SENSITIVITY OF SOLUTIONS OF LINEAR PDAE'S WITH RESPECT TO PERTURBATIONS

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Abstract. For initial boundary value problems (IBVPs) of linear partial differential–algebraic equations we study the influence of perturbations in the data on the numerical solution of the problems. First we investigate for initial value problems a specific solution representation from an analytic point of view. This can be extended to certain IBVPs with the result that we can easily determine the order of differentiation of the data which is needed to construct the solution of the IBVPs. Using this information we construct for several examples discretization schemes which reflect the theoretical results. These characterize the convergence properties of the numerical methods.

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1. Introduction

As is well known the solution of higher index differential–algebraic equations (DAEs) may not depend continuously on small perturbations in the equations.
During the numerical solution of DAEs this is usually reflected by the amplification of small errors arising, e.g., due to rounding effects. Several authors have investigated this phenomenon by means of a perturbation index which is a quantitative measure of the error amplification, see [2], [3], [4], [6], [8].

For partial differential–algebraic equations (PDAEs), a similar effect appears. This is studied in this paper for linear PDAEs of the form

$$Au_t + Bu_{xx} + Cu = f(t,x), \quad t > 0,\ x \in \Omega \ \subseteq\ \mathbb{R}, \tag{1.1}$$

where $u : (0,\infty) \times \Omega \to \mathbb{R}^n$, and $A,\ B,\ C$ are real constant $(n,n)-$matrices $(n > 1)$, and $f \in \mathbb{R}^n$ is a given vector (sufficiently smooth).
We suppose that none of the matrices A and B is the zero matrix and that at least one of them is singular. The domain Ω is chosen either as $\Omega = \mathbb{R}$ or as $\Omega = (0,1)$ which means that we consider either initial value problems (IVPs) or initial boundary value problems (IBVPs),

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respectively. In the first case (IVPs) we suppose that u and its derivatives (as far as needed) vanish for $x \to \pm\infty$. For IBVPs we assume that u can be chosen as zero for $x = 0$, $x = 1$ and all $t \geq 0$. In principle, this is possible when the so called consistent BVs are known. Furthermore, we add initial values (IVs)

$$u(0, x) = g(x), \qquad x \in \Omega, \tag{1.2}$$

where $g : \Omega \to \mathbb{R}^n$ is considered to be given too. If A is singular, then not all components of g can be prescribed arbitrarily, but we suppose that these are determined consistently (see the following section, more details about consistent data can be found, e.g., in [11]).

In order to motivate the investigation in this paper we illustrate the sensitivity of a numerical solution of the PDAE with respect to small perturbations by the following

Example 1.1. Solve on the intervals $I = (0, 1)$ and $\Omega = (0, 1)$ the simple PDAE

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u_t + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} u_{xx} = \begin{pmatrix} f_1(t, x) \\ f_2(t, x) \end{pmatrix}, \quad t \in I, \quad x \in \Omega, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{1.3}$$

with $f_1(t, x) = (1 - x)(e^x - 1) - 20tx^3$, $f_2(t, x) = x^5 - x$ and IVs $u_1(0, x) = u_2(0, x) = 0$. The exact solution is given by $u_1(t, x) = t(1 - x)(e^x - 1)$ and $u_2(t, x) = t(x^5 - x)$. Note that the uniform differential time index is $\nu_{dt} = 0$ and the differential spatial index is $\nu_{dx} = 3$ (for the definition of these indexes see [9], similar index definitions are used in [4], [5]). Since $\nu_{dx} = 3$ and f is given, arbitrary boundary values for u cannot be prescribed.
We solve this problem numerically by a simple finite difference method which discretizes the time derivative by the backward difference approximation (first order accurate) and the second space derivative by the usual centered second order difference. [1] However, to introduce a ("small") perturbation we compute the second space derivative of f_2 numerically with a reduced accuracy only (4 byte, all other quantities are calculated in an 8 byte arithmetic). Denoting the time and space step sizes by τ and h, respectively, and writing with a positive integer N $h = 1/N$ we compute for fixed $\tau = 0.1$ after 10 time steps (i.e. $t = 1$) the error e_1 (with respect to the exact solution) of the first component u_1. The result is given in the following table (the error of the second component u_2 which is of magnitude about 10^{-8} is not of interest here).

TABLE 1.1. Error e_1 and its numerically determined order p_1 with respect to $h = 1/N$ for fixed $\tau = 0.1$ at $t = 1$ and a perturbation in the discretization term of the second order space derivative of f_2

N	10	20	40	80	160	320	640	1280	2560
error e_1	4.95–1	1.31–2	3.30–3	8.46–4	7.26–4	3.26–3	1.35–2	5.36–2	2.15-1
order p_1	–	1.94	1.99	1.96	2.02	–0.94	–2.05	–1.99	–2.00

[1] The corresponding numerical method, also called BTCS (backward in time, centered in space), and its convergence is investigated for linear PDAE's in detail in [10], [11].

The table shows that the order p_1 of the error e_1 is for $N \lesssim 160$ nearly 2 (because the exact solution u_1 is linear in t) and for $N \gtrsim 640$ near -2. This order $p_1 \approx -2$ (which is the same also for smaller values of τ) is a direct consequence of the differential space index $\nu_{dx} = 3$ which means that u_1 is (in this example) expressed by a second space derivative of f_2. In fact, the analytic solution u_1 is given by

$$u_1(t,x) = \int_0^t f_1(\tau,x)\,d\tau + \int_0^t \int_0^\xi f_{2,xx}(\tau,x)\,d\tau\,d\xi. \tag{1.4}$$

The result $p_1 \approx -2$ for small values of h follows from $u_1 \sim f_{2,xx}$ and the spatial discretization used for $f_{2,xx}$ which is proportional to h^{-2}. Thus, the numerical solution approximating u is strongly influenced by perturbations in f_2. The appearance of $x-$derivatives of f in the solution u is a direct measure of the sensitivity for the numerical solution of the problem with respect to small perturbations on the right hand side of the PDAE. $\square$

As for DAEs, this sensitivity may be discussed by means of a perturbation index, see [4], [12] and section 3 of this paper. A *uniform* perturbation index is introduced and applied to a special linear PDAE in [1] illustrating the difficulties that may appear during the numerical solution of PDAE problems.
In general, one does not know the analytic solution (as in eq. (1.4)). However, in order to obtain a measure of the sensitivity it is enough to have estimates of the L^2-norm of u in terms of f (and the initial state g, see below) and its time and space derivatives. As is clear from eq. (1.4), in this example $\|u_1\|_{s2} \le C_0\,(\|f_1\|_{s2} + \|f_{2,xx}\|_{s2})$ where C_0 is some finite positive constant provided f_1 and $f_{2,xx}$ are members of $L^2(\Omega)$ ($\|w\|_{s2}$ is the $L^2(\Omega)-$norm of the scalar function w).

In this paper we study the influence of perturbations of the right hand side f (explained in section 3) by investigating the corresponding IVP by means of a Fourier and a Laplace transformation of the PDAE (section 4). The numerical solution of IBVPs adjoint in a certain sense to IVPs (see section 5) is studied in section 6.

2. Preliminaries for initial value problems

Let $\Omega = \mathbb{R}$ in this section. To characterize $u \in \mathbb{R}^n$ as solution of the PDAE, let $C_0^{i,j}$ be the set of real vector valued functions $v:\ (0,\infty)\times\Omega \to \mathbb{R}^n$ whose derivatives up to the $(i+j)^{th}$ order (time derivatives up to the i^{th} order and space derivatives up to the j^{th} order) are componentwise continuous and vanish for $x \to \pm\infty$. Then we look for a solution u of the initial value problem (IVP) (1.1), (2.1) where a solution of this problem is defined as follows.

Definition 2.1. A function $u = u(t,x) \in \mathbb{R}^n,\ t \ge 0,\ x \in \Omega$, is a classical solution of the IVP associated with the PDAE (1.1) if

1. $Au \in \left\{ v \in C_0^{1,0};\quad \frac{\partial^i v}{\partial t^i} \in L,\ \ i = 0,1 \right\}$,
   $Bu \in \left\{ v \in C_0^{0,2};\quad \frac{\partial^j v}{\partial x^j} \in L,\ j = 0,1,2 \right\}$,
   $Cu \in \left\{ v \in C_0^{0,0};\quad v \in L \right\}$, where for short $L := (L^1(\Omega))^n \cap (L^2(\Omega))^n$,
2. u is uniquely determined by its IVs

$$u_j(0,x) = g_j(x), \quad x \in \Omega,\ j \in \mathcal{M}_{IC}, \tag{2.1}$$

where $\mathcal{M}_{IC}$ is given below,

3. $u(t,x)$ solves the PDAE pointwise.

By means of the index set $\mathcal{M}_{IC} \subseteq \{1,2,...,n\}$ (details are given in [9]) we express the fact that, in general, arbitrary IVs cannot be prescribed for all components of u. $\mathfrak{M}_{IC}$ is defined formally by

Definition 2.2. If $j \in \mathcal{M}_{IC}$, then any IV can be given for u_j. If $j \notin \mathcal{M}_{IC}$, then an arbitrary IV cannot prescribed for u_j.

$g_j,\ j \in \mathcal{M}_{IC}$, in eq. (2.1) is a given sufficiently smooth scalar function (its smoothness is discussed below). We suppose in the following that the components $g_j,\ j \notin \mathcal{M}_{IC}$, are known, i.e. we suppose that they can be determined *consistently* from the PDAE and the IVs $g_j,\ j \in \mathcal{M}_{IC}$, see e.g. [7], [9].
Furthermore, in this paper we always assume that the problems investigated have one and only one solution.
Define $J_n := \{1,2,...,n\}$, and let $C^{l,m}_{sc}$ be the space of scalar real functions $w(t,x)$, $t \in (0,\infty)$ and $x \in \Omega$, whose derivatives up to the $(l+m)^{th}$ order (time derivatives up to the l^{th} order and space derivatives up to the m^{th} order) are continuous. Then the components f_k and $g_k,\ k \in J_n$, of the vectors f and g are assumed to be elements of sets

$$\mathcal{M}^{l,m} := \left\{ w \in C^{l,m}_{sc};\ \frac{\partial^{i+j}}{\partial t^i \partial x^j} w \in L^1(\Omega) \cap L^2(\Omega), \quad 0 \le i \le l,\ 0 \le j \le m \right\} \tag{2.2}$$

with some nonnegative integers $l,\ m$. In general, the IVP of the PDAE requires that the components of f and also of the initial state $g(x) = u(0,x),\ x \in \Omega$, must have a certain smoothness. Therefore, f_k and $g_k,\ k \in J_n$, are assumed to be from such sets for (later) specified nonnegative integers $l_{fk},\ m_{fk}$ and m_{gk}, respectively (of course, $l_{gk} = 0$). These integers are defined by (a superscript T means transposition)

Definition 2.3. Suppose there is a solution $u = (u_1,...,u_n)^T$ of the IVP (1.1), (2.1) which can be written in a certain representation. Then the nonnegative integers l_{fk} and m_{fk} are the highest orders of derivatives of f_k with respect to t and x, respectively, which appear in the representation considered. m_{gk} is the order of the highest $x-$derivative of g_k in the representation.
Upper bounds of $l_{fk},\ m_{fk}$ and m_{gk} are denoted by $L_{fk},\ M_{fk}$ and M_{gk}, respectively. □

This definition of $l_{fk},\ m_{fk},\ m_{gk}$ can be applied (in an obvious way) also to IBVPs (see section 5). For example, $u = (u_1,u_2)^T$ with u_1 as given in eq. (1.4) and $u_2 = f_2(t,x)$ is a representation of a solution of the PDAE (1.3) with $l_{f1} = l_{f2} = 0,\ m_{f1} = 0,\ m_{f2} = 2,\ m_{g1} = m_{g2} = 0$. The determination of integers $l_{fk},\ m_{fk},\ m_{gk}$ (or at least upper bounds of them) is one essential subject of sections 4.3, 4.4 and 5.

3. Relation to a perturbation index

Here, we consider (for example) IVPs. Beside the problem (1.1), (2.1) we pose the disturbed IVP for $\bar{u} = \bar{u}(t,x)$ as the solution of the IVP

$$A\bar{u}_t + B\bar{u}_{xx} + C\bar{u} = f(t,x) + f_p(t,x), \quad t > 0, \quad x \in \Omega, \tag{3.1}$$

$$\bar{u}(0,x) = g(x) + g_p(x), \quad x \in \Omega, \tag{3.2}$$

where the functions f and g are as in IVP (1.1), (2.1), and the components of the given perturbations $f_p,\ g_p$ are assumed to be in the set

$$\mathcal{M}_\eta^{l,m} = \Big\{ w \in \mathcal{M}^{l,m},\ \ \int_{\mathbb{R}} \Big|\frac{\partial^{i+j}}{\partial t^i x^j} w(t,x)\, dx\Big| \le \eta,\ \eta > 0,\ \ i \le l,\ j \le m \Big\} \tag{3.3}$$

for certain nonnegative integers l and m. (Of course, in this definition $l = 0$ for the components of g_p.) η is assumed to be small and, for simplicity, to be independent of t. We suppose that there is an unique solution of the IVP (3.1), (3.2).

As mentioned Campbell and Marszalek [4] proposed a definition of a perturbation index for IBVPs which is applied here analogously also for IVPs. Suppose there are estimates of the form

$$\|u - \bar{u}\|_2 \ \le C_1 \sum_{i=0}^{\bar{l}} \sum_{j=0}^{\bar{m}} \|\frac{\partial^{i+j}}{\partial t^i \partial x^j} f_p\|_\infty + C_2 \sum_{j=0}^{\bar{m}_0} \|\frac{\partial^j}{\partial x^j} g_p\|_\infty. \tag{3.4}$$

$\bar{l},\ \bar{m},\ \bar{m}_0$ are nonnegative integers, and $C_1,\ C_2$ are nonnegative constants independent of the perturbations $f_p,\ g_p$. By taking into account the set (2.2) the norm $\|\cdot\|_\infty$ is defined by

$$\|v\|_\infty := \max_{0 \le t' \le t} \{\|v(t',\cdot)\|_2\}\,. \tag{3.5}$$

$\|v\|_2$ is the familiar L^2-norm of the vector $v \in \mathbb{R}^n$. If such estimates (3.4) exist, then a perturbation index is introduced by

Definition 3.1.

$$\nu_p = 1 + \min\{\max\{\bar{l} + \bar{m}, \bar{m}_0\} : \quad (3.4) \quad \text{holds for} \quad (\bar{l}, \bar{m}, \bar{m}_0)\} \tag{3.6}$$

is called the perturbation index of the IVP (1.1), (2.1).

If the PDAE is a DAE (i.e. B is the zero matrix) and $g_p = 0$, then this definition of the perturbation index is identical with the perturbation index of DAEs [8].

4. Initial value problems

In this section we investigate for $\Omega = \mathbb{R}$ the dependence of the solution u from f and its derivatives with respect to t and x and from g and its derivatives with respect to x by means of the Fourier transformation (FT) and the Laplace transformation (LT). First we collect some basic relations.

4.1. **General relations.** We assume that the vector functions $f,\ g,\ u$ and their derivatives which are needed in the sequel can be Fourier transformed. Furthermore, we also suppose that f and u and their derivatives (as far as needed) can be Laplace transformed, i.e. there are positive constants C_0 and α such that

$$|\psi(t,x)| \le C_0 e^{\alpha t}, \qquad t \ge 0, \tag{4.1}$$

where ψ means f_j or u_j or derivatives of these functions.
Let v be a sufficiently smooth scalar function, e.g. $v \in \mathcal{M}^{l,m}$. We denote by $\mathcal{F}(v) \equiv \hat{v}(t,\omega)$

the FT of v and by $\mathcal{F}^{-1}(\hat{v}) = v(t,x)$ the inverse FT:

$$\mathcal{F}(v) = \hat{v}(t,\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\omega x} v(t,x)\, dx, \qquad \mathcal{F}^{-1}(\hat{v}) = v(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{v}(t,\omega)\, d\omega.$$

The LT of v and its inverse are denoted by $\mathcal{L}(v) \equiv v_\xi(x)$ and $\mathcal{L}^{-1}(v_\xi) \equiv v(t,x)$, respectively:

$$\mathcal{L}(v) = v_\xi(x) = \int_0^\infty e^{-\xi t} v(t,x)\, dt, \qquad \mathcal{L}^{-1}(v_\xi) = v(t,x) = \frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} e^{\bar{\xi} t} v_{\bar{\xi}}(x)\, d\bar{\xi},$$

for $\mathrm{Re}(\xi) > \alpha$ (the symbol $\hat{}$ and a lower index ξ at other variables have the same meaning). Of special interest are in the following the convolution formulas which we cite for convenience for general functions $v,\ w$ which can be transformed:

$$(v * w)_F(t,x) := \int_{-\infty}^{\infty} v(t, x-y) w(t,y)\, dy = \sqrt{2\pi} \mathcal{F}^{-1}(\hat{v}\hat{w}), \tag{4.2}$$

$$(v * w)_L(t,x) := \int_0^t v(t-\tau, x) w(\tau, x)\, d\tau = \mathcal{L}^{-1}(v_\xi w_\xi). \tag{4.3}$$

Furthermore, we mention also for $v \in L^2(\mathbb{R})$ Parseval's relation

$$\int_{-\infty}^{\infty} |v(t,x)|^2\, dx \ = \ \int_{-\infty}^{\infty} |\hat{v}(t,\omega)|^2\, d\omega. \tag{4.4}$$

Applying the FT componentwise, the IVP (1.1), (2.1) can be written as a DAE with parameter ω^2:

$$A \frac{d}{dt} \hat{u}(t,\omega) + (-\omega^2 B + C) \hat{u}(t,\omega) = \hat{f}(t,\omega) \tag{4.5}$$

$$\hat{u}_j(0,\omega) = \hat{g}_j(\omega), \quad j \in \mathcal{M}_{IC}. \tag{4.6}$$

The IVs for $\hat{u}_j,\ \ j \notin \mathcal{M}_{IC}$, are assumed to be consistent. Of course, eq. (4.5) can be Laplace transformed with the result

$$\hat{G}_\xi(\omega) \hat{u}_\xi(\omega) = \hat{f}_\xi(\omega) + A\hat{g}(\omega) \tag{4.7}$$

where $\hat{G}_\xi(\omega)$ is the $(n,n)-$matrix

$$\hat{G}_\xi(\omega) = \xi A - \omega^2 B + C. \tag{4.8}$$

This matrix is the basis of various investigations.

Remark 4.1. For fixed $\xi \in \mathbb{C}$ with $Re(\xi) > \alpha$ the matrix $\hat{G}_\xi(\omega)$ may be regular for all $\omega \in \mathbb{R}$ (Examples 4.3 and 4.4 below), or $\det(\hat{G}_\xi(\omega))$ may have zeros on the real $\omega-$line which implies that $\hat{G}_\xi^{-1}(\omega)$ has isolated singularities on the $\omega-$axis (see Example 4.5 where $\det(\hat{G}_\xi(\omega)) = \omega^6$). $\square$

4.2. **Fourier space solution.** Now we consider the determination of integers l_{fk}, m_{fk} for f_k, and m_{gk} for g_k from the point of view that we essentially work with the FT of u. Let the two matrix pencils

$$(A, -\omega^2 B + C), \quad \omega^2 \in \mathbb{R}, \tag{4.9}$$

$$(B, \xi A + C), \quad \mathrm{Re}(\xi) > \alpha, \tag{4.10}$$

be regular [11]. Then there are two regular $(n,n)-$matrices P_ω and Q_ω which transform A and $-\omega^2 B + C$ according to

$$P_\omega A Q_\omega = \begin{pmatrix} I_{n_1} & 0 \\ 0 & N_\omega \end{pmatrix}, \qquad P_\omega(-\omega^2 B + C)Q_\omega = \begin{pmatrix} R_\omega & 0 \\ 0 & I_{n_2} \end{pmatrix} \tag{4.11}$$

where R_ω and N_ω are $(n_1,n_1)-$ and $(n_2,n_2)-$matrices, respectively, and $n_1 + n_2 = n$. N_ω is a nilpotent matrix with nilpotency $\nu_\omega \geq 1$ which is identical with the uniform differential time index: $\nu_{dt} = \nu_\omega$. I_{n_j} is a unit matrix of order n_j. (If A is regular, then $n_2 = 0$, $I_{n_1} = I_n$ and $\left(\begin{smallmatrix} R_\omega & 0 \\ 0 & I_{n_2} \end{smallmatrix}\right)$ must be replaced by R_ω.) With this Kronecker transformation we get

Lemma 4.1. *Suppose*

1. $\nu_{dt} \geq 1$ *is the uniform differential time index of the PDAE (1.1),*
2. u *is solution of the IVP (1.1), (2.1).*

Then there is a representation of u (resulting from a FT) which is expressed, among other things, by time derivatives of components of f of maximal order $l_F := \nu_{dt} - 1$.

Proof. The transformations (4.11) imply that eq. (4.5) is equivalent to the decoupled system of equations

$$y'(t,\omega) + R_\omega y(t,\omega) = s_1(t,\omega) \tag{4.12}$$

$$N_\omega z'(t,\omega) + z(t,\omega) = s_2(t,\omega) \tag{4.13}$$

where $(y^T(t,\omega), z^T(t,\omega))^T := Q_\omega^{-1}\hat{u}(t,\omega)$ and $(s_1^T(t,\omega), s_2^T(t,\omega))^T := P_\omega \hat{f}(t,\omega)$. The solution of eq. (4.12) is given by

$$y(t,\omega) = e^{-R_\omega t} y(0,\omega) + \int_0^t e^{-R_\omega(t-\tau)} s_1(\tau,\omega)\, d\tau \tag{4.14}$$

while the solution of eq. (4.13) can be written

$$z(t,\omega) = \sum_{j=0}^{\nu_{dt}-1} (-N_\omega)^j s_2^{(j)}(t,\omega). \tag{4.15}$$

$s_2^{(j)}$ means the j^{th} time derivative of s_2. Because a time derivative is not influenced by the inverse FT, we see from the last equation that the maximal order of time derivatives of f is $\nu_{dt} - 1$. This proves the lemma. ⊠

Remark 4.2. The case $\nu_{dt} = 0$ excluded in Lemma 4.1 is not of interest here because eq. (4.5) is then a system of ordinary differential equations. In this case there are no $t-$derivatives of f in the representation of u. □

As is obvious, transforming the solution components y and z, eqs. (4.14) and (4.15), respectively, back to the original $(t,x)-$representation of u, we obtain a representation which

includes time derivatives of $f_k,\ k \in J_n$, of order l_{fk} where $l_{fk} \leq \nu_{dt} - 1$. But in this representation of u there may be also spatial derivatives of $f-$components. An example is given in eq. (1.4) where a second $x-$derivative of f_2 is under a time integral. Furthermore, in general, there can be also $x-$derivatives of components of the initial state g. The time and spatial derivatives of $f-$components may appear as single derivatives $\partial_t^i f_k,\ i \in [0, l_{fk}]$, and $\partial_x^j f_k,\ j \in [0, m_{fk}]$, or as mixed derivatives $\partial_t^i \partial_x^j f_k,\quad i \in [1, l_{fk}],\quad j \in [1, m_{fk}]$, for certain $k \in J_n$.
Obviously, in the representation of a solution u discussed here, $f_k,\ g_k$ must fulfill at least $f_k \in \mathcal{M}^{l_{fk}, m_{fk}}$ and $g_k \in \mathcal{M}^{0, m_{gk}}$.

Now we consider a more general determination of l_{fk} as well as m_{fk} and m_{gk}. According to eq. (4.7), we investigate in the $(t, \omega)-$plane the inverse LT $\mathcal{L}^{-1}\left(\hat{u}_\xi(\omega)\right)$ where

$$\hat{u}_\xi(\omega) = \hat{G}_\xi^{-1}(\omega)[\hat{f}_\xi(\omega) + A\hat{g}(\omega)]. \tag{4.16}$$

Therefore, first we consider componentwise equation

$$\hat{u}(t, \omega) = \mathcal{L}^{-1}\left(\hat{G}_\xi^{-1}(\omega)[\hat{f}_\xi(\omega) + A\hat{g}(\omega)]\right), \tag{4.17}$$

or with $A = (a_{ki})$ and $\hat{R}_k(\omega) := \sum_{i=1}^n a_{ki}\hat{g}_i(\omega)$

$$\hat{u}_j(t, \omega) = \mathcal{L}^{-1}\big(\hat{u}_{\xi j}(\omega)\big) = \sum_{k=1}^n \mathcal{L}^{-1}\left(\hat{G}_{\xi jk}^{-1}(\omega)\left[\hat{f}_{\xi k}(\omega) + \hat{R}_k(\omega)\right]\right), \quad j \in J_n. \tag{4.18}$$

The $(jk)^{th}$ element of the matrix $\hat{G}_\xi^{-1}(\omega)$ can be written

$$\hat{G}_{\xi jk}^{-1}(\omega) = \sum_{\beta, \gamma \in \mathbb{N}} p_{jk\beta\gamma}\xi^\beta \omega^\gamma / \hat{q}_{\xi jk\beta\gamma}(\omega) \tag{4.19}$$

where numerator and denominator are prime to each other. The scalar quantities $p_{jk\beta\gamma}$ (independent of ξ and ω) and $\hat{q}_{\xi jk\beta\gamma}(\omega)$ are known from the given PDAE (1.1). $\hat{q}_{\xi jk\beta\gamma}(\omega)$ is a polynomial in ξ and ω^2. Notice that in eq. (4.19) β and γ depend on the element with indices (jk), but to simplify the notation we avoid to write, e.g., β_{jk}.
There is a relation between differential indexes $\nu_{dt},\ \nu_{dx}$ and exponents $\beta,\ \gamma$: Given these indexes, for all $j, k \in J_n$ and all $\beta,\ \gamma$ (with $p_{jk\beta\gamma} \neq 0$) the limits

$$\lim_{\xi \to \infty} \frac{\xi^{\beta - \nu_{dt}}}{\hat{q}_{\xi jk\beta\gamma}(\omega)} \ \to\ 0 \qquad \text{and} \qquad \lim_{\omega \to \infty} \frac{\omega^{\gamma - \nu_{dx}}}{\hat{q}_{\xi jk\beta\gamma}(\omega)} \ \to\ 0 \tag{4.20}$$

hold, cf. [4], [11].

4.3. **Norm–estimates of u.** Now we consider for fixed indices $j,\ k,\ \beta,\ \gamma$ the representative member in the solution representation (4.16)

$$\hat{W}_{\xi k} \ :=\ \frac{\xi^\beta \omega^\gamma}{\hat{q}_\xi(\omega)} \hat{f}_{\xi k}(\omega), \qquad k \in J_n, \tag{4.21}$$

(for short, we omit the indices $jk\beta\gamma$ in this section) and look for an estimate of $\|W_k\|_{s2} = \|W_k(t, \cdot)\|_{s2}$ where $W_k(t, x) = \mathcal{F}^{-1}\mathcal{L}^{-1}\left(\hat{W}_{\xi k}(\omega)\right)$, and $\|\cdot\|_{s2}$ means the usual $L_2(\mathbb{R})-$norm of a scalar quantity. In this notation Parseval's relation (4.4) can be written as $\|v\|_{s2} = \|\hat{v}\|_{s2}$, and the identity

$$\|\omega^{\gamma}\hat{v}\|_{s2} \;=\; \|\partial_x^{\gamma} v\|_{s2} \tag{4.22}$$

holds for some non–negative integer γ.
Investigating all terms of the form (4.21) containing $\hat{f}_{\xi k}$ we find l_{fk} and m_{fk} (or at least upper bounds of them) as the maximal orders of time and space derivatives, respectively, by inspection of these terms.
For $\|W_k\|_{s2}$ we obtain the following estimates in which only the terms with the highest derivatives of f_k are shown (terms with lower derivatives are not of interest here).

Theorem 4.1. *Let C_0 be a non–negative constant which may be different in different expressions. Suppose $f_k \in \mathcal{M}^{l_{fk},m_{fk}}$ for some suitable non–negative integers l_{fk}, m_{fk} and that $\hat{q}_\xi(\omega) \neq 0$ for $\mathrm{Re}(\xi) > \alpha$ and all $\omega \in \mathbb{R}$. Furthermore, let $0 \le \beta \le l_{fk}$, $0 \le \gamma \le m_{fk}$. Then the following estimates hold:*

1. **$\hat{q}$ is independent of ξ.**
   (a) **$\hat{q}$ is independent of ω.** *Then*

$$\|W_k\|_{s2} \le C_0 \|\partial_t^{\beta}\partial_x^{\gamma} f_k\|_{s2} + ... \tag{4.23}$$

   (for terms "+..." see proof below).
   (b) **$\hat{q}$ depends on ω.** *Let $\delta \in \mathbb{N}$ be the highest power of ω in the polynomial $\hat{q}(\omega)$. Then*

$$\|W_k\|_{s2} \le C_0 \begin{cases} \|\partial_t^{\beta} f_k\|_{s2} + ... & \gamma < \delta \\ \|\partial_t^{\beta}\partial_x^{\gamma-\delta+\delta_0} f_k\|_{s2} + ... & \gamma \ge \delta \end{cases} \tag{4.24}$$

   where δ_0 can be chosen as 1 (however, sometimes $\delta_0 = 2$ may be suitable, e.g. because second order derivatives are in the PDAE).
2. **$\hat{q}$ depends on ξ.** *Suppose $\kappa \in \mathbb{N}$ is the highest power of ξ in $\hat{q}_\xi$, i.e. $\hat{q}$ is of the form $\hat{q}_\xi(\omega) = \xi^{\kappa}\omega^{\delta} + \hat{R}_\xi(\omega)$ where δ is some non–negative integer, and $\hat{R}$ is a polynomial in ξ and ω. Let for $\nu = \kappa - 1$ or $\nu = \beta < \kappa$*

$$\hat{h}_\nu(t,\omega) := \mathcal{L}^{-1}\left(\frac{\xi^{\nu}}{\xi^{\kappa} + \omega^{-\delta}\hat{R}_\xi(\omega)}\right), \tag{4.25}$$

$$\hat{\psi}_\nu(t,\omega) := \int_0^t |\hat{h}_\nu(t-\tau,\omega)|^2\, d\tau, \tag{4.26}$$

   and assume that $\hat{\psi}_\nu(t,\omega)$ is bounded for $t \ge 0$ and all $\omega \in \mathbb{R}$. Furthermore, suppose
   (I) *$|(-i\omega)^{\gamma-\delta}\partial_\tau^{\beta-\kappa+1}\hat{f}_k(\tau,\omega)|^2$ (for $\beta \ge \kappa$) or $|(-i\omega)^{\gamma-\delta}\hat{f}_k(\tau,\omega)|^2$ (for $\beta < \kappa$) is continuous for $\tau \in [0,t]$, $\omega \in \mathbb{R}$,*
   (II) *$\int_{-\infty}^{\infty} |\omega^{\gamma-\delta}\partial_\tau^{\beta-\kappa+1}\hat{f}_k(\tau,\omega)|^2\, d\omega$ (for $\beta \ge \kappa$) or $\int_{-\infty}^{\infty} |\omega^{\gamma-\delta}\hat{f}_k(\tau,\omega)|^2\, d\omega$ (for $\beta < \kappa$) is uniformly convergent with respect to $\tau \in [0,t]$.*
   Then for $\gamma - \delta \ge 0$

$$\|W_k\|_{s2} \le \begin{cases} \left[\int_0^t \left(\int_{-\infty}^{\infty} \hat{\psi}_{\kappa-1}(t,\omega)|\partial_\tau^{\beta-\kappa+1}\omega^{\gamma-\delta}\hat{f}_k(\tau,\omega)|^2\, d\omega\right) d\tau + ...\right]^{1/2} & \beta \ge \kappa \\ \left[\int_0^t \left(\int_{-\infty}^{\infty} \hat{\psi}_{\beta}(t,\omega)|\omega^{\gamma-\delta}\hat{f}_k(\tau,\omega)|^2\, d\omega\right) d\tau\right]^{1/2} & \beta < \kappa. \end{cases} \tag{4.27}$$

If $|\hat h_\nu(t,\omega)| \le h^0_\nu$ *(independent of* t *and* ω*) is assumed, then*

$$\|W_k\|_{s2} \le \begin{cases} \Big[(h^0_{\kappa-1})^2 \ t \ \int_0^t \|\partial_\tau^{\beta-\kappa+1}\partial_x^{\gamma-\delta} f_k(\tau,\cdot)\|^2_{s2}\, d\tau + ...\Big]^{1/2} & \beta \ge \kappa \\ h^0_\beta \ t^{1/2}\Big[\int_0^t \|\partial_x^{\gamma-\delta} f_k(\tau,\cdot)\|^2_{s2}\, d\tau\Big]^{1/2} & \beta < \kappa. \end{cases} \tag{4.28}$$

Proof. First we cite two well–known basic formulas which are valid at least for $0 \le r \le l_{fk}, \ 0 \le s \le m_{fk}$:

$$\mathcal{L}^{-1}\left(\xi^r \hat f_{\xi k}(\omega) - \hat S_{\xi k r}\right) = \partial_t^r \hat f_k(t,\omega), \quad \hat S_{\xi k r} := \sum_{\mu=0}^{r-1} \xi^\mu \partial_t^{r-1-\mu} \hat f_k(0,\omega), \tag{4.29}$$

$$\mathcal{F}^{-1}\left(\omega^s \hat f_{\xi k}(\omega)\right) = \frac{1}{(-i)^s}\partial_x^s f_{\xi k}(x). \tag{4.30}$$

1. (a) Since $\hat q$ is independent of ξ and ω, $\hat W_{\xi k}$ is of the form $\hat W_{\xi k} = \xi^\beta \omega^\gamma \hat f_{\xi k}$ which can be written $\hat W_{\xi k} = \omega^\gamma[\xi^\beta \hat f_{\xi k} - \hat S_{\xi k \beta}] + \omega^\gamma \hat S_{\xi k \beta} = \omega^\gamma[\xi^\beta \hat f_{\xi k} - \hat S_{\xi k\beta}] + ...$ (The term omitted is not of interest here. The same is used often in the rest of the proof.) Applying (4.29), (4.30) we get the estimate in (4.23).
   (b) In this case

   $$\hat W_{\xi k} = \frac{\xi^\beta \omega^\gamma}{\omega^\delta + \hat R(\omega)} \hat f_{\xi k} = \xi^\beta \hat f_{\xi k} \begin{cases} \frac{\omega^\gamma}{\omega^\delta + \hat R(\omega)} & \gamma < \delta \\ \frac{\omega^{\delta-\delta_0}}{\omega^\delta + \hat R(\omega)} \omega^{\gamma-\delta+\delta_0} & \gamma \ge \delta. \end{cases}$$

   Since by assumption $\omega^\delta + \hat R(\omega) \ne 0$ for all $\omega \in \mathbb{R}$, we can apply formula (4.29) and the Fourier convolution theorem (4.2) combined with (4.30). This gives the estimate in (4.24).
2. To prove the estimates (4.27), (4.28) we write

   $$\hat W_{\xi k} = \frac{\xi^\beta \omega^\gamma}{\xi^\kappa \omega^\delta + \hat R_\xi(\omega)} \hat f_{\xi k} = \omega^{\gamma-\delta} \begin{cases} \frac{\xi^{\kappa-1}}{\xi^\kappa + \omega^{-\delta}\hat R_\xi(\omega)} \xi^{\beta-\kappa+1} \hat f_{\xi k} & \beta \ge \kappa \\ \frac{\xi^\beta}{\xi^\kappa + \omega^{-\delta}\hat R_\xi(\omega)} \hat f_{\xi k} & \beta < \kappa \end{cases} \tag{4.31}$$

   and obtain (by inverse LT)

   $$\hat W_k(t,\omega) = \omega^{\gamma-\delta} \begin{cases} \int_0^t \hat h_{\kappa-1}(t-\tau,\omega)\partial_\tau^{\beta-\kappa+1} \hat f_k(\tau,\omega)\, d\tau + ... & \beta \ge \kappa \\ \int_0^t \hat h_\beta(t-\tau,\omega) \hat f_k(\tau,\omega)\, d\tau & \beta < \kappa. \end{cases} \tag{4.32}$$

   For short, let $\nu = \kappa - 1$ or $\nu = \beta$, and consider for these two cases

   $$\hat I_{k\nu}(t,\omega) := \omega^{\gamma-\delta} \int_0^t \hat h_\nu(t-\tau,\omega)\partial_\tau^{\beta-\nu} \hat f_k(\tau,\omega)\, d\tau. \tag{4.33}$$

   $|\hat I_{k\nu}|^2$ can be estimated by means of Cauchy-Schwartz' inequality

   $$|\hat I_{k\nu}(t,\omega)|^2 \le \left(\int_0^t |\hat h_\nu(t-\tau,\omega)|^2\, d\tau\right)\left(\int_0^t |\omega^{\gamma-\delta}\partial_\tau^{\beta-\nu} \hat f_k(\tau,\omega)|^2\, d\tau\right).$$

Because

$$\|\hat{I}_{k\nu}(t,\cdot)\|_{s2}^2 := \int_{-\infty}^{\infty} |\hat{I}_{k\nu}(t,\omega)|^2\, d\omega \le \int_{-\infty}^{\infty} \left(\int_0^t \hat{\psi}_\nu(t,\omega) |\omega^{\gamma-\delta} \partial_\tau^{\beta-\nu} \hat{f}_k(\tau,\omega)|^2\, d\tau \right) d\omega,$$

Parseval's identity and assumptions (I) and (II) of the Theorem (by these the two integrations can be interchanged, i.e. $\int_{-\infty}^{\infty} \left(\int_0^t ...\, d\tau \right) d\omega = \int_0^t \left(\int_{-\infty}^{\infty} ...\, d\omega \right) d\tau$) give the estimate (4.27). The second estimate (4.28) is obtained when the assumption $|\hat{h}_\nu(t,\omega)| \le h_\nu^0$ is used:

$$\|I_{k\nu}(t,\cdot)\|_{s2}^2 = \|\hat{I}_{k\nu}(t,\cdot)\|_{s2}^2 \le (h_\nu^0)^2 t \int_0^t \left(\int_{-\infty}^{\infty} |\omega^{\gamma-\delta} \hat{\partial}_\tau^{\beta-\nu} f_k(\tau,\omega)|^2\, d\omega \right) d\tau.$$

This is the relation (4.28).
The proof of the Theorem is complete. ⊠

This Theorem shows that an estimate of $\|u\|_2$ can be determined solely by means of the matrix $\hat{G}_\xi^{-1}(\omega)$. Note that the estimate (4.27) may be sharper than the estimate (4.28).

Often the estimates (4.27), (4.28) can be improved by reducing the exponent $\gamma - \delta$. Before we give a general result we illustrate it by the following

Example 4.1. Let $\hat{W}_{\xi k} = \frac{\xi\omega^2}{\xi+\omega^2} \hat{f}_{\xi k}$ (compare eq. (4.21)). The corresponding eq. (4.32) reads with $\beta = \kappa = 1,\ \gamma = 2,\ \delta = 0,\ \hat{R}_\xi(\omega) = \omega^2,\ \hat{h}(t,\omega) = \exp(-\omega^2 t)$

$$\hat{W}_k(t,\omega) = \int_0^t e^{-\omega^2(t-\tau)} \omega^2 \hat{f}_{k,\tau}(\tau,\omega)\, d\tau + ... =: \hat{I}_k(t,\omega) + ...$$

(with $k = 2$, this relation appears below also in Example 4.4 on the right of the first line of eq. (4.45)). Here, the term ω^2 as a factor of $\hat{f}_{k,\tau}$ can be eliminated by partial integration:

$$\hat{I}_k(t,\omega) = \hat{f}_{k,t}(t,\omega) - e^{-\omega^2 t} \hat{f}_{k,t}(0,\omega) - \int_0^t e^{-\omega^2(t-\tau)} \hat{f}_{k,\tau\tau}(\tau,\omega)\, d\tau.$$

This implies the estimate

$$\begin{aligned}\|W_k(t,\cdot)\|_{s2} =& \|\hat{W}_k(t,\cdot)\|_{s2} \le \|\hat{f}_{k,t}(t,\cdot) - e^{-\omega^2 t} \hat{f}_{k,t}(0,\cdot)\|_{s2} + \Big\| \int_0^t e^{-\omega^2(t-\tau)} \hat{f}_{k,\tau\tau}(\tau,\cdot)\, d\tau \Big\|_{s2} + ... \\ &\le \|\hat{f}_{k,t}(t,\cdot) - e^{-\omega^2 t} \hat{f}_{k,t}(0,\cdot)\|_{s2} + \left[\int_{-\infty}^{\infty} \left(\int_0^t |\hat{f}_{k,\tau\tau}(\tau,\omega)|^2\, d\tau \right) d\omega \right]^{1/2} + ...\end{aligned}$$

Therefore, we have an estimate with a second time derivative of $\hat{f}_k$ (and without $x-$derivatives). Furthermore, one $\tau-$derivative may be removed under some conditions (see Theorem 4.3 below).

In the general case, we consider again $\hat{W}_{\xi k} = \frac{\xi^\beta \omega^\gamma}{\hat{q}_\xi(\omega)}$ where $\hat{q}_\xi(\omega)$ is of the form given in eq. (4.31).

Theorem 4.2. *Suppose*

(I) *the general assumptions defined at the beginning of Theorem 4.1 are valid,*

(II) *the two assumptions 2. (I), (II) in Theorem 4.1 are valid,*

(III) *for* $\nu = \kappa - 1$ *or* $\nu = \beta$, $\hat{h}_\nu$ *(defined in eq. (4.25) is of the form such that* $\int \hat{h}_\nu(t-\tau,\omega)\, d\tau$ *can be written*

$$\int \hat{h}_\nu(t-\tau,\omega)\, d\tau = \frac{\hat{h}_\nu^{(1)}(t-\tau,\omega)}{\omega^{\alpha_\nu}}$$

where α_ν *is a positive integer, and* $\hat{h}_\nu, \hat{h}_\nu^{(1)}$ *are bounded,* $|\hat{h}_\nu(t-\tau,\omega)| \le h_{0\nu}$, $|\hat{h}_\nu^{(1)}(t-\tau,\omega)| \le h_{0\nu}^{(1)}$ *for* $\tau \in [0,t]$ *and all* $\omega \in \mathbb{R}$.

(IV) $\gamma - \delta - \alpha_\nu \ \ge\ 0$.

Under these assumptions the following two results hold.

(1) *If* $\gamma - \delta \ >\ 0$, *then* $\hat{I}_{k\nu}(t,\omega)$, *(4.33), can be estimated for* $\nu = \kappa - 1$ *or* $\nu = \beta$ *by means of space derivatives of* f_k *of order* $\gamma - \delta - \alpha_\nu$ *only:*

$$\|\hat{I}_{k\nu}(t,\cdot)\|_{s2} \ \le \|\hat{A}(t)\|_{s2} + h_{0\nu}^{(1)} \left[\int_{-\infty}^{\infty} \left(\int_0^t |\hat{B}(\tau)|\, d\tau\right)^2 d\omega\right]^{1/2} \tag{4.34}$$

$$\le \|\hat{A}(t)\|_{s2} + \ h_{0\nu}^{(1)} t^{1/2} \left[\int_{-\infty}^{\infty} \left(\int_0^t |\hat{B}(\tau)|^2\, d\tau\right) d\omega\right]^{1/2} \tag{4.35}$$

$$= \|\hat{A}(t)\|_{s2} + h_{0\nu}^{(1)} t^{1/2} \left[\int_0^t \|\hat{B}(\tau)\|_{s2}^2\, d\tau\right]^{1/2} \tag{4.36}$$

where $\hat{A}(t) := \omega^{\gamma-\delta-\alpha_\nu}\left[\hat{h}_\nu^{(1)}(0,\omega)\partial_t^{\beta-\nu}\hat{f}_k(t,\omega) - \hat{h}_\nu^{(1)}(t,\omega)\partial_t^{\beta-\nu}\hat{f}_k(0,\omega)\right]$ *and*

$$\hat{B}(\tau) := \omega^{\gamma-\delta-\alpha_\nu}\partial_\tau^{\beta-\nu+1}\hat{f}_k(\tau,\omega). \tag{4.37}$$

(2) *If* $\gamma - \delta \ =\ 0$, *then*

$$\|\hat{I}_{k\nu}(t,\cdot)\|_{s2} \ \le h_{0\nu} \left[\int_{-\infty}^{\infty} \left(\int_0^t |\hat{D}(\tau)|\, d\tau\right)^2 d\omega\right]^{1/2} \tag{4.38}$$

$$\le h_{0\nu} t^{1/2} \left[\int_{-\infty}^{\infty} \left(\int_0^t |\hat{D}(\tau)|^2\, d\tau\right) d\omega\right]^{1/2} \ =\ h_{0\nu} t^{1/2} \left[\int_0^t \|\hat{D}(\tau)\|_{s2}^2\, d\tau\right]^{1/2}$$

with $\hat{D}(\tau) := \partial_\tau^{\beta-\nu}\hat{f}_k(\tau,\omega)$.

Proof. If $\gamma - \delta - \alpha_\nu \ge 0$, a partial integration in the expression for $\hat{I}_{k\nu}$ can be performed. Assumption (III) implies

$$\hat{I}_{k\nu}(t,\omega) \ = A(t) - \int_0^t \hat{h}_\nu^{(1)}(t-\tau,\omega)\omega^{\gamma-\delta-\alpha_\nu}\partial_\tau^{\beta-\nu+1}\hat{f}_k(\tau,\omega)\, d\tau.$$

The expression on the right hand side has the form (1.1) in Appendix A. The definition of the norm in $\omega-$space and assumption $|\hat{h}_\nu^{(1)}| \le h_{0\nu}^{(1)}$ (assumed in (III)) lead immediately to (4.34) (see also ineq. (1.4) in Appendix A). Using the estimate (1.2) in Appendix A and again the boundedness of $\hat{h}_\nu^{(1)}$ we obtain the estimate (4.35). Relation (4.36) follows from (4.35) under assumption (II). The inequalities in (4.38) are special cases of the foregoing estimates (omit the partial integration, i.e. replace $A(t)$ by zero and $h_{0\nu}^{(1)}$ by $h_{0\nu}$). ⊠

Under some further conditions, the estimates (4.34) and (4.38) can be sharpened. To show this we decompose $\hat{f}_k$ in its real and imaginary part by writing $\hat{f}_k = \hat{f}_{kr} + i f_{ki}$ and estimate first for $\gamma - \delta > 0$ (see (4.34) and (4.37)) $|\hat{B}(\tau)| \leq |\omega^{\gamma-\delta-\alpha_\nu}|(|\partial_\tau^p \hat{f}_{kr}| + |\partial_\tau^p \hat{f}_{ki}|)$,

$$\int_0^t |\hat{B}(\tau)|\, d\tau \leq |\omega^{\gamma-\delta-\alpha_\nu}|(\hat{J}_r(t,\omega) + \hat{J}_i(t,\omega))$$

where for short $p := \beta - \nu + 1$, $\hat{J}_\lambda := \int_0^t |\hat{j}_\lambda(\tau,\omega)|\, d\tau$, $\hat{j}_\lambda := \partial_\tau^p \hat{f}_{k\lambda}(\tau,\omega)$, $\lambda = r,\ i$. For $p > 0$, we reduce the number of $\tau-$derivatives under the integral by one. The following cases (assumptions for Theorem 4.3) will be considered:

(B1) $\hat{j}_\lambda(\tau,\omega)$, $\lambda = r,\ i$, does not change its sign for $\tau \in [0,t]$ and $\omega \in \mathbb{R}$. Then (by partial integration)

$$\hat{J}_\lambda(t,\omega) \ = \ \operatorname{sgn}(\hat{j}_\lambda(t,\omega))\partial_t^{p-1}\left[\hat{f}_{k\lambda}(t,\omega) - \hat{f}_{k\lambda}(0,\omega)\right]. \tag{4.39}$$

(B2) If the first case does not hold, suppose there are for $t > 0$ $n_\lambda^+ \geq 1$ intervals $\Delta t_{\lambda i}^+ = [a_{\lambda i}^+, b_{\lambda i}^+]$, $i = 1, ..., n_\lambda^+$, $n_\lambda^+ < \infty$, where $\hat{j}_\lambda \geq 0$ and $n_\lambda^- \geq 1$ intervals $\Delta t_{\lambda k}^- = [a_{\lambda k}^-, b_{\lambda k}^-]$, $k = 1, ..., n_\lambda^-$, $n_\lambda^- < \infty$, where $\hat{j}_\lambda < 0$ (in general, all numbers n_λ^+, n_λ^- and $a_{\lambda i}^+$, $b_{\lambda i}^+$, $a_{\lambda k}^-$, $b_{\lambda k}^-$ may depend on ω). Assume $\left(\cup_{i=1}^{n_\lambda^+}\Delta t_{\lambda i}^+\right) \cup \left(\cup_{k=1}^{n_\lambda^-}\Delta t_{\lambda k}^-\right) = [0,t]$. For simplicity, suppose further that the decomposition considered here is independent of $\omega \in \mathbb{R}$. Then it is for $\lambda = r,\ i$

$$\hat{J}_\lambda(t,\omega) = \sum_{i=1}^{n_\lambda^+}\int_{a_{\lambda i}^+}^{b_{\lambda i}^+} \hat{j}_\lambda(\tau,\omega)\, d\tau - \sum_{i=1}^{n_\lambda^-}\int_{a_{\lambda i}^-}^{b_{\lambda i}^-} \hat{j}_\lambda(\tau,\omega)\, d\tau$$

$$= \sum_{i=1}^{n_\lambda^+} \partial_t^{p-1}\left[\hat{f}_{k\lambda}(b_{\lambda i}^+,\omega) - \hat{f}_{k\lambda}(a_{\lambda i}^+,\omega)\right] - \sum_{i=1}^{n_\lambda^-} \partial_t^{p-1}\left[\hat{f}_{k\lambda}(b_{\lambda i}^-,\omega) - \hat{f}_{k\lambda}(a_{\lambda i}^-,\omega)\right]. \tag{4.40}$$

Thus, because $p - 1 = \beta - \nu$ we have proved part (1) of the following Theorem. Part (2) can be proved analogously (for this replace $\hat{B}(\tau)$ by $\hat{D}(\tau)$, set formally $p = \beta - \nu$ and replace $\omega^{\gamma-\delta-\alpha_\nu}$ by 1).

Theorem 4.3. *Suppose the assumptions of Theorem 4.2 and (B1) or (B2) hold. Then we obtain:*

(1) *For $\gamma - \delta > 0$, the estimate given in (4.34) can be sharpened by*

$$\|I_{k\nu}(t,\cdot)\|_{s2} \ \leq \|A(t)\|_{s2} + h_{0\nu}^{(1)}\left[\int_{-\infty}^{\infty}\left(\omega^{\gamma-\delta-\alpha_\nu}\left(\hat{J}_r(t,\omega) + \hat{J}_i(t,\omega)\right)\right)^2 d\omega\right]^{1/2} \tag{4.41}$$

where (according to (4.39) or (4.40)) the order of the highest time derivative of $\hat{f}_k$ is $\beta - \nu$ only. In other words: The L_2-norm of $I_{k\nu}$ can be estimated by time derivatives of f_k of order $\beta - \nu$ and by space derivatives of reduced order $\gamma - \delta - \alpha_\nu$ (assumed to be non–negative).

(2) *For* $\gamma-\delta=0,\ \beta-\nu\geq 1,$ *the estimate in the first line of (4.38) can be sharpened by*

$$\|I_{k\nu}(t,\cdot)\|_{s2}\ \leq h_{0\nu}\left[\int_{-\infty}^{\infty}\left(\hat{J}_r(t,\omega)+\hat{J}_i(t,\omega)\right)^2 d\omega\right]^{1/2}. \tag{4.42}$$

Here the order of the highest time derivative of $\hat{f}_k$ *is* $\beta-\nu-1$.

We illustrate this Theorem by

Example 4.2. Let $\hat{W}_{\xi k}=\frac{\xi\omega^2}{\xi+\omega^2}\hat{f}_{\xi k}$ considered already in Example 4.1. $\beta=\kappa=1,\ \gamma=2$ and $\delta=0$ imply $\nu=0,\ \alpha=2,\ p=2,$ and $\hat{I}_k(t,\omega)=\int_0^t e^{-\omega^2(t-\tau)}\omega^2\partial_\tau\hat{f}_k(\tau,\omega)\,d\tau$. We may use part (1) of Theorem 4.3 ($\gamma-\delta=2$). Let f_k be of the form $f_k(t,x)=\sqrt{\frac{\pi}{2}}t^q e^{-|x|}$ with some $q>1$. This implies $\hat{f}_k(t,\omega)=\hat{f}_{kr}(t,\omega)=t^q(1+\omega^2)^{-1}$. Thus, assumption (B1) is valid. For convenience, here we repeat the estimates in the proof of Theorem 4.3 (avoiding the auxiliary quantities $p,\ \hat{j}_\lambda,\ \hat{J}_\lambda$): Defining $A(t):=\partial_t\hat{f}_k(t,\omega)-e^{-\omega^2 t}\partial_t\hat{f}_k(0,\omega)$ we get

$$\begin{aligned}
\hat{I}_{k\nu}(t,\omega)\ &=A(t)-\int_0^t e^{-\omega^2(t-\tau)}\partial_\tau^2\hat{f}_k(\tau,\omega)\,d\tau\\
|\hat{I}_k(t,\omega)|\ &\leq|A(t)|+\int_0^t|\partial_\tau^2\hat{f}_k(\tau,\omega)|\,d\tau=|A(t)|+\int_0^t\partial_\tau^2\hat{f}_k(\tau,\omega)\,d\tau\\
&=|A(t)|+\partial_t\hat{f}_k(t,\omega)-\partial_t\hat{f}_k(0,\omega),
\end{aligned}$$

i.e. $\|I_k(t,\cdot)\|_{s2}$ can be estimated by a time derivative of first order of $\hat{f}_k$ ($\beta-\nu=1,\ \gamma-\delta-\alpha=0$ in (1) of Theorem 4.3).

4.4. **Representation of the solution: Examples.** In this section we present examples illustrating the ideas of the foregoing subsection to obtain norm estimates of solution components of u in terms of $f-$components and their time and space derivatives and in terms of $g-$components and their space derivatives. Our main interest is the determination of non–negative integers $l_{fi},\ m_{fi},\ m_{gi}$ (or at least of upper bounds $L_{fi},\ M_{fi},\ M_{gi}$ of them) for $i=1,...,n$ because these characterize the sensitivity of the solution with respect to perturbations of f and g.

Example 4.3. Let $n=3$ and the matrices in the PDAE (1.1) be given by
$A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad B=\begin{pmatrix}0&-1&0\\0&0&-1\\0&0&0\end{pmatrix},\quad C=\begin{pmatrix}1&1&0\\0&0&1\\0&1&0\end{pmatrix}$. This implies

$$\hat{G}_\xi^{-1}(\omega)=\begin{pmatrix}\frac{1}{1+\xi}&0&-\frac{1+\omega^2}{1+\xi}\\0&0&1\\0&\frac{1}{1+\omega^2}&-\frac{\xi}{1+\omega^2}\end{pmatrix},\qquad \hat{G}_\xi^{-1}(\omega)A=\begin{pmatrix}\frac{1}{1+\xi}&0&0\\0&0&0\\0&\frac{1}{1+\omega^2}&0\end{pmatrix}.$$

The uniform differential time index ν_{dt} is 2, and the differential spatial index ν_{dx} is 3. An IV can be prescribed only for u_1, i.e. $\mathcal{M}_{IC}=\{1\}$. Obviously, here the cases 2. for $k=1$ and 1.(b) for $k=2,3$ of Theorem 4.1 are valid (for $k=1$ apply the second estimate in (4.28) with $\beta=\delta=\gamma=0$ and $\kappa=1$). The structure of $\hat{G}_\xi^{-1}$ implies for $t\geq 0$ $l_{f1}=m_{f1}=0,\ l_{f2}=m_{f2}=0,\ l_{f3}=1,\ m_{f3}=2$. These integers specify the smoothness of the components f_i which must be in the set $\mathcal{M}^{l_{fi},m_{fi}}$, see (2.2). Furthermore, it is easy to see that $m_{gi} = 0,\ i = 1,2,3,\ t \geq 0$. To discuss some further aspects we consider, e.g., the representation of $\hat{u}_3$. Eq. (4.18) is here

$$\hat{u}_3(t,\omega) = = \frac{1}{1+\omega^2}\left[\hat{f}_2(t,\omega) - \mathcal{L}^{-1}\left(\xi\hat{f}_{\xi 3}(\omega) - \hat{f}_3(0,\omega)\right) + \mathcal{L}^{-1}\left(-\hat{f}_3(0,\omega) + \hat{g}_2(\omega)\right)\right].$$

(According to (4.29) we added and subtracted the term $\hat{S}_{\xi kr}$ for $k = 3, r = 1$.) The inverse LT can be performed with the result

$$\hat{u}_3(t,\omega) = \frac{1}{1+\omega^2}\left[\hat{f}_2(t,\omega) - \partial_t\hat{f}_3(t,\omega)\right].$$

Estimates of $\|u_3\|_{s2}$ in terms of $\|f_2\|_{s2},\ \|\partial_t f_3\|_{s2}$ follow directly from this representation or with the help of the first line of (4.24) (because $\delta = 2,\ \gamma = 0$).
By the way, we notice that the consistent IV for u_2 can be obtained very easily from $\hat{u}_3$. Because $-\hat{f}_3(0,\omega)+\hat{g}_2(\omega)$ must be zero, the consistent IV is $g_2(x) = f_3(0,x)$. For completness, we also give the consistent consistent IV for $u3$:

$$g_3(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\left[f_2(0,y) - \partial_t f_3(0,y)\right] dy. \quad \Box$$

Example 4.4. An interesting case is the PDAE $(n = 3)$[2]

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} u_t + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} u_{xx} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} u = f(t,x), \quad t > 0,\ \ x \in \Omega = \mathbb{R}, \tag{4.43}$$

with IV $u_1(0,x) = g_1(x)$, i.e. we choose $\mathcal{M}_{IC} = \{1\}$. (There is yet another possibility for $\mathcal{M}_{IC}$. Notice that when f is given, g_2 and g_3 cannot be chosen arbitrarily.) The PDAE has indexes $\nu_{dt} = \nu_{dx} = 1$. Since $\nu_{dt} = 1$ we know from Lemma 4.1 that there is a representation of u (generated by FT) which needs no time derivatives of $f-$components.
We find $\hat{G}_\xi^{-1}(\omega) = \frac{1}{\xi+\omega^2}\begin{pmatrix} 1 & \omega^2 & 1 \\ -\omega^2 & \xi\omega^2 & \xi \\ 1 & -\xi & 1 \end{pmatrix}$, $\hat{G}_\xi^{-1}(\omega)A = \frac{1}{\xi+\omega^2}\begin{pmatrix} 1 & 0 & 0 \\ -\omega^2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $\hat{q}_{\xi jk\beta\gamma}(\omega) = \hat{q}_\xi(\omega) = \xi + \omega^2$ (i.e. $\hat{q}$ is independent of j,k,β,γ). Therefore, we can apply the second case of Theorem 4.1 considered also in Theorem 4.2, and according to (4.25) $\hat{h}_0(t,\omega) = \mathcal{L}^{-1}\left(\frac{1}{\hat{q}_\xi(\omega)}\right) = e^{-\omega^2 t}$, i.e. the function $\hat{\psi}_0$ is bounded (as assumed in Theorem 4.1). $\hat{\psi}_0$ is given by $\hat{\psi}_0(t,\omega) = (1 - e^{-2\omega^2 t})/(2\omega^2)$.
The easiest way to apply the technique of subsection 4.3 is to write down the three equations

[2]This PDAE was constructed by J. Wensch, Martin–Luther–Universität Halle.

(4.18), $j=1,2,3$, in detail:

$$\hat u_1(t,\omega) = e^{-\omega^2 t}\hat g_1(\omega) + \int_0^t e^{-\omega^2(t-\tau)}[\hat f_1 + \omega^2 \hat f_2 + \hat f_3](\tau,\omega)\,d\tau \tag{4.44}$$

$$\hat u_2(t,\omega) = e^{-\omega^2 t}[\omega^2 \hat f_2(0,\omega) + \hat f_3(0,\omega) - \omega^2 \hat g_1(\omega)] + \int_0^t e^{-\omega^2(t-\tau)}[-\omega^2 \hat f_1 + \omega^2 \hat f_{2,\tau} + \hat f_{3,\tau}](\tau,\omega)\,d\tau \tag{4.45}$$

$$= [\omega^2 \hat f_2 + \hat f_3](t,\omega) - \omega^2 e^{-\omega^2 t}\hat g_1(\omega) + \int_0^t e^{-\omega^2(t-\tau)}[-\omega^2 \hat f_1 - \omega^4 \hat f_2 - \omega^2 \hat f_3](\tau,\omega)\,d\tau$$

$$\hat u_3(t,\omega) = e^{-\omega^2 t}[-\hat f_2(0,\omega) + \hat g_1(\omega)] + \int_0^t e^{-\omega^2(t-\tau)}[\hat f_1 - \hat f_{2,\tau} + \hat f_3](\tau,\omega)\,d\tau \tag{4.46}$$

$$= -\hat f_2(t,\omega) + e^{-\omega^2 t}\hat g_1(\omega) + \int_0^t e^{-\omega^2(t-\tau)}[\hat f_1 + \omega^2 \hat f_2 + \hat f_3](\tau,\omega)\,d\tau.$$

Each term in these equations is of the type considered in Theorems 4.1 – 4.3. (Comparing with eq. (4.31) we get the parameters $\delta = 0$, $\kappa = 1$, $\hat R_\xi(\omega) = \omega^2$, and for all elements of the matrix $\hat G_\xi^{-1}$ it is $\gamma - \delta \geq 0$ (γ is 0 or 2).

From the representation (4.44) – (4.46), the estimate (1.4) in Appendix A and the identity (4.22) we get as upper bounds for l_{fk}, m_{fk}

$$L_{f1} = 0,\ L_{f2} = 1,\ L_{f3} = 1,\ M_{f1} = M_{f2} = 2,\ M_{f3} = 0.$$

However, using the estimates of Theorem 4.2 we obtain

$$L_{f1} = 1,\ L_{f2} = 2,\ L_{f3} = 1,\ M_{fk} = 0 \quad \text{for} \quad k = 1,2,3.$$

Further bounds sharper as these follow for vectors f which satisfy the assumptions of Theorem 4.3. In fact, according to Definition 2.3 we obtain from this Theorem (see also Example 4.2) the orders of derivatives of $f-$components appearing in a Fourier decomposition:

$$l_{f1} = l_{f3} = 0,\ l_{f2} = 1, \quad m_{fk} = 0,\ k = 1,2,3. \tag{4.47}$$

As is clear from the representation (4.44) – (4.46), the consistent IVs of $\hat u_2$ and $\hat u_3$ are $\hat g_2(\omega) = \omega^2 \hat f_2(0,\omega) + \hat f_3(0,\omega) - \omega^2 \hat g_1(\omega)$ and $\hat g_3(\omega) = -\hat f_2(0,\omega) + \hat g_1(\omega)$, respectively. Hence,

$$m_{g1} = \begin{cases} 2, & t = 0, \\ 0, & t > 0, \end{cases} \qquad \text{and} \qquad m_{g2} = m_{g3} = 0, \quad t \geq 0. \qquad \square$$

Example 4.5. An example of another type is given for $n = 3$ by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.48}$$

The corresponding system (1.1) has indexes $\nu_{dt} = 3$, $\nu_{dx} = 0$. If f is given, IVs for u cannot be prescribed arbitrarily. The matrices $\hat G_\xi^{-1}$ and $\hat G_\xi^{-1} A$ are

$$\hat G_\xi^{-1}(\omega) = \frac{1}{\omega^6}\begin{pmatrix} \omega^4 & -(1+\xi)\omega^2 & (\xi^2+\xi-\omega^2) \\ 0 & \omega^4 & -\xi\omega^2 \\ 0 & 0 & \omega^4 \end{pmatrix}, \qquad \hat G_\xi^{-1}(\omega) A = \frac{1}{\omega^6}\begin{pmatrix} 0 & \omega^4 & -(1+\xi)\omega^2 \\ 0 & 0 & \omega^4 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.49}$$

The peculiarity is that the elements of these matrices become infinite for $\omega \to 0$. Nevertheless, the integers $l_{fi},\ m_{fi}$ can be determined by means of $\hat{G}_\xi^{-1}$. They are for $t \geq 0$:

$$l_{f1} = 0,\ l_{f2} = 1,\ l_{f3} = 2 \qquad \text{and} \qquad m_{fi} = 0,\ i \in J_3. \tag{4.50}$$

The numbers m_{gi} are $m_{gi} = 0$ for $i \in J_3$. $\square$

5. Initial boundary value problems

It is possible to extend the investigations of IVPs in section 4 to certain IBVPs. To be specific we consider on, say $\Omega_0 = (0,1)$, the IBVP

$$Au_t + Bu_{xx} + Cu = f^0(t,x), \qquad t > 0, \quad x \in \Omega_0, \tag{5.1}$$

$$u(0,x) = g^0(x), \qquad x \in \bar{\Omega}_0, \tag{5.2}$$

$$u(t,0) = u(t,1) = 0, \qquad t \geq 0, \tag{5.3}$$

where $f^0,\ g^0$ are given sufficiently smooth functions satisfying $f^0(t,0) = f^0(t,1) = g^0(0) = g^0(1) = 0$. A finite $x-$interval is of importance when considering the numerical solution of problems being modelled by PDAEs of the type (5.1). The problem of determining *consistent* IVs and BVs is discussed elsewhere, see e.g. [7], [11], [12].
We suppose that there is one and only one (classical) solution of the IBVP where a solution is defined as follows (compare also Definition 2.1).

Definition 5.1. $u \in \mathbb{R}^n$ is called classical solution of the IBVP (5.1) – (5.3) if

1. it is sufficiently smooth,
2. u is uniquely determined by the conditions (5.2) – (5.3),
3. u solves the PDAE (5.1) pointwise.

Remark 5.1. Three IBVPs of the form (5.1)–(5.3) are discussed (and solved numerically) in section 6: (i) the PDAE (4.43) and f^0 in (6.12), (ii) the PDAE (4.43) with f^0 in (6.13), (iii) the PDAE with matrices (4.48) and f^0 in (6.21). $\square$
In analogy to the FT (see section 4.1), we assume that $f^0,\ g^0$ and the solution u of the IBVP can be written as Fourier series with respect to the orthonormal scalar system $\phi_\omega(x) := \sqrt{2}\sin(\pi\omega x),\ \omega = 1, 2, ...,$

$$f^0(t,x) = \sum_{\omega=1}^{\infty} \hat{f}^0(t,\omega)\phi_\omega(x), \qquad u(t,x) = \sum_{\omega=1}^{\infty} \hat{u}(t,\omega)\phi_\omega(x) \tag{5.4}$$

with Fourier coefficients, e.g.

$$\hat{f}^0(t,\omega) = \int_0^1 f^0(t,x)\phi_\omega(x)\,dx. \tag{5.5}$$

It is easily shown that the vector valued coefficients $\hat{u}$ must satisfy the following system of equations which are DAEs for a singular matrix A:

$$A\frac{d}{dt}\hat{u}(t,\omega) + (\mu_\omega B + C)\hat{u}(t,\omega) = \hat{f}^0(t,\omega)), \quad \hat{u}(0,\omega) = \hat{g}^0(\omega), \quad \omega = 1, 2, ..., \tag{5.6}$$

with $\mu_\omega := -(\pi\omega)^2$. Applying a Laplace transform to eq. (5.6) (again this is supposed to be possible) the IBVP for $\hat{u}_\xi$ leads to

$$\hat{u}_\xi(\omega) = \hat{G}_\xi^{-1}(\omega)[\hat{f}_\xi^0(\omega) + A\hat{g}^0(\omega)] \tag{5.7}$$

where $\hat{G}_\xi(\omega) := \xi A + \mu_\omega B + C$ is required to be invertible for all ξ with $Re(\xi) \geq \alpha$ and for all ω. As a result we obtain

Lemma 5.1. *Suppose*

(I) *the IVP (1.1), (1.2) (i.e. $\Omega = \mathbb{R}$) has a solution representation which needs the derivatives $\partial_t^{l_{fk}} f_k,\ \partial_x^{m_{fk}} f_k,\ \partial_x^{m_{gk}} g_k,\ k \in J_n$,*

(II) *$f_k \in \mathcal{M}^{l_{fk}, m_{fk}},\ g_k \in \mathcal{M}^{0, m_{gk}},\ k \in J_n$.*

Then the IBVP (5.1) – (5.3) has a corresponding representation of the solution which needs the same order of derivatives of $f_k^0,\ g_k^0$:

$$l_{f^0 k} = l_{fk}, \quad m_{f^0 k} = m_{fk}, \quad m_{g^0 k} = m_{gk}, \quad k \in J_n.$$

Proof. The proof rests on the analogy of

1. Fourier transforms (section 4.1) and Fourier expansions (5.4),
2. the equations for $\hat{u}_\xi(\omega),\ \omega \in \mathbb{R}$, and $\hat{u}_\xi(\tilde{\omega}),\ \tilde{\omega} = 1, 2, ...,$ given in (4.16) and (5.7), respectively. ⊠

Remark 5.2. Lemma 5.1 is valid also in the reverse direction. □

After this Lemma we suppress in the following the index 0 on the right of eqs. (5.1) – (5.3) (i.e. $f \equiv f^0,\ g \equiv g^0$ in the following).

The Theorems 4.1 – 4.3 are valid (with some immaterial modifications) also for the IBVPs considered here. To illustrate this we write down the analogue to Theorem 4.3. First we need some definitions. Let for $k \in J_n,\ \omega = 1, 2, ...,\ \hat{W}_{\xi k}(\omega)$ be of the form (compare with eq. (4.31))

$$\hat{W}_{\xi k} = \frac{\xi^\beta (\pi\omega)^\gamma}{\xi^\kappa (\pi\omega)^\delta + \hat{R}_\xi(\omega)} \hat{f}_{\xi k} = (\pi\omega)^{\gamma-\delta} \begin{cases} \frac{\xi^{\kappa-1}}{\xi^\kappa + (\pi\omega)^{-\delta}\hat{R}_\xi(\omega)} \xi^{\beta-\kappa+1} \hat{f}_{\xi k} & \beta \geq \kappa \\ \frac{\xi^\beta}{\xi^\kappa + (\pi\omega)^{-\delta}\hat{R}_\xi(\omega)} \hat{f}_{\xi k} & \beta < \kappa \end{cases}, \tag{5.8}$$

and define for $\nu = \kappa - 1$ or $\nu = \beta$

$$\begin{aligned} \hat{h}_\nu(t, \omega) :=& \mathcal{L}^{-1}\left(\frac{\xi^\nu}{\xi^\kappa + (\pi\omega)^{-\delta}\hat{R}_\xi(\omega)}\right), \qquad \int \hat{h}_\nu(t-\tau, \omega)\, d\tau = \frac{\hat{h}_\nu^{(1)}(t-\tau, \omega)}{(\pi\omega)^{\alpha_\nu}}, \\ \hat{A}(t, \omega) :=& (\pi\omega)^{\gamma-\delta-\alpha_\nu}\left[\hat{h}_\nu^{(1)}(0, \omega)\partial_t^{\beta-\nu}\hat{f}_k(t, \omega) - \hat{h}_\nu^{(1)}(t, \omega)\partial_t^{\beta-\nu}\hat{f}_k(0, \omega)\right], \\ \hat{B}(\tau, \omega) :=& (\pi\omega)^{\gamma-\delta-\alpha_\nu}\partial_\tau^{\beta-\nu+1}\hat{f}_k(\tau, \omega), \quad \hat{D}(\tau, \omega) := \partial_\tau^{\beta-\nu}\hat{f}_k(\tau, \omega). \end{aligned}$$

The norm of a scalar function $v = v(x)$ is now $\|v\|_{s2} := \left[\int_0^1 |v(x)|^2\, dx\right]^{1/2}$. Then the following Theorem holds for the IBVP:

Theorem 5.1. *Suppose*

1. *$\hat{W}_{\xi k},\ k \in J_n$, has the form (5.8),*
2. *$f(t, x)$ and $\hat{h}_\nu(t, \omega)$ are sufficiently smooth on their domains of definition (compare with the first assumption of Theorem 4.2),*

3. *assumptions (B1) or (B2) (defined above for Theorem 4.3) hold for the Fourier coefficients* $\hat{f}_k^0(t,\omega)$. *(Note that these are real here, i.e. it is* $\hat{f}_{ki} = \hat{j}_i = \hat{J}_i = 0$ *and* $\hat{f}_{kr} = \hat{f}_k$ *in (B1), (B2).)*

Then $\hat{W}_k(t,\omega) = \hat{I}_{k\nu}(t,\omega) + ...,\ k \in J_n,\ \nu = \kappa - 1$ *or* $\nu = \beta$, *and*

$$\hat{I}_{k\nu}(t,\omega) := (\pi\omega)^{\gamma-\delta} \int_0^t \hat{h}_\nu(t-\tau,\omega)\partial_\tau^{\beta-\nu}\hat{f}_k(\tau,\omega)\,d\tau$$

can be estimated

(1) *for* $\gamma - \delta > 0$ *by*

$$\|I_{k\nu}(t,\cdot)\|_{s2} \le \|A(t)\|_{s2} + h_{0\nu}^{(1)} \left[\sum_{\omega=1}^{\infty}\Big((\pi\omega)^{\gamma-\delta-\alpha_\nu}\hat{J}_r(t,\omega)\Big)^2\right]^{1/2} \tag{5.9}$$

where the order of the highest time derivative of $\hat{f}_k$ *which appears in* $\hat{J}_r$ *(defined explicitly in eq. (4.39) or (4.40) with* $\lambda = r$*) is of order* $\beta - \nu$,

(2) *for* $\gamma - \delta = 0,\ \beta - \nu \ge 1$ *by*

$$\|I_{k\nu}(t,\cdot)\|_{s2} \le h_{0\nu} \left[\sum_{\omega=1}^{\infty}\Big(\hat{J}_r(t,\omega)\Big)^2\right]^{1/2}. \tag{5.10}$$

Here, the order of the highest time derivative of $\hat{f}_k$ *is* $\beta - \nu - 1$. □

Proof. If one takes into account that Parsevals relation (4.4) now reads $\int_0^1 |v(t,x)|^2\,dx = \sum_\omega |\hat{v}(t,\omega)|^2$, then the proof is (with this modification) the same as for Theorem 4.3. ⊠

Example 5.1. In special cases, the determination of J_r may be easy. To give an example we assume that f_k is of product form, $f_k(t,x) = \varphi_k(t)\psi_k(x)$ ($\varphi_k,\ \psi_k$ real). The Fourier coefficient $\hat{f}_k$ is $\hat{f}_k(t,\omega) = \varphi_k(t)\hat{\psi}_k(\omega)$, and

$$\hat{J}_r = \int_0^t |\partial_\tau^p \hat{f}_k(\tau,\omega)|\,d\tau = |\hat{\psi}_k(\omega)| \int_0^t |\partial_\tau^p \varphi_k(\tau)|\,d\tau.$$

The integration may be performed by means of this formula, for example, when $\varphi_k(t) = t^\alpha$ with some $\alpha \ge p-1 \ge 0:\ \int_0^t |\partial_\tau^p\varphi_k(\tau)|\,d\tau = \int_0^t |\partial_\tau^p \tau^\alpha|\,d\tau = \partial_t^{p-1}t^\alpha = \partial_t^{p-1}\varphi_k(t)$. An application of this type is considered in section 6.1.

6. Numerical solution under perturbations

The numerical solution of PDAEs of the form (1.1) without perturbations has been considered, e.g., in [11]. In this section we study the influence of perturbations of f when such problems are solved numerically. Perturbations may have their origin in rounding errors, but there may be also perturbations due to some lack in the mathematical model.

In order to demonstrate the influence of perturbations we investigate here finite difference solutions of IBVPs of PDAEs considered already in examples 4.4 and 4.5. In each case we construct a difference scheme which converges. Then we try to solve by that numerical scheme a problem where to the right hand side vector f a small perturbation is added (see also section

3). We calculate errors of the numerical solution (with respect to the exact solution) and determine their order numerically. We will find a strong dependence of this order on certain integers l_{fi} and m_{fi} as considered in the foregoing sections.

To discretize the PDAE we introduce a time step size τ and a space step size $h = 1/N$ where $N \geq 2$ is an integer. With τ and h a grid

$$\Omega_{\tau,h} := \{(t_m, x_k) = (m\tau, kh), \quad m = 0, 1, ..., \quad k = 0, 1, ..., N\}$$

is defined. We replace on inner grid points (labeled by m, k) $m = 1, 2, ...,\ k = 1, ..., N-1$, $u_{j,xx}(t_m, x_k)$ by the usual second order approximation

$$u_{j,xx}(t_m, x_k) \approx \frac{1}{h^2}\left(u^m_{j,k-1} - 2u^m_{j,k} + u^m_{j,k+1}\right), \qquad j \in J_n, \tag{6.1}$$

and $u_{j,t}(t_m, x_k)$ by the backward first order formula

$$u_{j,t}(t_m, x_k) \approx \frac{1}{\tau}\left(u^m_{j,k} - u^{m-1}_{j,k}\right), \qquad j \in J_n, \tag{6.2}$$

or by the central second order formula

$$u_{j,t}(t_m, x_k) \approx \frac{1}{2\tau}\left(u^{m+1}_{j,k} - u^{m-1}_{j,k}\right), \qquad j \in J_n. \tag{6.3}$$

Besides these relations we introduce the familiar $(N-1, N-1)-$matrices

$$P_{N-1} = \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & & \ddots & \\ & & 1 & -2 \end{pmatrix} \qquad \text{and} \qquad Q_{N-1} = I_{N-1} - \frac{\tau}{h^2}P_{N-1}.$$

(The eigenvalues of P_{N-1} are negative, hence $\frac{1}{h^2}P_{N-1}$ and Q_{N-1} are invertible.) Furthermore, let $\|G\|$ be the discrete L^2-norm

$$\|G\| = \left[h\sum_{j=1}^{N-1}|G_j|^2\right]^{1/2}, \quad G = (G_1, G_2, ..., G_{N-1})^T, \quad G_j \in \mathbb{R},$$

Then with the corresponding induced matrix norm the estimates

$$\|\left(\frac{1}{h^2}P_{N-1}\right)^{-1}\| \leq \tilde{C} \qquad \text{and} \qquad \|Q^{-1}_{N-1}\| \leq 1 \tag{6.4}$$

hold ($\tilde{C} > 0$ is independent of N).

Remark 6.1. By means of the approximations (6.1) and (6.2) we solve the IBVP by the BTCS scheme considered for PDAEs without perturbations of f in [10], [11]. □

6.1. **Numerical solution of a problem with differential indexes 1, 1.** We consider on $\Omega = (0, 1)$ the PDAE in (4.43) which has differential indexes $\nu_{dt} = \nu_{dx} = 1$. Let in the undisturbed case $f = (f_1,\ f_2,\ f_3)^T$ be given. For simplicity, we pose the following assumptions:

(A1) the analytic solution (which we denote by $v = v(t, x)$) is known,
(A2) v satisfies zero boundary conditions $v(t, 0) = v(t, 1) = 0$.

To solve the IBVP by means of the BTCS scheme we write eq. (4.43) in component form

$$u_{1,t} - u_2 = f_1, \qquad u_1 - u_3 = f_2, \qquad -u_{3,xx} + u_2 = f_3. \tag{6.5}$$

This system of equations is discretized by means of (6.1) and (6.2). Defining for $j \in J_3$ $(N-1)-$dimensional vectors $U_j^m := (u_{j,1}^m, u_{j,2}^m, ..., u_{j,N-1}^m)^T$, $F_j^m := (f_{j,1}^m, f_{j,2}^m, ..., f_{j,N-1}^m)^T$ (where $f_{j,k}^m = f_j(t_m, x_k)$), a finite dimensional approximation of the IBVP can be written

$$\frac{1}{\tau}(U_1^m - U_1^{m-1}) = U_2^m + F_1^m, \qquad U_1^m - U_3^m = F_2^m, \qquad \frac{1}{h^2}P_{N-1}U_3^m = U_2^m - F_3^m. \tag{6.6}$$

This can be rearranged easily to give for $m = 1, 2, ...$

$$\begin{aligned} U_1^m =& Q_{N-1}^{-1}\left(U_1^{m-1} - \frac{\tau}{h^2}P_{N-1}F_2^m + \tau(F_1^m + F_3^m)\right), \\ U_2^m =& \frac{1}{\tau}\left(U_1^m - U_1^{m-1}\right) - F_1^m, \quad U_3^m = U_1^m - F_2^m. \end{aligned} \tag{6.7}$$

From the first equation we calculate U_1^m, and then U_2^m and U_3^m.

First we must show that the scheme (6.7) converges. Let V_j^m be the vector of the exact solution restricted to the grid, i.e. $V_j^m = \left(v_{j,1}^m, v_{j,2}^m, ..., v_{j,N-1}^m\right)^T$ where $v_{j,k}^m := v_j(t_m, x_k)$. For $j \in J_3$ and a positive integer m we consider the error $E_j^m := U_j^m - V_j^m$ and its norm $\|E_j^m\|$ under the condition $\tau = t_0/m$ where $t_0 > 0$ is assumed to be a fixed time.

Lemma 6.1. *Let*

(I) $f_1,\ f_3,\ v_2 \in C_{sc}^{2,\infty}$,
(II) $f_2,\ v_1,\ v_3 \in C_{sc}^{3,\infty}$,
(III) $E_1^0 = 0$,
(IV) $\tau = t_0/m, \quad t_0 > 0$,
(V) $h = 1/N$.

Then the BTCS scheme (6.7) converges for $m \to \infty,\ N \to \infty$ componentwise with orders $(1,2)$, i.e.

$$\|E_j^m\| = O(\tau) + O(h^2), \qquad j \in J_3.$$

Proof. Using assumptions (I) and (II) we can insert the exact solution $V_j^m,\ j \in J_3$, into the system (6.7). Especially we obtain for $j = 1$ and $\tau,\ h \to 0$

$$\begin{aligned} Q_{N-1}V_1^m =& V_1^{m-1} - \frac{\tau}{h^2}P_{N-1}F_2^m + \tau(F_1^m + F_3^m) \\ & - 2\frac{\tau}{h^2}\sum_{l=2}^{\infty}\frac{h^{2l}}{(2l)!}\frac{\partial^{2l}}{\partial x^{2l}}(V_1^m - F_2^m) - \frac{\tau^2}{2}\frac{d^2V_1^m}{dt^2} + O(\tau^3). \end{aligned}$$

Combining this with the first equation in (6.7), the global discretization error for U_1 is in the limits $\tau,\ h \to 0$

$$\begin{aligned} E_1^m =& Q_{N-1}^{-1}\left(E_1^{m-1} + 2\frac{\tau}{h^2}\sum_{l=2}^{\infty}\frac{h^{2l}}{(2l)!}\frac{\partial^{2l}}{\partial x^{2l}}(V_1^m - F_2^m) + \frac{\tau^2}{2}\frac{d^2V_1^m}{dt^2} + O(\tau^3)\right) \\ =& Q_{N-1}^{-1}\left(E_1^{m-1} + O_1^m(\tau^2) + O_2^m(\tau h^2)\right). \end{aligned} \tag{6.8}$$

Assumptions (III) and (IV) imply

$$\|E_1^m\| = O(\tau) + O(h^2) \tag{6.9}$$

(here, we suppressed the indices at the $O-$symbols). The same order of convergence holds for $\|E_3^m\|$ (this follows from the third equation in (6.7) and assumption (I)).
To get an estimate for $\|E_2^m\|$ we use the second equation of (6.7). The global discretization error for U_2 is $E_2^m = \frac{1}{\tau}(E_1^m - E_1^{m-1}) + O_1(\tau)$. By means of relation (6.8) we obtain

$$\begin{aligned} E_1^m - E_1^{m-1} =& Q_{N-1}^{-1}\Big(E_1^{m-1} - E_1^{m-2} + 2\frac{\tau}{h^2}\sum_{l=2}^{\infty}\frac{h^{2l}}{(2l)!}\frac{\partial^{2l}}{\partial x^{2l}}(V_1^m - V_1^{m-1} - F_2^m + F_2^{m-1}) \\ &+ \frac{\tau^2}{2}\frac{d^2(V_1^m - V_1^{m-1})}{dt^2} + O(\tau^3)\Big). \end{aligned}$$

Assumption (II) implies that this expression can be rewritten

$$\begin{aligned} E_1^m - E_1^{m-1} =& Q_{N-1}^{-1}\Big(E_1^{m-1} - E_1^{m-2} + 2\frac{\tau^2}{h^2}\sum_{l=2}^{\infty}\frac{h^{2l}}{(2l)!}\frac{\partial^{2l}}{\partial x^{2l}}\frac{d(V_1^m - F_2^m)}{dt} + O_1^m(\tau^3) + O_2^m(\tau^3 h^2)\Big) \\ =& Q_{N-1}^{-1}\left(E_1^{m-1} - E_1^{m-2} + O_1^m(\tau^2 h^2) + O_2^m(\tau^3)\right). \end{aligned}$$

Using assumptions (III), (IV) and the second estimate in (6.4) we get

$$\|E_1^m - E_1^{m-1}\| = O(\tau h^2) + O(\tau^2)$$

(again the indices at the $O-$symbols are suppressed). Since $E_2^m = \frac{1}{\tau}\left(E_1^m - E_1^{m-1}\right) + O_1(\tau)$ the result of the Lemma for $j = 2$ follows. ⊠

Remark 6.2. It can be shown that, when $v_1,\ v_3 \in C^{3,2p+2},\ f_2 \in C^{3,2p+2},\ p \geq 1$, then the order relation for the global discretization error E_2 is

$$\|E_2^m\| = O(\tau) + O(h^2) + O(\frac{h^{2p}}{\tau}) \quad \text{for} \quad \tau,\ h \to 0,$$

i.e. the discretization scheme is conditionally convergent. □

Remark 6.3. We note that beside the scheme (6.7) there is yet another scheme for solving the IBVP of this example, namely

$$\begin{aligned} U_3^m =& Q_{N-1}^{-1}\left(U_3^{m-1} - F_2^m + F_2^{m-1} + \tau(F_1^m + F_3^m)\right), \\ U_1^m =& U_3^m + F_2^m, \quad U_2^m = \frac{1}{\tau}\left(U_1^m - U_1^{m-1}\right). \end{aligned} \tag{6.10}$$

Formally, we get this finite dimensional method e.g. by inserting $U_1^m = U_3^m + F_2^m$ into the first equation of the system in (6.7). □

Looking at section 3 we consider now a problem in which $f_1 = f_3 = f_{1p} = f_{3p} = 0$, and f_2 (different from zero) is substituted by $f_2 + f_{2p}$ where f_{2p} is a small perturbation. This means that F_2^m is replaced by $F_2^m + F_{2p}^m$. (For simplicity, g_p is chosen also as zero.) Assuming that

$f_2(t,x)$ is for $t \in (0,t_0)$ and $x \in \Omega$ of (maximal) order 1 we take for the k^{th} component of F_{2p}^m e.g.

$$F_{2p,k}^m = 10^{-z_0}[1-(-1)^{m+1}][1-(-1)^k] \tag{6.11}$$

where $z_0 > 0$.
We apply the numerical scheme (6.7). In order to discuss the convergence properties of the numerical solution U_j^m or $\bar{U}_j^m$ with respect to the exact solution V_j^m we set $\tau = K_0 h^2$ where K_0 is a positive constant. Then we calculate for $h \to 0$ error norms

$$\|E_{j,up}^m\| := \|U_j^m - V_j^m\| \qquad \text{and} \qquad \|E_{j,p}^m\| := \|\bar{U}_j^m - V_j^m\|, \quad j = 1,2.$$

U_j^m and $\bar{U}_j^m$ are the numerical solutions corresponding to F_2^m and $F_2^m + F_{2p}^m$, respectively. As usual, we represent the convergence properties by order relations of the error, i.e. we write $E_j = O(h^{p_j})$ and $E_j = O(\tau^{q_j})$ for $j = 1,2$, and determine the orders $p_j,\ q_j$ numerically (here, E_j means $E_{j,up}$ or $E_{j,p}$, and we write, e.g., $E_j = O(h^{p_j})$ when $\|E_j\| = O(h^{p_j})$).

For a first example we take the PDAE in (4.43) with

$$f_1 = f_3 = 0, \qquad f_2(t,x) = t(x-x^2)(1-2x) + \frac{x}{60} - \frac{x^3}{6} + \frac{x^4}{4} - \frac{x^5}{10}. \tag{6.12}$$

The only (analytic) solution of the IBVP with zero boundary values (and IVs as given by this solution) is

$$v_1 = t(x-x^2)(1-2x), \quad v_2 = (x-x^2)(1-2x), \quad v_3 = -\frac{x}{60} + \frac{x^3}{6} - \frac{x^4}{4} + \frac{x^5}{10}.$$

(Notice that the two assumptions $(A1)$, $(A2)$ are fulfilled.) Since f_2 is a sum of two product terms considered in Example 5.1 we easily can apply Theorem 5.1 to obtain estimates of $\|u_j\|_{s2},\ j \in J_3$.
We present results for parameters $K_0 = 0.1,\ t_0 = 0.1,\ z_0 = 9$. The $h-$order of the errors $E_{j,up},\ j = 1,2$, in the undisturbed case is (to a good approximation) 2 while the $h-$orders in the disturbed case are given for the first two components of the solution vector in Table 6.1.

TABLE 6.1. Numerical solution of the model (4.43) with $\Omega = (0,1)$ and (6.12) by means of the scheme (6.7): $h-$order of errors $E_{1,p},\ E_{2,p}$ for $\tau = \frac{1}{10}h^2$ and the perturbation F_{2p}, (6.11), with $z_0 = 9$

N	15	20	25	30	40	60	80	120	180
p_1	-	1.9967	1.9982	1.9988	1.9993	1.9996	1.9998	1.9999	1.9999
p_2	-	2.0930	1.9269	1.4097	-0.1237	-1.7262	-1.9899	-2.0034	-2.0020

The numerical result $E_{j,up} \approx O(h^2),\ j = 1,2$, is expected from Lemma 6.1 (because $\tau = K_0 h^2$). On the other hand, the relation $E_{1,p} \approx O(h^2)$ is in agreement with the theoretical result given by (1) in Theorem 5.1 (where $\beta = 0,\ \gamma = 2,\ \delta = 0,\ \nu = \beta = 0,\ \alpha_\nu = 2$) which means that $\|u_1(t,\cdot)\|_{s2}$ can be estimated without any derivatives of f_2. (In other words: there is for E_1 no essential difference of the undisturbed and the disturbed case). The interesting numerical result $E_{2,p} \approx O(h^{-2})$ for small h is also in agreement with the theoretical investigation. Again we can apply Theorem 5.1 (where now $\beta = 1,\ \gamma = 2,\ \delta = 0,\ \kappa = 1,\ \nu = \kappa - 1 = 0,\ \alpha_\nu = 2$), i.e. $\|u_2(t,\cdot)\|_{s2}$ can be estimated by the norm of $f_{2,t}$ (because $\beta-\nu=1$ in (1) of Theorem 5.1). Therefore, we expect for a perturbation $E_{2,p}\approx O(\tau^{-1})$. Because $\tau=K_0h^2$ we obtain $E_{2,p}\approx O(h^{-2})$.

In a second example for the PDAE in (4.43) let

$$f_1=f_3=0,\quad f_2(t,x)=t^3(x-x^2)(1-2x)+3t^2\Big(\frac{x}{60}-\frac{x^3}{6}+\frac{x^4}{4}-\frac{x^5}{10}\Big). \tag{6.13}$$

The exact solution with zero initial values of v and under assumption (A1) is

$$v_1=t^3(x-x^2)(1-2x),\quad v_2=3t^2(x-x^2)(1-2x),\quad v_3=3t^2\Big(-\frac{x}{60}+\frac{x^3}{6}-\frac{x^4}{4}+\frac{x^5}{10}\Big).$$

Here, we take parameters $K_0=0.01,\ t_0=0.4$ and the perturbation again in the form (6.11)

TABLE 6.2. Numerical solution of the model (4.43) with $\Omega=(0,1)$ and (6.13) by means of the scheme (6.7): $\tau-$order of the errors $E_{1,p},\ E_{2,p}$ for $\tau=\frac{1}{100}h^2$ and the perturbation (6.11) with $z_0=7$

m	10^3	$4\cdot10^3$	$9\cdot10^3$	$1.6\cdot10^4$	$6.4\cdot10^4$	$2.56\cdot10^5$	$5.76\cdot10^5$
q_1	–	0.9774	0.9961	0.9973	0.9964	0.9555	0.7495
q_2	–	0.9552	0.3980	–0.7371	–0.9916	-1.0044	-1.0025

with $z_0=7$. We solve this IBVP numerically by means of the scheme (6.7) with $\tau=K_0h^2$, but now we represent the $\tau-$order of the errors $E_{1,p},\ E_{2,p}$. The $\tau-$order in the undisturbed case is to a good approximation equal to 1. The $\tau-$order of $E_{j,p},\ j=1,2$, is given in Table 6.2. The numerical order relation $E_{j,up}\approx O(\tau),\ j=1,2$, is again in agreement with Lemma 6.1 (because $h^2=\tau/K_0$). The two relations (see Table 6.2) $E_{1,p}\approx O(\tau)$ and $E_{2,p}\approx O(\tau^{-1})$ for small τ are in agreement with the theoretical investigation mentioned already before.

6.2. **Numerical solution of a problem with differential indexes 3, 0.** The PDAE investigated on $\Omega=(0,1)$ in this section is (1.1) with matrices (4.48). The differential indexes are $\nu_{dt}=3,\ \nu_{dx}=0$. When considering perturbations of f the most interesting component is f_3. This shows the matrix $\hat{G}_\xi^{-1}$ in (4.49). Therefore, we suppose f in the form $f_1=f_2=0$ and $f_3\neq0$. Again we pose the assumptions (A1), (A2) defined in section 6.1. Writing the PDAE in component form

$$u_{2,t}-u_{1,xx}+u_2+u_3=0,\qquad u_{3,t}-u_{2,xx}=0,\qquad -u_{3,xx}=f_3$$

and using the approximations (6.1), (6.3) we define a numerical method by

$$U_3^m=-\,R^{-1}F_3^m,\qquad U_2^m=\frac{1}{2\tau}R^{-1}\left(U_3^{m+1}-U_3^{m-1}\right), \tag{6.14}$$

$$U_1^m=R^{-1}\Big(U_2^m+U_3^m+\frac{1}{2\tau}(U_2^{m+1}-U_2^{m-1})\Big) \tag{6.15}$$

where for short $R:=\frac{1}{h^2}P_{N-1}$. As in section 6.1 it can be shown that this scheme is convergent.

Lemma 6.2. *Let*

(I) $v_1,\ v_3,\ f_3 \in C_{sc}^{5,\infty}$,

(II) $v_2 \in C_{sc}^{6,\infty}$,

(III) $h = 1/N$.

Then the scheme (6.14), (6.15) converges componentwise for $m \to \infty,\ N \to \infty$ with orders

$$\|E_j^m\| = O(\tau^2) + O(h^2) \quad for \quad j = 1, 2, \qquad \|E_3^m\| = O(h^2).$$

Proof. Because $v_3 \in C_{sc}^{5,\infty}$ we have by Taylor expansion

$$v_{3,xx}|_k^m = \frac{1}{h^2}(v_{3,k+1}^m - 2v_{3,k}^m + v_{3,k-1}^m) - 2\sum_{l=1}^{\infty} \frac{h^{2l}}{(2l+2)!} \frac{\partial^{2l+2} v_{3,k}^m}{\partial x^{2l+2}}.$$

Inserting this in eq. $v_{3,xx}|_k^m = -f_{3,k}^m$ we can write in matrix notation

$$RV_3^m - 2\sum_{l=1}^{\infty} \frac{h^{2l}}{(2l+2)!} \frac{\partial^{2l+2}}{\partial x^{2l+2}} V_3^m = -F_3^m.$$

Subtracting this equation from the first equation in (6.14) we get ($E_3^m := U_3^m - V_3^m$))

$$RE_3^m = 2\sum_{l=1}^{\infty} \frac{h^{2l}}{(2l+2)!} \frac{\partial^{2l+2}}{\partial x^{2l+2}} V_3^m \tag{6.16}$$

and, of course, $||E_3^m|| = O(h^2)$ (by the first inequality in (6.4)). For an estimate of $||E_2^m||$ we use eq. (6.16) to obtain

$$E_3^{m+1} - E_3^{m-1} = 2R^{-1} \sum_{l=1}^{\infty} \frac{h^{2l}}{(2l+2)!} \frac{\partial^{2l+2}}{\partial x^{2l+2}} (V_3^{m+1} - V_3^{m-1}).$$

The difference $V_3^{m+1} - V_3^{m-1}$ can be expanded (since by assumption $v_3 \in C_{sc}^{5,\infty}$) at the time level $t_m = m\tau$

$$V_3^{m+1} - V_3^{m-1} = 2\left(\tau \frac{\partial}{\partial t} V_3^m + \frac{\tau^3}{3!} \frac{\partial^3}{\partial t^3} V_3^m + O(\tau^5)\right). \tag{6.17}$$

Therefore,

$$E_3^{m+1} - E_3^{m-1} = 4R^{-1} \sum_{l=1}^{\infty} \frac{h^{2l}}{(2l+2)!} \frac{\partial^{2l+2}}{\partial x^{2l+2}} \left(\tau \frac{\partial}{\partial t} V_3^m + \frac{\tau^3}{3!} \frac{\partial^3}{\partial t^3} V_3^m + O(\tau^5)\right). \tag{6.18}$$

Now we insert the exact solution components $v_2,\ v_3$ into the second equation of (6.14) (which we write here $\frac{1}{2\tau}(U_3^{m+1} - U_3^{m-1}) - RU_2^m = 0$):

$$\frac{1}{2\tau}(V_3^{m+1} - V_3^{m-1}) - RV_2^m = \frac{\tau^2}{3!} \frac{\partial^3}{\partial t^3} V_3^m + O(\tau^4) - 2\sum_{l=1}^{\infty} \frac{h^{2l}}{(2l+2)!} \frac{\partial^{2l+2}}{\partial x^{2l+2}} V_2^m.$$

Subtracting this equation from $\frac{1}{2\tau}(U_3^{m+1} - U_3^{m-1}) - RU_2^m = 0$ we get

$$E_2^m = R^{-1}\left(\frac{1}{2\tau}(E_3^{m+1} - E_3^{m-1}) + \frac{\tau^2}{3!} \frac{\partial^3}{\partial t^3} V_3^m + O(\tau^4) - 2\sum_{l=1}^{\infty} \frac{h^{2l}}{(2l+2)!} \frac{\partial^{2l+2}}{\partial x^{2l+2}} V_2^m\right). \tag{6.19}$$

By means of (6.18) we have the following result: The first term on the right hand side is of order $O(h^2)$, the second term is of order $O(\tau^2)$, the third term is of order $O(\tau^4)$, and the last term is of order $O(h^2)$, i.e. $||E_2^m|| = O(\tau^2) + O(h^2)$ which is the statement of the Lemma for $j = 2$.
For an estimate of $||E_1^m||$ we first note that equation (6.15) implies

$$E_1^m = R^{-1}\Big(E_2^m + E_3^m + \frac{1}{2\tau}(E_2^{m+1} - E_2^{m-1}) + O^m(\tau^2) + O^m(h^2)\Big), \qquad \tau,\ h \to 0. \tag{6.20}$$

Since the order of $||E_2^m||$, $||E_3^m||$ is known already we must estimate $||E_2^{m+1} - E_2^{m-1}||$ only. From eq. (6.19) we see that $E_2^{m+1} - E_2^{m-1}$ can be written

$$E_2^{m+1} - E_2^{m-1} = R^{-1}\Big(\frac{1}{2\tau}(E_3^{m+2} - 2E_3^m + E_3^{m-2}) + \frac{\tau^2}{3!}\frac{\partial^3}{\partial t^3}(V_3^{m+1} - V_3^{m-1}) + O(\tau^4)\Big).$$

Since $E_3^{m+2} - 2E_3^m + E_3^{m+2} = (2\tau)^2 \frac{\partial^2}{\partial t^2} E_3^m + O(\tau^4)$ and $V_3^{m+1} - V_3^{m-1} = 2\left(\tau \frac{\partial}{\partial t} V_3^m + O(\tau^3)\right)$ (see (6.17)), we have the order relation (because according to (6.16) $\frac{\partial^2}{\partial t^2} E_3^m = O(h^2)$) $||E_2^{m+1} - E_2^{m-1}|| = O(\tau h^2) + O(\tau^3)$, and eq. (6.20) gives the estimate of the Lemma for $j = 1$. This completes the proof. ⊠

Since the scheme (6.14), (6.15) is convergent under the conditions of Lemma 6.2 we can study a disturbed problem. Let the right hand side vector f be given by

$$f_1 = f_2 = 0, \qquad f_3 = f_3(t, x) = t^3(-x + 3x^2 - 2x^3), \tag{6.21}$$

and choose the solution

$$v_1 = K_1(t)x + \frac{2t + t^2}{80}\Big(\frac{x^3}{63} - \frac{x^5}{30} + \frac{x^7}{21} - \frac{x^8}{28} + \frac{x^9}{126}\Big) + \frac{t^3}{60}\Big(- \frac{x^3}{6} + \frac{x^5}{2} - \frac{x^6}{2} + \frac{x^7}{7}\Big) \tag{6.22}$$

$$\text{with} \quad K_1(t) = \frac{2t + t^2}{80}\Big(- \frac{1}{63} + \frac{1}{30} - \frac{1}{21} + \frac{1}{28} - \frac{1}{126}\Big) + \frac{t^3}{2520},$$

$$v_2 = \frac{t^2}{20}\Big(\frac{x}{42} - \frac{x^3}{6} + \frac{x^5}{2} - \frac{x^6}{2} + \frac{x^7}{7}\Big), \qquad v_3 = \frac{t^3}{2}\Big(- \frac{x}{30} + \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\Big). \tag{6.23}$$

The IVs and BVs (see eqs. (5.2), (5.3)) necessary for a numerical solution are taken from these functions. As a small perturbation of F_3^m we use here

$$F_{3p}^m = 10^{-z_0}\big[1 - (-1)^{m+1}\big] \tag{6.24}$$

which is independent of x_k, $k = 1, ..., N - 1$. (A dependence of the perturbation on x is not of interest because the space index of the PDAE is zero in this example. By the same reason a perturbation g_p need not be considered, hence we take $g_p = 0$.) We solve this IBVP numerically under the condition $\tau = K_0 h$ with parameters $K_0 = 0.25$, $t_0 = 0.5$, $z_0 = 8$ by the scheme (6.14), (6.15). First, at a time $t_0 = m\tau$ we calculate the $\tau-$order of the errors $E_{j,up}$, $j = 1, 2, 3$. In this undisturbed case we find for $K_0 > 0$ numerically in a good approximation $E_{j,up} \approx O(\tau^2)$, $j = 1, 2, 3$, which is in agreement with Lemma 6.2. If the perturbation (6.24) is added to F_3^m, we get for the $\tau-$order of $E_{1,p}$ and $E_{2,p}$ the results given in Table 6.3. (The $\tau-$order of the error E_3 is not of interest because there is no time error in the first equation of (6.14).)

TABLE 6.3. Numerical solution of the model (1.1), (4.48) with $\Omega = (0, 1)$ and (6.21) by means of the scheme (6.14), (6.15): $\tau-$order of the errors $E_{1,p}$, $E_{2,p}$ for $\tau = h/4$ and the perturbation (6.24) with $z_0 = 9$

m	20	30	40	60	80	100	160	240	360	800	1600
q_1	-	2.042	1.066	-1.150	-1.859	-1.966	-1.995	-1.996	-1.999	-1.999	-1.999
q_2	-	2.018	2.009	2.004	2.002	1.909	1.268	-0.123	-0.990	-0.999	-0.999

Table 6.3 offers two essential results: $E_{1,p} \approx O(\tau^{-2})$ and $E_{2,p} \approx O(\tau^{-1})$ for small τ. From the theoretical point of view, this can be explained, e.g., by means of the matrix $\hat{G}_\xi^{-1}$ defined in (4.49) in conjunction with eq. (4.18) which is valid also for IBVPs (as explained in section 5, especially Lemma 5.1). According to (4.18), among others the components u_1 and u_2 are expressed by a second time derivative of f_3 and by a first time derivative of f_3, respectively. This is the reason that we find numerically for the IBVP $E_{1,p} \approx O(\tau^{-2})$ and $E_{2,p} \approx O(\tau^{-1})$.

7. Conclusion

It was shown in this paper that there is also for PDAEs a phenomenon which is well known from the theory of DAEs [8]: perturbations in the data (in the right hand side function f and in the initial data g) may influence the numerical solution considerably. Often this is described by means of a perturbation index. A definition of this index for PDAEs (1.1) has been given by Campbell and Marszalek (see section 3).
A difficulty to solve IVPs of DAEs and IBVPs of PDAEs numerically is caused by the fact that certain derivatives of the data appear in the solution representation. Knowing for PDAEs the order of these derivatives one can characterize the convergence properties of any finite dimensional discretization method for solving IBVPs. Under some assumtions, it was shown in this paper that for PDAEs of the type (1.1) it is possible, for example, to determine the order of derivatives of the components of f (which are needed to construct a certain solution representation of the IBVP) from the simple $(n, n)-$matrix $\hat{G}_\xi^{-1}(\omega)$. Among others, by several examples we have shown that indeed the convergence properties of numerical schemes can be characterized by means of results extracted from $\hat{G}_\xi^{-1}$.
Of course, one should like to know whether these convergence properties (or a perturbation index) can be determined by means of the differential indexes ν_{dt} and ν_{dx} alone. Up to now we did not obtain a general answer for this question. For certain PDAEs the influence of perturbations on the numerical solution can be characterized by the indexes alone (as in Examples 4.3 and 4.5), but there are other PDAEs for which this is not known (Example 4.4 is of this type).

Appendix A.

Let $\omega \in \mathbb{R}$ and $\hat{I} = \hat{I}(\omega)$, $\hat{a} = \hat{a}(\omega)$, $\hat{b}(\tau) = \hat{b}(\tau, \omega)$, $b = |\hat{b}(\tau)|$, $b' = |\hat{b}(\tau')|$, and suppose the functions are such that all the following integral operations make sense. Then we prove the following

Lemma: If

$$\hat{I} = \hat{a} + \int_0^t \hat{b}(\tau)\, d\tau, \tag{1.1}$$

then

$$\|\hat{I}\|_{s2} \;\leq\; \|\hat{a}\|_{s2} + \; t^{1/2} \left[\int_{-\infty}^{\infty} \left(\int_0^t |\hat{b}(\tau)|^2 \, d\tau \right) \, d\omega \right]^{1/2}. \tag{1.2}$$

If the two integrations can be interchanged, then

$$\|\hat{I}\|_{s2} \;\leq\; \|\hat{a}\|_{s2} + t^{1/2} \left[\int_0^t \|\hat{b}(\tau)\|_{s2}^2 \, d\tau \right]^{1/2}. \quad \Box \tag{1.3}$$

Proof: It is $\|\hat{I}\|_{s2} \leq \|\hat{a}\|_{s2} + \| \int_0^t \hat{b}(\tau)\, d\tau \|_{s2}$. To estimate the last term on the right we first note that

$$\| \int_0^t \hat{b}(\tau)\, d\tau \|_{s2}^2 = \int_{-\infty}^{\infty} \left| \int_0^t \hat{b}(\tau)\, d\tau \right|^2 \, d\omega \leq \int_{-\infty}^{\infty} \left(\int_0^t |\hat{b}(\tau)| \, d\tau \right)^2 \, d\omega. \tag{1.4}$$

This upper bound can be written

$$\int_{-\infty}^{\infty} \left(\int_0^t b \, d\tau \right) \left(\int_0^t b' \, d\tau' \right) \, d\omega = \int_{-\infty}^{\infty} \left(\int_0^t \left[\int_0^t bb' \, d\tau' \right] \, d\tau \right) \, d\omega.$$

Because $bb' \leq \frac{1}{2}(b^2 + b'^2)$ the last upper bound can be estimated further by

$$\frac{1}{2} \int_{-\infty}^{\infty} \left(\int_0^t \left[\int_0^t (b^2 + b'^2) \, d\tau' \right] \, d\tau \right) \, d\omega = \frac{1}{2} \int_{-\infty}^{\infty} \left(t \int_0^t b^2 \, d\tau + t \int_0^t b'^2 \, d\tau' \right) \, d\omega.$$

Therefore, $\int_{-\infty}^{\infty} \left(\int_0^t |\hat{b}(\tau)| \, d\tau \right)^2 \, d\omega \;\leq\; t \int_{-\infty}^{\infty} \left(\int_0^t b^2 \, d\tau \right) \, d\omega$. This implies the first estimate of the Lemma. The second follows from the assumption and the definition of the L_2-norm in $\omega-$space. $\Box$

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