Recognition Conditions for $B_{p,p}$ Singularities

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Abstract. In this paper, we prove a theorem that gives recognition conditions for the $B_{p,p}$ singularities, that are corank 2 singularities of Boardman symbol $\sum^{2,0}$.

Theorem. *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ be a map-germ, $f \sim (x_1, \dots, x_{n-2}, y^p, y^2 + z^2) : IV_p$, or $f \sim (x_1, \dots, x_{n-2}, y^p \pm z^p, yz) : I_{p,p}$, where $\pm$ agree for p odd (p even: $II_{p,p}$). ($\sim$ means $\mathcal{K}$-equivalent). (Denote $f \sim B_{p,p}$ for short).*

(i) Then, there exist $\vec{u}, \vec{v} \in Ker(df(x))$ linearly independent vectors such that $Ker(df(x)) = span\{\vec{u}, \vec{v}\}$. Moreover, for $p \geq 3$, up to permutations in the source and target, there exists $\lambda \in GL(n; \mathcal{E}_n)$ such that:

(ii) For all $\vec{w} \in Ker(df(x))$, $\vec{w} \neq \vec{0}$, we have $H \sim A_k$, $k \geq p-2$ or $k = \infty$, where

$$\begin{cases} H(x) = d(\lambda \cdot f)(x) \cdot \vec{w} + (i \circ \pi \circ f)(x) \\ i : \mathbb{R}^{n-2} \hookrightarrow \mathbb{R}^n : \text{ canonical inclusion} \\ \pi : \mathbb{R}^n \to \mathbb{R}^{n-2}, \ (x_1, x_2, \dots, x_n) \mapsto (x_1, x_2, \dots, x_{n-2}) \end{cases}$$

(iii) For $\vec{w} \in Ker(df(x))$ generic, we have $H \sim A_{p-2}$. Moreover, write $\vec{w} = (w_1, w_2, \dots, w_{n-2}, \alpha, \beta)$. Then, there exist $\gamma_1, \gamma_2, \dots, \gamma_r \in \mathbb{R}$, $r \leq p$ such that for all $\vec{0} \neq \vec{w} \in Ker(df(x))$ with $\beta = \gamma.\alpha$, $\gamma \in \mathbb{R}$, where $\gamma \notin \{\gamma_1, \gamma_2, \dots, \gamma_r\}$, we have $H \sim A_{p-2}$.

For the proof we will need first a Lemma that says what is the meaning of differentiating in a general Kernel direction.

Lemma 1. *$[df(x) \cdot \vec{w}] \circ \psi^{-1} = dh(X) \cdot (0, \cdots, 0, a, b)$, where*

$$\begin{cases} h = f \circ \psi^{-1}, \; f \; as \; above \\ \vec{w} = (w_1, w_2, \cdots, w_{n-2}, a, b) \; is \; a \; general \; direction \; in \; Ker(df)(x) \\ \psi(x) = (f_1(x), \cdots, f_{n-2}(x), x_{n-1}, x_n), \; where \; x = (x_1, \cdots, x_n) \end{cases}$$

Proof. The normal forms IV_p, $I_{p,p}$ and $II_{p,p}$ have Boardman symbol $\sum^{2,0}$. Since the Boardman symbol is $\mathcal{K}$-invariant, see [1], we have that $f \in \sum^{2,0}$, and in particular $f \in \sum^2$. Hence, $dim\ Ker(df(x)) = 2$. Since f is equidimensional, it is equivalent to say that $corank(f) = 2$. This means that there exists a submatrix A of $df(x)$ that whithout loss of generality is given by $A = (\partial f_i / \partial x_j),\ i, j = 1, \cdots, n-2$, so that $|A| \neq 0$. Now, we determine a general direction $\vec{w} \in Ker(df(x)),\ \vec{w} = (w_1, \cdots, w_{n-2}, a, b),\ a, b \in \mathbb{R}$. Since $\psi(x) = (f_1(x), \cdots, f_{n-2}(x), x_{n-1}, x_n),\ where\ x = (x_1, \cdots, x_n)$, we have that $h(X) = (X_1, \cdots, X_{n-2}, f_{n-1} \circ \psi^{-1}(X), f_n \circ \psi^{-1}(X))$. The direction $\vec{w}$ is given by solving the linear system:

$$(\star) \begin{cases} \sum_{i=1}^{n-2} w_i \; f_{1,x_i} + a \; f_{1,x_{n-1}} + b \; f_{1,x_n} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{i=1}^{n-2} w_i \; f_{n-2,x_i} + a \; f_{n-2,x_{n-1}} + b \; f_{n-2,x_n} = 0 \\ a = a \\ b = b \end{cases}$$

where $f_{i,x_j} = \partial f_i / \partial x_j$. But this is equivalent to write:

$$\begin{pmatrix} f_{1,x_1} & f_{1,x_2} & \cdots & f_{1,x_{n-2}} & f_{1,x_{n-1}} & f_{1,x_n} \\ f_{2,x_1} & f_{2,x_2} & \cdots & f_{2,x_{n-2}} & f_{2,x_{n-1}} & f_{2,x_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ f_{n-2,x_1} & f_{n-2,x_2} & \cdots & f_{n-2,x_{n-2}} & f_{n-2,x_{n-1}} & f_{n-2,x_n} \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{n-2} \\ a \\ b \end{pmatrix} =$$

$(0\ 0\ \cdots\ 0\ a\ b)^T$. Hence, by Cramer's Rule, we have

$$w_j = \frac{(-1)^{n-j-1} a |C_1 \cdots \hat{C_j} \cdots C_{n-1} \hat{C_n}| + (-1)^{n-j-1} b |C_1 \cdots \hat{C_j} \cdots \widehat{C_{n-1}} C_n|}{|C_1 C_2 \cdots C_{n-2}|},$$

$1 \leq j \leq n-2$, where $C_i = (f_{1,x_i}\ f_{2,x_i}\ \cdots\ f_{n-2,x_i})^T,\ 1 \leq i \leq n.$

Therefore,

$df(x) \cdot (w_1, \cdots, w_{n-2}, a, b) = (0, \cdots, 0, \sum_{j=1}^{n-2} w_j \; f_{n-1,x_j} \; + a \; f_{n-1,x_{n-1}}$

$+b\ f_{n-1,x_n}, \sum_{j=1}^{n-2} w_j\ f_{n,x_j}\ + a\ f_{n,x_{n-1}} + b\ f_{n,x_n}) =$
$= (0, \cdots, 0, a[f_{n-1,x_{n-1}}\ + \sum_{j=1}^{n-2}(-1)^{n-j-1}\ f_{n-1,x_j}\ \frac{|C_1 \cdots \hat{C_j} \cdots C_{n-1}\hat{C_n}|}{|C_1 C_2 \cdots C_{n-2}|}]$
$+b[f_{n-1,x_n}\ + \sum_{j=1}^{n-2}(-1)^{n-j-1}\ f_{n-1,x_j}\ \frac{|C_1 \cdots \hat{C_j} \cdots \hat{C_{n-1}} C_n|}{|C_1 C_2 \cdots C_{n-2}|}]$, "*analogous with* f_n").

On the other hand,

$\partial/\partial X_{n-1}\ (f_{n-1} \circ \psi^{-1})(X) = \partial/\partial X_{n-1}\ f_{n-1}(\pi_1 \circ \psi^{-1}(X), \cdots, \pi_{n-2} \circ \psi^{-1}(X),$
$X_{n-1}, X_n) = (f_{n-1,x_{n-1}}\ \circ \psi^{-1}(X)) \cdot 1\ + \sum_{j=1}^{n-2}(f_{n-1,x_j}\ \circ \psi^{-1}(X))$
$\partial/\partial X_{n-1}\ (\pi_j \circ \psi^{-1})(X)) = (f_{n-1,x_{n-1}} \circ \psi^{-1}(X)) + \sum_{j=1}^{n-2}(f_{n-1,x_j} \circ \psi^{-1}(X)) \cdot$
$\underbrace{\pi_j((d\psi^{-1})(X) \cdot (0, \cdots, 0, 1, 0))}_{\alpha_{j,n-1},\ 1 \leq j \leq n-2}.$

We have that

$$(d\psi^{-1}) = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n-2} & \alpha_{1,n-1} & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n-2} & \alpha_{2,n-1} & \alpha_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \alpha_{n-2,1} & \alpha_{n-2,2} & \cdots & \alpha_{n-2,n-2} & \alpha_{n-2,n-1} & \alpha_{n-2,n} \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

and hence

$$(d\psi) \cdot \begin{pmatrix} \alpha_{1,n-1} \\ \vdots \\ \alpha_{n-2,n-1} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

In other words,

$$\begin{pmatrix} f_{1,x_1} & f_{1,x_2} & \cdots & f_{1,x_{n-2}} \\ f_{2,x_1} & f_{2,x_2} & \cdots & f_{2,x_{n-2}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n-2,x_1} & f_{n-2,x_2} & \cdots & f_{n-2,x_{n-2}} \end{pmatrix} \cdot \begin{pmatrix} \alpha_{1,n-1} \\ \alpha_{2,n-1} \\ \vdots \\ \alpha_{n-2,n-1} \end{pmatrix} = \begin{pmatrix} -f_{1,x_{n-1}} \\ -f_{2,x_{n-1}} \\ \vdots \\ -f_{n-2,x_{n-1}} \end{pmatrix}$$

Hence,

$$\alpha_{j,n-1} = \frac{|\cdots C_{j-1} - C_{n-1} C_{j+1} \cdots|}{|C_1 C_2 \cdots C_{n-2}|} = (-1)\frac{|\cdots C_{j-1} C_{n-1} C_{j+1} \cdots|}{|C_1 C_2 \cdots C_{n-2}|} =$$

$$= (-1)(-1)^{n-2-j}\frac{|C_1\cdots\hat{C}_j\cdots C_{n-1}\hat{C}_n|}{|C_1C_2\cdots C_{n-2}|} = (-1)^{n-j-1}\frac{|C_1\cdots\hat{C}_j\cdots C_{n-1}\hat{C}_n|}{|C_1C_2\cdots C_{n-2}|}$$

Analogous formulas hold for $\partial/\partial X_n\ (f_{n-1}\circ\psi^{-1})(X),\ \partial/\partial X_{n-1}\ (f_n\circ\psi^{-1})(X),\ \partial/\partial X_n\ (f_n\circ\psi^{-1})(X)$. This proves Lemma 1. $\square$

Corollary 1. (of the proof of Lemma 1). *Suppose* $f\in\mathcal{E}(n,n),\ f\in\sum^c$, *such that* $|(\partial f_i/\partial x_j)|\neq 0$, *where* $i,j=1,\cdots,n-c,\ 1\le c\le n-1$. *Let* $\vec{w}=(w_1,w_2,\cdots,w_{n-c},a_1,\cdots,a_c)$ *a general direction in* $Ker(df)(x)$, *and* $\psi(x)=(f_1(x),\cdots,f_{n-c}(x),x_{n-c+1},\cdots,x_n)$, *where* $x=(x_1,\cdots,x_n)$. *Then,*

$$[df(x)\cdot\vec{w}]\circ\psi^{-1}=dh(X)\cdot(0,\cdots,0,a_1,\cdots,a_c),$$

where $h=f\circ\psi^{-1}$

Remark. In reality, we only need that $|(\partial f_i/\partial x_j)|\neq 0$, where $i,j=1,\cdots,n-c$. In other words, f can have $rank\ge n-c$, and so $\vec{w}=(w_1,w_2,\cdots,w_{n-c},a_1,\cdots,a_c)$ needs not to be a direction in $Ker(df)(x)$, but needs only to satisfy a linear system corresponding to $(\star)$ of the proof of Lemma 1.

Corollary 2. (Kernel Lemma) *Corollary 1 holds for higher order derivatives, i.e., with the same hypothesis of Corollary 1, we have for all* $k\in\mathbb{N}$ *that*

$$[d^kf(x)\cdot(\vec{w})^k]\circ\psi^{-1}=d^kh(X)\cdot(0,\cdots,0,a_1,\cdots,a_c)^k,$$

where $h=f\circ\psi^{-1}$.

Proof. (by induction on k). From Corollary 1, the statement above is valid for $k=1$. So, suppose Corollary 2 holds for all integers $\le k$. We have that

$$d^{k+1}f(x)\cdot(\vec{w})^{k+1}=d(\underbrace{d^kf(x)\cdot(\vec{w})^k}_{g(x)})\cdot\vec{w}$$

with $g(x)=(0,\cdots,0,g_{n-c+1}(x),\cdots,g_n(x))$ $(*)$

Consider now $m(x)=(f_1(x),\cdots,f_{n-c}(x),g_{n-c+1}(x),\cdots,g_n(x))$. By Corollary 1 (see remark above), we have that $[dm(x)\cdot\vec{w}]\circ\psi^{-1}=d(m\circ\psi^{-1})\cdot(0,\cdots,0,a_1,\cdots,a_c)=d(X_1,\cdots,X_{n-c},(g_{n-c+1}\circ\psi^{-1})(X),\cdots,(g_n\circ\psi^{-1})(X))\cdot(0,\cdots,0,a_1,\cdots,a_c)\overset{Ind.Hyp.}{=}d(X_1,\cdots,X_{n-c},$ "$\underbrace{d^kh(X)\cdot(0,\cdots,0,a_1,\cdots,a_c)^k}_{last\ c\ components\ of}$")$\cdot(0,\cdots,0,a_1,\cdots,a_c)=d^{k+1}h(X)\cdot(0,\cdots$

$,0,a_1,\cdots,a_c)^{k+1}$. On the other hand, we have by $(*)$ that $[dm(x)\cdot\vec{w}]\circ\psi^{-1} = [d^{k+1}f(x)\cdot(\vec{w})^{k+1}]\circ\psi^{-1}$. This proves Corollary 2. $\square$

Corollary 3. (Recognition conditions for A_k singularity) *Let* $f\sim(x_1,\cdots,x_{n-1},x_n^{k+1})$, $k\in\mathbb{N}$. *Then,*

$$[d^j f(x)\cdot(\vec{w})^j](0)=0,\ 1\le j\le k,\ \textit{and}\ [d^{k+1}f(x)\cdot(\vec{w})^{k+1}](0)\neq 0,$$

where $\vec{w}\in Ker(df(x))$.

Proof. This is a straightforward application of Corollary 2, since we can suppose without loss of generality that $|(\partial f_i/\partial x_j)|\neq 0$, where $i,j=1,\cdots,n-1$, and then a general Kernel direction is given by $\vec{w}=(w_1,\cdots,w_{n-1},1)$. $\square$

Proof of Theorem. (Part 1). We have that $f\in\sum^{2,0}$, and hence does $h=f\circ\psi^{-1}$. In particular, $h\notin\sum^{2,2}$ and then $j^2(f_{n-1}\circ\psi^{-1})\neq 0$ or $j^2(f_n\circ\psi^{-1})\neq 0$. So, assume $j^2(f_n\circ\psi^{-1})\neq 0$. More than this, we may assume that $j^2(f_n\circ\psi^{-1}(0,\cdots,0,X_{n-1},X_n))\neq 0$, since h is $\mathcal{C}$-equivalent to $(X_1,\cdots,X_{n-2},f_{n-1}\circ\psi^{-1}(0,\cdots,0,X_{n-1},X_n),f_n\circ\psi^{-1}(0,\cdots,0,X_{n-1},X_n))$ $=: q$. Moreover, $dh(X)\cdot(0,\cdots,0,a,b)+i\circ\pi\circ h(X)$ is $\mathcal{C}$-equivalent to $dq(X)\cdot(0,\cdots,0,a,b)+i\circ\pi\circ q(X)$. So we can replace h by q. Now, consider $u(X_{n-1},X_n)=$ *last two components of* q We have that q is a suspension of u, and hence $q\sim u$, in the extended sense of [3]. In other words, studying the $\mathcal{K}$-type of q is the same thing as studying the $\mathcal{K}$-type of u. Moreover, $dq(X)\cdot(0,\cdots,0,a,b)+i\circ\pi\circ q(X)$ is $\mathcal{C}$-equivalent to $du(X_{n-1},X_n)\cdot(a,b)$, in the sense above. Therefore, we can simplify our work considering h given by $h(x,y)=(h_1(x,y),h_2(x,y))$, with $j^2h_2\neq 0$, and $h\sim\ (x^p,x^2+y^2)$, or $h\sim\ (x^p\pm y^p,xy)$.

(Part 2). Lemma 1 implies that $[d([\lambda]\cdot f(x))\cdot\vec{w}]\circ\psi^{-1}=d([\lambda]\cdot f\circ\psi^{-1}(X))\cdot(0,\cdots,0,a,b)$, where $\vec{w}=(w_1,\cdots,w_{n-2},a,b)\in Ker(df(x))$, and $\lambda\in GL(n;\mathcal{E}_n)$,

$$\lambda(x)=\begin{pmatrix} I_{n-2} & 0 & 0\\ 0 & 1 & g(x)\\ 0 & 0 & 1\end{pmatrix}$$

From Part 1, we can see $[\lambda]\cdot f\circ\psi^{-1}$ as a map $\widetilde{h}:\mathbb{R}^2\to\mathbb{R}^2$, $\widetilde{h}=[\lambda]\cdot h$, $\widetilde{h}=(\widetilde{h_1},h_2)$, and

$$\lambda(x)=\begin{pmatrix} 1 & g(x)\\ 0 & 1\end{pmatrix}$$

We have $f\sim B_{p,p}$, $p\ge 3$, and then h_2 is of Morse type, implying that h_2 is $2-\mathcal{R}$-determined, and so h_2 is $\mathcal{R}$-equivalent to j^2h_2, with $|Hess(h_2)|\neq 0$.

Now, we will show that we can find λ such that $O(\widetilde{h_1}) = p$. First, we recall a basic fact about the elements of the group $\mathcal{R}$ (germs of diffeomorphisms in the source, that preserve the origin), namely:

Proposition 1. *Let $f \in \mathcal{M}_n$ such that $j^{k-1}f = 0$, and $j^k f \neq 0$. Then, $j^{k-1}(f \circ \psi) = 0$, and $j^k(f \circ \psi) \neq 0$, for all $\psi \in \mathcal{R}$.*

Now, let us analyse separately the cases $|Hess(h_2)| > 0$ and $|Hess(h_2)| < 0$.

(1) If $|Hess(h_2)| > 0$, there exists $\varphi_1^{-1} \in \mathcal{R}$ such that $h_2 \circ \varphi_1^{-1}(X,Y) = XY$. Thus, $h \circ \varphi_1^{-1}(X,Y) = (h_1 \circ \varphi_1^{-1}(X,Y), XY)$. Now, we can pick $\lambda_1 \in GL(2;\mathcal{E}_2)$ as above, such that $[\lambda_1] \cdot (h \circ \varphi_1^{-1})(X,Y) = (\sum_i \alpha_i X^i + \sum_j \beta_j Y^j, XY)$. Now, suppose that $O(([\lambda_1 \circ \varphi_1] \cdot h)_1) \neq p$, say $= k$. We have that $([\lambda_1 \circ \varphi_1] \cdot h) = (h_1 + [g_1 \circ \varphi_1] \cdot h_2, h_2)$. Then, $([\lambda_1 \circ \varphi_1] \cdot h) \circ \varphi_1^{-1}(X,Y) = (h_1 \circ \varphi_1^{-1}(X,Y) + g_1(X,Y).XY, XY)$.

(i) We have by Proposition 1 that $O(h_1 \circ \varphi_1^{-1}(X,Y) + g_1(X,Y).XY) = O(h_1(x,y) + [g_1 \circ \varphi_1(x,y)] \cdot h_2(x,y)) = k$.
(ii) $h_1 \circ \varphi_1^{-1}(X,Y) + g_1(X,Y).XY$ has no multiples of XY in $\mathcal{E}_2$.

Then, (i) and (ii) imply that $[\lambda_1] \cdot (h \circ \varphi_1^{-1})(X,Y) = (\sum_{i \geq k} A_i X^i + \sum_{j \geq k} B_j Y^j, XY)$, with $A_k^2 + B_k^2 \neq 0$. Hence, $h \not\sim B_{p,p}$, by [1] or by [2]. Therefore, $O(([\lambda_1 \circ \varphi_1] \cdot h)_1) = p$.

(2) If $|Hess(h_2)| < 0$, there exists $\varphi_2^{-1} \in \mathcal{R}$ such that $h_2 \circ \varphi_2^{-1}(X,Y) = X^2 + Y^2$. Thus, $h \circ \varphi_2^{-1}(XY) = (h_1 \circ \varphi_2^{-1}(X,Y), X^2 + Y^2)$. Now, we can pick $\lambda_2 \in GL(2;\mathcal{E}_2)$ as above, such that $[\lambda_2] \cdot (h \circ \varphi_2^{-1})(X,Y) = (\sum_i \alpha_i X^i + \sum_j \beta_j Y X^j, X^2 + Y^2)$. Now, using the same reasoning as above, we conclude that $O(([\lambda_2 \circ \varphi_2] \cdot h)_1) = p$.

So, we have proved in both cases that there exists $\lambda \in GL(2;\mathcal{E}_2)$ such that $\underbrace{\lambda \cdot h(x,y)}_{\widetilde{h}(x,y)} = (\sum_{i=0}^p a_i x^i y^{p-i} + O(p+1), b_2 x^2 + b_1 xy + b_0 y^2 + O(3))$, with $b_1^2 \neq 4b_0 b_2$, $\sum_{i=0}^p a_i^2 \neq 0$.

Remark. Let $v(x,y) = d\widetilde{h}(x,y) \cdot (c,d) = (m(x,y), (2b_2 c + b_1 d)x + (2b_0 d + b_1 c)y + O(2))$. Suppose that for some (c_0, d_0) we have $c_0 b_1 + 2d_0 b_0 = 0$ and $d_0 b_1 + 2c_0 b_2 = 0$. Then, $-2d_0 b_0 / c_0 = b_1 = -2c_0 b_2 / d_0$, implying $b_1^2 = 4b_0 b_2$. Hence, $\widetilde{h}$ would have Boardman symbol $\sum^{2,1}$, (if $p \geq 3$). Therefore,

$\forall (c,d) \in \mathbb{R}^2$, $\underbrace{cb_1 + 2db_0}_{B_0} \neq 0$ or $\underbrace{db_1 + 2cb_2}_{B_1} \neq 0$. So, assume $B_1 \neq 0$.

Then, v_2 is $1 - \mathcal{R}$-determined, and in particular $\exists \phi \in \mathcal{R}$ such that $v_2 \circ \phi(X,Y) = X$. Then, we have that $O(m \circ \phi) = O(m) \geq p-1$. In other words, $v \sim A_k,\ k \geq p-2$, or $k = \infty$. This proves item (ii) of the Theorem.

(Part 3). From Part 2, we can write $\widetilde{h}(x,y) = (\sum_{j=0}^{p} a_j x^j y^{p-j} + O(p+1), b_0 y^2 + b_1 xy + b_2 x^2 + O(3))$, with $b_1^2 \neq 4b_0 b_2,\ \sum_{i=0}^{p} a_i^2 \neq 0$. Now, let

$$\begin{cases} j^p \widetilde{h_1}(x,y) =: \alpha(x,y) \\ j^2 h_2(x,y) =: \beta(x,y) \end{cases}$$

We have that $v(x,y) := d\widetilde{h}(x,y) \cdot (c,d) = (\sum_{j=1}^{p} (cja_j + da_{j-1}(p-j+1))x^{j-1}y^{p-j} + O(p), (db_1 + 2b_2 c)x + (cb_1 + 2db_0)y + O(2))$. Now, let

$$\begin{cases} j^{p-1} v_1(x,y) =: \alpha'(x,y) \\ j^1 v_2(x,y) =: \beta'(x,y) \end{cases}$$

Claim. *β' does not divide α'.*

Proof. Suppose $\beta' \mid \alpha'$. Then, $\alpha'(x,y) = (\sum_{k=0}^{p-2} \lambda_k x^k y^{p-2-k})\beta'(x,y)$. Now, equating the coefficients and taking into account that $\beta' \mid \alpha'$ for all $(c,d) \in \mathbb{R}^2$, we obtain, after some calculations, that

$$\begin{cases} a_0 = (2/p)\lambda_0 b_0 \\ a_1 = (2/p)(\lambda_0 b_1 + \lambda_1 b_0) \\ a_{p-1} = (2/p)(\lambda_{p-3} b_2 + \lambda_{p-2} b_1) \\ a_p = (2/p)(\lambda_{p-2} b_2) \\ a_k = (2/p)(\lambda_k b_0 + \lambda_{k-1} b_1 + \lambda_{k-2} b_2),\ 2 \leq k \leq p-2 \end{cases}$$

Now, it is easy to see that $\beta \mid \alpha$. For, let $\gamma_k = (2/p)\lambda_k,\ 0 \leq k \leq p-2$. Then,

$$\begin{cases} a_0 = \gamma_0 b_0 \\ a_1 = \gamma_0 b_1 + \gamma_1 b_0 \\ a_{p-1} = \gamma_{p-3} b_2 + \gamma_{p-2} b_1 \\ a_p = \gamma_{p-2} b_2 \\ a_k = \gamma_k b_0 + \gamma_{k-1} b_1 + \gamma_{k-2} b_2,\ 2 \leq k \leq p-2 \end{cases}$$

Hence, $(\sum_{j=0}^{p-2} \gamma_j x^j y^{p-2-j})\beta(x,y) = \alpha(x,y)$. Therefore, $\beta' \mid \alpha'$ implies $\beta \mid \alpha$, which is an absurd, since $\widetilde{h} \sim B_{p,p}$. $\square$

Now, from Part 2, we know that $v \sim A_k$, $k \geq p-2$ or $k = \infty$. So, let us compute $[d^{p-1}v(x,y) \cdot (\xi_1,1)^{p-1}](0)$, where $(\xi_1,1)$ is a general direction in $Ker(dv(x,y))$. From Corollary 2, we have that $(d^{p-1}v(x,y) \cdot (\xi_1,1)^{p-1}) \circ \psi^{-1} = d^{p-1}(v \circ \psi^{-1}(X,Y) \cdot (0,1)^{p-1}$, where $\psi(x,y) = (v_2(x,y),y)$. But, $[d^{p-1}v(x,y) \cdot (\xi_1,1)^{p-1}](0) = [(d^{p-1}v(x,y) \cdot (\xi_1,1)^{p-1}) \circ \psi^{-1}](0)$, by Proposition 1. Hence, $[d^{p-1}v(x,y) \cdot (\xi_1,1)^{p-1}](0) = [d^{p-1}(v \circ \psi^{-1}(X,Y) \cdot (0,1)^{p-1}](0)$. We have that $\widehat{\psi^{-1}} = ((X - B_0Y)/B_1 \; + 0(2), Y)$, where $\widehat{f}$ denotes the Taylor series of f. Write $v(x,y) = (\sum_{i=0}^{p-1} A_i x^i y^{p-1-i} + O(p), B_1x + B_0y + O(2))$. Then, $\widehat{(v \circ \psi^{-1})} = \widehat{v} \circ \widehat{\psi^{-1}} = (\sum_{i=0}^{p-1} A_i((X - B_0Y)/B_1 \; + O(2))^i Y^{p-1-i} + O(p), X) = (\sum_{i=0}^{p-1} A_i(-B_0/B_1)^i Y^{p-1} + g(X,Y) + O(p), X)$, where $g(0,Y) \equiv 0$, $g \in \mathcal{M}_2$. Hence, $[d^{p-1}(v \circ \psi^{-1}(X,Y) \cdot (0,1)^{p-1}](0) = ((p-1)! \sum_{i=0}^{p-1} A_i(-B_0/B_1)^i, 0) = (((p-1)!/B_1^{p-1}) \sum_{i=0}^{p-1} A_i(-B_0)^i B_1^{p-1-i}, 0)$. We have that $\sum_{i=0}^{p-1} A_i(-B_0)^i B_1^{p-1-i}$ is a homogeneous polynomial of degree p in the variables c, d. So, write $\sum_{i=0}^{p-1} A_i(-B_0)^i B_1^{p-1-i} = P(c,d)$. Therefore, $\mathcal{V}(P) \subset \mathbb{R}^2$ (the variety of P) is a measure-zero set in $\mathbb{R}^2$, as long as $P \not\equiv 0$. But, $P \equiv 0 \Leftrightarrow P/B_1^{p-1} \equiv 0 \Leftrightarrow j^{p-1}v_1((-B_0/B_1)y, y) \equiv 0 \Leftrightarrow x = (-B_0/B_1)y$ is a root of $j^{p-1}v_1(x,y) \Leftrightarrow (x + (B_0/B_1)y) \mid j^{p-1}v_1 \Leftrightarrow (B_1x + B_0y) \mid j^{p-1}v_1 \Leftrightarrow j^1v_2 \mid j^{p-1}v_1$, i.e., $\beta' \mid \alpha'$. Therefore, $P \not\equiv 0$ and the first part of item (iii) of the Theorem is then proved. The second part follows imediately from the homogeneity of P. $\square$

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