s)} = \bigwedge^r \mathcal{F}^{(s)}$,
$(S^r\mathcal{F})^{(s)} = S^r\mathcal{F}^{(s)}$, $\left(\mathcal{F}^T\right)^{(s)} = \left(\mathcal{F}^{(s)}\right)^T$, $\mathcal{F}^{(0)} = \mathcal{F}$, $\left(\mathcal{F}^{(s)}\right)^{(t)} = \mathcal{F}^{(s+t)}$.
If $\mathcal{F}$ is any sheaf of germs of differential forms on Y then one gets the sheaf $\mathcal{F}^{(s)}$ by using the p^s-th powers $(df)^{p^s}$ instead of the differentials df [6].
For instance, the dimensions $\dim H^0(Y, (\Omega^T)^{(s)})$ and $\dim H^0(Y, (\Omega^T)^{(s)} \otimes {\omega_Y}^{\otimes t})$ with the canonical line bundle ω_Y and with $t \in \mathbb{N}$ are birational invariants of Y .
For each Young tableau T with depth $T \leq n = \dim Y$ and with the row lengths $l_1, l_2, \ldots$ let T^* be the Young tableau with the following row lengths:
$l_j^* = l_1 - l_{n+1-j}$ $\forall j \in \{1, 2, \ldots, n\}$ [9]. Then by Serre duality we obtain
$\dim H^i(Y, (\Omega^T)^{(s)}(t)) = \dim H^{n-i}(Y, (\Omega^{T^*})^{(s)}(-t) \otimes {\omega_Y}^{\otimes(1-l_1 \cdot q)})$.
Examples of algebraic hypersurfaces Y show that the dimensions of the cohomology groups $H^i(Y, (\Omega^T)^{(s)}(t))$ are independent of the familiar invariants. They depend not only on the degree of Y, but also on the defining equation of Y [6].

The aim of this paper is to prove a Lefschetz type theorem and a vanishing theorem concerning the sheaf $(\Omega^T)^{(s)}$ on a smooth complete intersection Y. With this earlier results concerning the sheaf Ω^T are proven in a simple and clear way [3]. Note that the Hilbert polynomials $\chi(Y, (\Omega_Y^T)^{(s)}(t))$ are calculated in [6].

2. The Lefschetz type theorem

Let us start with

Lemma 1. *Let* $0 \longrightarrow \mathcal{F}_0 \xrightarrow{\psi_0} \mathcal{F}_1 \xrightarrow{\psi_1} \dots \xrightarrow{\psi_{m-1}} \mathcal{F}_m \longrightarrow 0$ *be an exact sequence of coherent sheaves on the algebraic variety* Y *. Then one has for each integer* $i \geq 0$*:*
a) $H^i(Y, \mathcal{F}_0) = 0$ *if* $H^{i-j+1}(Y, \mathcal{F}_j) = 0 \ \ \forall j \in \{1, 2, \dots, \min\{i+1, m\}\}$
b) $H^i(Y, \mathcal{F}_m) = 0$ *if* $H^{i+j-1}(Y, \mathcal{F}_{m-j}) = 0 \ \ \forall j \in \{1, 2, \dots, m\}$.

Proof. The lemma immediatly ensues from the short exact sequences
$0 \longrightarrow \mathrm{im}\psi_{j-1} \longrightarrow \mathcal{F}_j \longrightarrow \mathrm{im}\psi_j \longrightarrow 0 \quad (j \in \{1, 2, \dots, m-1\})$
and the corresponding long exact cohomology sequences [11], [13]. □

Theorem 1. *Let* X *and* Y *be smooth projective manifolds such that* Y *is an ideal theoretically complete intersection of* X *by algebraic hypersurfaces of multidegree* $(m_1, m_2, \dots, m_c)$ $(n = \dim Y \ , \ N = \dim X \ , \ c = N - n = \mathrm{codim}_X Y)$.
Let T *be a Young tableau with* r *boxes and with the column lengths* $d_1 \geq d_2 \geq \dots$ $(d_j = 0$ *if* $j > \text{length } T)$ *and let* $\mu := \sum_{j=1}^{c} d_j$ *be the number of boxes inside the* c *front columns of* T*. Assume* $\mu \leq \dim Y$ *and let* $t_0 \in \mathbb{Z}$ *be a fixed integer.*
If $m := \min\{m_1, m_2, \dots, m_c\}$ *is large enough then the restriction map*

$$\varphi^* : \ H^i(X, (\Omega_X^T)^{(s)}(t)) \longrightarrow H^i(Y, (\Omega_Y^T)^{(s)}(t))$$

- *is a monomorphism for all* $t, i \in \mathbb{Z}$ *with* $t \leq t_0$, $0 \leq i \leq \dim Y - \mu$,
- *is an isomorphism for all* $t, i \in \mathbb{Z}$ *with* $t \leq t_0$, $0 \leq i < \dim Y - \mu$.

Proof. These results are obtained from an exact sequence of sheaves which can be described as follows: Let $l = \mathrm{length} T$ be the number of columns of T and let $d_1, d_2, \dots, d_l \ (> 0)$ be the column lengths. Assume $M(T)$ to be the set of all integer matrices $A = ((d_{i,j}))$ with $c+1$ rows, $l = \text{length } T$ columns and with the properties:

- $d_{1,j} = d_j \quad \forall j \in \{1, 2, \dots, l\}$,
- $d_{i,l} \geq d_{i+1,l} \geq 0 \quad \forall i \in \{1, 2, \dots, c\}$,
- $d_{i,j} \geq d_{i+1,j} \geq d_{i,j+1} \ \ \forall i \in \{1, 2, \dots, c\} \ \ \forall j \in \{1, 2, \dots, l-1\}$.

The i-th row sum of such a matrix A will be denoted by $\varrho_i(A) := \sum_{j=1}^{l} d_{i,j}$.
In particular, we set $\varrho(A) := \varrho_{c+1}(A)$. Then $r - \mu \leq \varrho(A) \leq r$ for all $A \in M(T)$.
Let $T'(A)$ denote the Young tableau with $\varrho(A)$ boxes and with the column lengths $d_{c+1,1}$, $d_{c+1,2}$, ... , $d_{c+1,l}$, i.e., $T'(A)$ depends only on the last row of A .

We set $M_j(T) := \{A \in M(T) \ : \ \varrho(A) = r - j\}$ for each $j \in \{0, 1, \dots, \mu\}$ and $t(A) = q \cdot \sum_{\nu=1}^{c} (\varrho_{\nu+1}(A) - \varrho_\nu(A)) \cdot m_\nu \quad (q = p^s)$.
Then the exact sequence is the following (cp. [5] , [6]):

$$0 \longrightarrow E_\mu^T \longrightarrow E_{\mu-1}^T \longrightarrow \dots \longrightarrow E_2^T \longrightarrow E_1^T \xrightarrow{\gamma} (\Omega_{X|Y}^T)^{(s)} \longrightarrow (\Omega_Y^T)^{(s)} \longrightarrow 0 \tag{1}$$

with $E_j^T = \bigoplus_{A \in M_j(T)} (\Omega_{X|Y}^{T'(A)})^{(s)}(t(A))$, $(\Omega_{X|Y}^{T'(A)})^{(s)} = \mathcal{O}_Y \otimes (\Omega_X^{T'(A)})^{(s)}$ if $\varrho(A) > 0$

and $(\Omega_{X|Y}^{T'(A)})^{(s)} = \mathcal{O}_Y$ in the case $\varrho(A) = 0$.

Furthermore, an exact sequence can be derived from the Koszul complex:

$$\begin{aligned} &0 \to (\Omega_X^{T'})^{(s)}(-m_1 - m_2 - \dots - m_c) \longrightarrow \bigoplus_{1 \le j \le c} (\Omega_X^{T'})^{(s)}(-m_1 - \dots - \widehat{m_j} - \dots - m_c) \longrightarrow \\ &\dots \longrightarrow \bigoplus_{1 \le j \le c} (\Omega_X^{T'})^{(s)}(-m_j) \xrightarrow{\alpha} (\Omega_X^{T'})^{(s)} \longrightarrow (\Omega_{X|Y}^{T'})^{(s)} \longrightarrow 0 \ . \end{aligned} \tag{2}$$

We consider corresponding exact cohomology sequences

$H^i(X, (\mathrm{im}\alpha)(t)) \to H^i(X, (\Omega_X^T)^{(s)}(t)) \xrightarrow{\beta} H^i(Y, (\Omega_{X|Y}^T)^{(s)}(t)) \to H^{i+1}(X, (\mathrm{im}\alpha)(t))$,

$H^i(Y, (\mathrm{im}\gamma)(t)) \to H^i(Y, (\Omega_{X|Y}^T)^{(s)}(t)) \xrightarrow{\delta} H^i(Y, (\Omega_Y^T)^{(s)}(t)) \to H^{i+1}(Y, (\mathrm{im}\gamma)(t))$.

If $H^i(X, (\mathrm{im}\alpha)(t))$, $H^{i+1}(X, (\mathrm{im}\alpha)(t))$, $H^i(Y, (\mathrm{im}\gamma)(t))$ and $H^{i+1}(Y, (\mathrm{im}\gamma)(t))$ are trivial then $\varphi^* = \delta \circ \beta$ is an isomorphism. Because of lemma 1 we have: If $H^{i+j-1}(X, (\Omega_X^T)^{(s)}(t - m_{k_1} - m_{k_2} - \dots - m_{k_j})) = 0$ for all $j \in \{1, 2, \dots, c\}$ and for all combinations $1 \le k_1 < k_2 < \dots < k_j \le c$ then $H^i(X, (\mathrm{im}\alpha)(t)) = 0$.
The same implication holds true for $i+1$ instead of i .
From $i \le \dim Y - \mu$, $j \le c$ ensues $i < \dim Y$ and $i + j < \dim Y + c = \dim X$.
That means: If m is large enough then all cohomology groups $H^{i+j-\varepsilon}(X, (\Omega_X^T)^{(s)}(t - m_{k_1} - m_{k_2} - \dots - m_{k_j}))$ with $\varepsilon \in \{0, 1\}$ and $0 \le i \le \dim Y - \mu$ and $1 \le j \le c$ and $t \le t_0$ are trivial, i.e., the homomorphism β is an isomorphism for all $t, i \in \mathbb{Z}$ with $t \le t_0$, $0 \le i \le \dim Y - \mu$ (cp. [13], III, 5, Thm.3).
Now we observe the homomorphism $\delta : H^i(Y, (\Omega_{X|Y}^T)^{(s)}(t)) \longrightarrow H^i(Y, (\Omega_Y^T)^{(s)}(t))$:
If $H^{i+j-1}(Y, (\Omega_{X|Y}^{T'(A)})^{(s)}(t + t(A))) = 0$ for all $j \in \{1, 2, \dots, \mu\}$ and all $A \in M_j(T)$ then we have $H^i(Y, (\mathrm{im}\gamma)(t)) = 0$.
If $H^{i+j-1+\tau}(X, (\Omega_X^{T'(A)})^{(s)}(t + t(A) - m_{k_1} - m_{k_2} - \dots - m_{k_\tau})) = 0 \quad \forall \tau \in \{0, 1, \dots, c\}$

then $H^{i+j-1}(Y,(\Omega_{X|Y}^{T'(A)})^{(s)}(t+t(A)))=0$.
$t(A)$ denotes the integer $t(A)=q\cdot\sum_{\nu=1}^{c}(\varrho_{\nu+1}(A)-\varrho_\nu(A))\cdot m_\nu$. Since $A\in M_j(T)$, $\varrho_{\nu+1}(A)-\varrho_\nu(A)\le 0\ \ \forall\nu\in\{1,2,\ldots,c\}$ and $\sum_{\nu=1}^{c}(\varrho_{\nu+1}(A)-\varrho_\nu(A))=\varrho(A)-r=-j$ we obtain $t(A)\le -q\cdot j\cdot m\le -m$. From $i\le\dim Y-\mu$, $j\le\mu$, $\tau\le c$ ensues $i+j-1+\tau<\dim Y+c=\dim X$. That means: If m is large enough then all cohomology groups $H^{i+j-1+\tau}(X,(\Omega_X^{T'(A)})^{(s)}(t+t(A)-m_{k_1}-m_{k_2}-\ldots-m_{k_\tau}))$ with $0\le i\le\dim Y-\mu$, $1\le j\le\mu$, $A\in M_j(T)$, $0\le\tau\le c$ are trivial.
Therefore, if m is large enough then the groups $H^i(Y,(\mathrm{im}\gamma)(t))$ are trivial and δ is a monomorphism for all $t,i\in\mathbb{Z}$ with $t\le t_0$, $0\le i\le\dim Y-\mu$.
Under the stronger condition $i<\dim Y-\mu$ even the groups $H^{i+1}(Y,(\mathrm{im}\gamma)(t))$ are trivial for all $t\le t_0$ and δ is an epimorphism for all $t\le t_0$, if m is large enough. □

Corollary 1. *If* $m=\min\{m_1,m_2,\ldots,m_c\}$ *is large enough then the restriction map* $\varphi^*: H^0(X,(\Omega_X^T)^{(s)})\longrightarrow H^0(Y,(\Omega_Y^T)^{(s)})$ *of the global sections is a monomorphism in the case* $\mu=\dim Y$ *and is an isomorphism in the case* $\mu<\dim Y$.

Corollary 2. *Let* X *and* T *be as above and assume* $\dim X>\text{depth }T+1$. *Then there exists a smooth complete intersection* $Y\subset X$ *with* $\dim Y=\text{depth }T+1$ *and* $H^0(X,(\Omega_X^T)^{(s)})\cong H^0(Y,(\Omega_Y^T)^{(s)})$.

Proof. Because of the Bertini theorem there are projective hypersurfaces $H_1,H_2,\ldots$ such that for every $j<\dim X-$ depth T the intersection $X\cap H_1\cap H_2\cap\ldots\cap H_j$ becomes a smooth irreducible complete intersection in X of dimension $\dim X-j$. □

3. The sheaf $(\Omega^T)^{(s)}$ on complete intersections

Firstly we deal with the projective space $\mathbb{P}^N$ (Remember $q=p^s$):

Lemma 2. *Let* T *be a Young tableau with* r *boxes and with* $d=\text{depth }T\le N$.

(3) *If* depth $T<i<N$ *then* $H^i(\mathbb{P}^N,(\Omega^T)^{(s)}(t))=0\quad\forall t\in\mathbb{Z}$.
(4) *If* $0\le i<N$ *and* $t<(r-i)\cdot q$ *then* $H^i(\mathbb{P}^N,(\Omega^T)^{(s)}(t))=0$.
(5) $H^0(\mathbb{P}^N,(\Omega^T)^{(s)}(t))=0$ *if and only if* $t<(r+\text{ length }T)\cdot q$.

Proof. Let $l_1\ge l_2\ge\ldots$ be the row lengths of T. Because of $H^0(\mathbb{P}^N,\mathcal{O}(t))=0$ for all $t<0$ and $H^i(\mathbb{P}^N,\mathcal{O}(t))=0\ \forall i\in\{1,2,\ldots,N-1\}\quad\forall t\in\mathbb{Z}$ the statements

(3) and (4) ensue via lemma 1 from the free resolution

$0 \to (\Omega^T)^{(s)} \longrightarrow \mathcal{O}(-rq)^{\oplus a_0} \longrightarrow \mathcal{O}((1-r)q)^{\oplus a_1} \longrightarrow \ \ldots \ \longrightarrow \mathcal{O}((d-r)q)^{\oplus a_d} \to 0$

with well known integers $a_0, a_1, \ldots, a_d$ [5].

Note that in the complex case $\boldsymbol{k} = \mathbb{C}$ (that implies $s = 0, q = 1$) the dimensions $\dim H^i(\mathbb{P}^N, \Omega^T(t))$ are completely known [1], [5].

(5) will be proven by induction on N. In the case of $N = 1$ we obtain depth $T = 1$, $r =$ length T, $(\Omega^T)^{(s)} = (S^r\Omega^1)^{(s)} = S^{rq}\Omega^1 = S^{rq}(\mathcal{O}(-2)) = \mathcal{O}(-2rq)$ and therefore, $H^0(\mathbb{P}^1, (\Omega^T)^{(s)}(t)) = 0$ if and only if $t < 2rq = (r+ \text{ length } T) \cdot q$.

Now assume $N \geq 2$ and depth $T < N$. In this case one can show:

$H^0(\mathbb{P}^N, (\Omega^T)^{(s)}(t)) = 0$ if and only if $H^0(\mathbb{P}^{N-1}, (\Omega^T_{\mathbb{P}^{N-1}})^{(s)}(t)) = 0$. In fact, if $\omega \neq 0$ is a global section of $(\Omega^T)^{(s)}(t)$ on $\mathbb{P}^N$ then a hyperplane $E = \mathbb{P}^{N-1}$ will exist such that the restriction of ω onto E is different from zero as well [2].

Conversely, let $E = \mathbb{P}^{N-1}$ be an arbitrary hyperplane, let $P \in \mathbb{P}^N \setminus E$ be any point, let $\psi : \mathbb{P}^N \setminus P \longrightarrow E$ be the projection from P and let $\overline{\omega} \neq 0$ be a global section of $(\Omega^T_E)^{(s)}(t)$. Then $\psi^*(\overline{\omega}) \neq 0$ is regular on $\mathbb{P}^N \setminus P$ with $N \geq 2$, hence regular on the whole space $\mathbb{P}^N$. Actually, if $\psi^*(\overline{\omega})$ were not regular then a pole divisor of $\psi^*(\overline{\omega})$ through the point P would exist. Therefore by induction condition:

$H^0(\mathbb{P}^N, (\Omega^T)^{(s)}(t)) = 0$ iff $t < (r+ \text{ length } T) \cdot q$.

Now assume $N \geq 2$ and depth $T = N$ and $l_1 > l_N$. Then the Young tableau T consists of a rectangle with N rows and l_N columns and a Young tableau T' with $r' = r - N \cdot l_N$ boxes, with depth $T' < N$ and length $T' =$ length $T - l_N = l_1 - l_N$.

Then one has $(\Omega^T)^{(s)} = (\Omega^{T'})^{(s)}(-(N+1) \cdot l_N \cdot q)$ [3] and therefore

$H^0(\mathbb{P}^N, (\Omega^T)^{(s)}(t)) = 0$ if and only if $t - (N+1) \cdot l_N \cdot q < (r' + \text{ length } T') \cdot q$,

in other words: $H^0(\mathbb{P}^N, (\Omega^T)^{(s)}(t)) = 0$ if and only if $t < (r+ \text{ length } T) \cdot q$.

Finally we can assume that $N \geq 2$ and depth $T = N$ and $l_1 = l_N$. Then the Young tableau T is a rectangle with N rows and l_1 columns, i.e., $(\Omega^T)^{(s)} = \mathcal{O}(-(N+1) \cdot l_1 \cdot q)$ and $r = N \cdot l_1$.

Therefore, $H^0(\mathbb{P}^N, (\Omega^T)^{(s)}(t)) = 0$ if and only if $t < (r+ \text{ length } T) \cdot q$. $\square$

Theorem 2. *Let $Y \subseteq \mathbb{P}^N$ be a smooth ideal theoretically complete intersection by algebraic hypersurfaces of multidegree $(m_1, m_2, \ldots, m_c)$ $(c = \operatorname{codim} Y)$. Assume $2 \leq m_1 \leq m_2 \leq \ldots \leq m_c$. Let T be a Young tableau as in* theorem 1 *with r boxes and with the column lengths $d_1, d_2, \ldots$ $(d_1 = \text{depth } T \ , \ 0 < r \geq d_1 \geq d_2 \geq \ldots)$. Let $\mu := \sum_{j=1}^{c} d_j$ be the number of boxes inside the c front columns of T $(\mu \leq r)$. Assume* $\operatorname{char} \boldsymbol{k} = p > r$ or (*$\operatorname{char} \boldsymbol{k} = 0$ and $s = 0$, $q = 1$*).

(6) *If* $\text{depth } T < i < \dim Y - \mu$ *then* $H^i(Y, (\Omega^T)^{(s)}(t)) = 0 \quad \forall t \in \mathbb{Z}$.

(7) *If* $0 \leq i \leq \text{depth } T$ *and* $i < \dim Y - \mu$ *and* $t < (r - i - \lambda) \cdot q + \sum_{\nu=1}^{\lambda} m_\nu$ *for all* $\lambda \in \{0, 1, \ldots, \min\{c, d_1 - i\}\}$ *then* $H^i(Y, (\Omega^T)^{(s)}(t)) = 0$.

Corollary 3. *If* $\mu < \dim Y$ *and* $t < (r - \lambda) \cdot q + \sum_{\nu=1}^{\lambda} m_\nu$ *for all* $\lambda \in \{0, 1, \ldots, \min\{c, d_1\}\}$ *then* $H^0(Y, (\Omega^T)^{(s)}(t)) = 0$.

Corollary 4. *If* $\mu < \dim Y$ *then* $H^0(Y, (\Omega^T)^{(s)}) = 0$.

Corollary 5. *If* $\operatorname{codim} Y < \dim Y$ then $H^0(Y, (S^r\Omega^1)^{(s)}) = 0$.

Corollary 6. *If* $r < \dim Y$ *then* $H^0(Y, (\bigwedge^r \Omega^1)^{(s)}) = 0$.

Corollary 7. *If* $r < \dim Y$ *then* $H^0(Y, ((\Omega^1)^{\otimes r})^{(s)}) = 0$.

Proof. The arguments are almost the same as in the proof of theorem 1 with $\mathbb{P}^N$ instead of X , but in this case we need in addition the fact that for each matrix $A \in M(T)$ the Young tableau $T'(A)$ can be embedded into the top-left of T . This ensues from $d_{c+1,j} \leq d_j \quad \forall j \in \{1, 2, \ldots, l\}$. Therefore, the number $\mu(T'(A))$ of boxes inside the c front columns of $T'(A)$ is lower than or equal to $\mu = \mu(T)$. In particular, we will use $\text{depth } T'(A) = d_{c+1,1} \ \leq \ d_1 = \ \text{depth } T$:
From the exact sequence (1) with $X = \mathbb{P}^N$ and from lemma 1 we obtain:
If $H^{i+j}(Y, (\Omega_{X|Y}^{T'(A)})^{(s)}(t+t(A))) = 0$ for all $j \in \{0, 1, \ldots, \mu\}$ and for all $A \in M_j(T)$ then $H^i(Y, (\Omega^T)^{(s)}(t)) = 0$. From (2) we know: $H^{i+j}(Y, (\Omega_{X|Y}^{T'(A)})^{(s)}(t + t(A))) = 0$ if $H^{i+j+\sigma}(X, (\Omega_X^{T'(A)})^{(s)}(t + t(A) - m_{k_1} - m_{k_2} - \ldots - m_{k_\sigma})) = 0 \quad \forall \sigma \in \{0, 1, \ldots, c\}$ and for all combinations $1 \leq k_1 < k_2 < \ldots < k_\sigma \leq c$. Assume in the following

i, j, A, σ to be fixed with $i < \dim Y - \mu$, $j \in \{0, 1, \ldots, \mu\}$, $A \in M_j(T)$ and $\sigma \in \{0, 1, \ldots, c\}$. Then we have $i + j + \sigma < \dim Y - \mu + \mu + c = \dim X = N$.
To prove (6) assume $d_1 = \text{depth } T < i$.
Then depth $T'(A) \leq$ depth $T < i+j+\sigma < \dim X$ and from lemma 2 with $\mathbb{P}^N = X$ $H^{i+j+\sigma}(X, (\Omega_X^{T'(A)})^{(s)}(t + t(A) - m_{k_1} - m_{k_2} - \ldots - m_{k_\sigma})) = 0$. (6) is proven.
Now to prove (7) let it be $0 \leq i \leq d_1$, $i < \dim Y - \mu$ and
$t < (r - i - \lambda) \cdot q + \sum_{\nu=1}^{\lambda} m_\nu$ for all $\lambda \in \{0, 1, \ldots, \min\{c, d_1 - i\}\}$.
The number of boxes of the Young tableau $T'(A)$ equals $\varrho(A) = r - j$.
If $\sigma > d_1 - i$ then depth $T'(A) \leq$ depth $T = d_1 < i + \sigma \leq i + j + \sigma < \dim X$ and therefore $H^{i+j+\sigma}(X, (\Omega_X^{T'(A)})^{(s)}(t + t(A) - m_{k_1} - m_{k_2} - \ldots - m_{k_\sigma})) = 0$. Now assume $\sigma \leq d_1 - i$, i.e., $0 \leq \sigma \leq \min\{c, d_1 - i\}$. Since $m_1 \geq 2$ we have $t(A) \leq -qjm_1 \leq -2qj$ (cf. proof of theorem 1), hence $t + t(A) - m_{k_1} - m_{k_2} - \ldots - m_{k_\sigma} <$
$< (r - i - \sigma) \cdot q + \sum_{\nu=1}^{\sigma} m_\nu - 2jq - m_{k_1} - m_{k_2} - \ldots - m_{k_\sigma} \leq (r - i - \sigma) \cdot q - 2jq =$
$= (\varrho(A) - i - j - \sigma) \cdot q$. With lemma 2 we obtain
$H^{i+j+\sigma}(X, (\Omega_X^{T'(A)})^{(s)}(t + t(A) - m_{k_1} - m_{k_2} - \ldots - m_{k_\sigma})) = 0$. (7) is shown.
Corr.3 ensues from (7) ($i = 0$), it implies Corr.4 ($t = 0$). One obtains Corr.5 and Corr.6 from Corr.4 because of $S^r(\Omega^1)^{(s)} = (S^r\Omega^1)^{(s)} = (\Omega^T)^{(s)}$ with depth $T = 1$ and $\bigwedge^r (\Omega^1)^{(s)} = (\bigwedge^r \Omega^1)^{(s)} = (\Omega^r)^{(s)} = (\Omega^{T'})^{(s)}$ with length $T' = 1$, depth $T' = r$.
Finally Corr.7 ensues from Corr.4 since $((\Omega^1)^{\otimes r})^{(s)} = \bigoplus_T (\Omega^T)^{(s)}$. □

Remarks: Until now we did not use the equivalence (5) which allows us to formulate the following improvement of Corr.3:
If $\mu < \dim Y$, $t < (r + m_1 - 2) \cdot q$, $t < (r + \text{length } T) \cdot q$ and $t < (r - \lambda) \cdot q + \sum_{\nu=1}^{\lambda} m_\nu$ for all $\lambda \in \{1, 2, \ldots, \min\{c, d_1\}\}$ then $H^0(Y, (\Omega^T)^{(s)}(t)) = 0$.
In the special case $s = 0$, $q = 1$ from it we obtain one of the main results of [3]:
If $\mu < \dim Y$, $t < r + m_1 - 2$ and $t < r +$ length T then $H^0(Y, \Omega^T(t)) = 0$.
In fact, in this case $t < r - \lambda + \sum_{\nu=1}^{\lambda} m_\nu$ for all $\lambda \in \{1, 2, \ldots, \min\{c, d_1\}\}$ ensues from $t < r + m_1 - 2$.
To prove the above-mentioned assertion, we have to repeat the proof of (7) for the special case of $i = 0$. Assume the above-mentioned conditions and furthermore, $j \in \{0, 1, \ldots, \mu\}$, $A \in M_j(T)$, $\sigma \in \{0, 1, \ldots, c\}$, $1 \leq k_1 < k_2 < \ldots < k_\sigma \leq c$.
We will prove that $H^{j+\sigma}(X, (\Omega_X^{T'(A)})^{(s)}(t + t(A) - m_{k_1} - m_{k_2} - \ldots - m_{k_\sigma})) = 0$:
In the case of $\sigma > 0$ we take word-to-word the proof of (7).

In the case of $\sigma = 0$, $j \geq 1$ we obtain $t < (r + m_1 - 2) \cdot q$, $t(A) \leq -2qj$, hence $t + t(A) < (r + m_1 - 2) \cdot q - 2qj \leq (\varrho(A) - j) \cdot q$. This implies with lemma 2 $H^j(X, (\Omega_X^{T'(A)})^{(s)}(t + t(A))) = 0$. Finally in the case of $\sigma = 0$, $j = 0$ we use (5): In fact, from $j = 0$ ensues $\varrho(A) = r$, $T'(A) = T$, $t(A) = 0$, that means $H^0(X, (\Omega_X^T)^{(s)}(t)) = 0$ because of $t < (r+ \text{length } T) \cdot q$. □

Example: Assume $p = \text{char } \boldsymbol{k} = 7$. Then $Y : x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0$ and $Z : x_0^4 \cdot x_1 + x_0 \cdot x_1^4 + x_2^5 + x_3^5 = 0$ are smooth projective surfaces in $\mathbb{P}^3$. Since these hypersurfaces have the same degree one cannot tell the one apart from the other using only conventional birational invariants.
In particular $\dim H^i(Y, \Omega_Y^r(t)) = \dim H^i(Z, \Omega_Z^r(t))$ $\forall\, i, r, t$ [7].
The multiplicity of the canonical class of Y or Z equals 1 , i.e., for instance the dimension $\dim H^0(Y, (\Omega_Y^r)^{(s)}(t)) = \dim H^0(Y, (\Omega_Y^r)^{(s)} \otimes \omega_Y^t)$ is a birational invariant of Y for each integer $t \geq 0$. In the special case $s = 1$, $r = 1$, $t = 13$ one has the values $\dim H^0(Y, (\Omega_Y^1)^{(1)}(13)) = 1$ and $\dim H^0(Z, (\Omega_Z^1)^{(1)}(13)) = 5$ [6].
For this reason Y and Z are not birationally equivalent.

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