Asymptotic Behaviour of Linear Evolutionary Integral Equations

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ABSTRACT: The asymptotic behaviour of bounded solutions of evolutionary integral equations in a Banach space X

$$\dot{u}(t) = \int_0^\infty A_0(\tau)\dot{u}(t-\tau)\, d\tau + \int_0^\infty dA_1(\tau)u(t-\tau) \; + f(t), \qquad t \in \mathbb{R},$$

on the real line and of

$$\dot{v}(t) = \left(\int_0^t A(t-\tau)v(\tau)\, d\tau\right)^{\cdot} + g(t), \qquad t \in \mathbb{R}_+,$$

on the half-line are studied. Assuming that the inhomogeneity f (resp. g) belongs to a given homogeneous subspace $\mathcal{E}$ of $BUC(\mathbb{R};X)$ (resp. $BUC(\mathbb{R}_+;X)$) it is shown that given bounded solutions u (resp. v) belong also to $\mathcal{E}$ provided the spectra of these equations are countable. The results are applied to an equation of scalar type which is of importance in applications like viscoelasticity.

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0 Introduction

Let X be a Banach space, A the generator of a C_0-semigroup, and consider the evolution equation

$$\dot{u}(t) = Au(t) + f(t), \qquad t \in \mathbb{R}, \tag{0.1}$$

where $f \in L^\infty(\mathbb{R};X)$, as well as the abstract Cauchy problem

$$\dot{v}(t) = Av(t) + g(t), \qquad v(0) = v_0, \qquad t \in \mathbb{R}_+, \tag{0.2}$$

where $g \in L^\infty(\mathbb{R}_+;X)$, $v_0 \in X$, and the dot indicates the derivative with respect to time. As is well known, the unique mild solution of (0.2) is then given by

$$v(t) = S(t)v_0 + \int_0^t S(\tau)g(t-\tau)\, d\tau, \qquad t \in \mathbb{R}_+, \tag{0.3}$$

where $(S(t))_{t\geq 0} = (e^{tA})_{t\geq 0}$ denotes the semigroup generated by A, and in case the type $\omega_0(A)$ of e^{tA} is negative, the unique mild solution of (0.1) is

$$u(t) = \int_0^\infty S(\tau)f(t-\tau)\, d\tau, \qquad t \in \mathbb{R}. \tag{0.4}$$

Such variation of parameter formulae arise also in the context of linear evolutionary integral equations which include (0.1) and (0.2) as special cases.

$$(0.5) \qquad \dot{u}(t) = \int_0^\infty A_0(\tau)\dot{u}(t-\tau)\, d\tau + \int_0^\infty dA_1(\tau)u(t-\tau) \; + f(t), \quad t \in \mathbb{R},$$

on the real line and

$$(0.6) \qquad \dot{v}(t) = \Big(\int_0^t A(t-\tau)v(\tau)\, d\tau\Big)^{\cdot} + g(t), \quad v(0) = v_0, \qquad t \in \mathbb{R}_+,$$

on the half-line. In this case $(S(t))_{t\geq 0}$ denotes the so-called resolvent family, and (0.3) and (0.4) make sense if the resolvent family exists and is integrable. As a general reference for evolutionary integral equations we refer to the second author's monograph [18].

The asymptotic behaviour of the solutions of (0.5) and (0.6) in the *stable case*, i.e. in case the resolvent family exists and is integrable, is well-known; see Prüss [18] and Prüss and Ruess [19]. In fact, it was shown there that convolutions with operator-valued BV-functions are leaving invariant homogeneous spaces on the line and that solutions u and v are asymptotic to each other provided f and g have this property.

The study of the asymptotic behaviour of solutions of (0.5) and (0.6) in case $(S(t))_{t\geq 0}$ is not integrable or does not even exist is much more difficult and only few general results are so far known. I was begun in the paper of the second author [17]. Next we want to mention the paper Arendt and Prüss [4] who used vector-valued Tauberian theorems to study the asymptotic behaviour of the resolvent $(S(t))_{t\geq 0}$ in the so-called scalar case $A(t) = a(t)A$, where A denotes an unbounded closed linear operator in X and $a \in L^1_{loc}(\mathbb{R}_+)$ a scalar kernel. We also want to mention the results in [18, Sections 11, 12]. There, the so-called *non-resonant* case for (0.5) is studied in detail. Non-resonant means that the spectrum Λ_0 of (0.5) does not intersect the Carleman spectrum of the right hand side f; see below for the definition of these concepts. If in addition $\Lambda_0 = \emptyset$ and (0.6) admits a resolvent family it is shown that solutions u and v are asymptotic to each other, provided f and g are so, and certain regularity conditions are satisfied.

More recently, considerable progress concerning the asymptotic behaviour of bounded solutions of evolution equations (0.1) and (0.2) has been made; cf. Arendt and Batty [2], [3], Batty, van Neerven and Räbiger [9], [10], Chill [12], Ruess and Vu [21]. To mention one of the most prominent results in this new line of research, suppose that f is almost periodic, $\sigma(A) \cap i\mathbb{R}$ is countable, and u

is a bounded, uniformly continuous and uniformly ergodic (see Section 1 for the definition) solution of (0.1). Then u is almost periodic.

It is the purpose of this article to extend results of this type to evolutionary integral equations (0.5) and (0.6). It turns out that such extensions are valid in a fairly general setting, however, require proofs different from those for evolution equations, and which are due to enlarged generality more sophisticated. They heavily employ recent results from harmonic analysis of bounded vector-valued functions.

The plan of this paper is as follows. In the preliminary Section 1 we present some necessary definitions, like that of the Fourier transform, of almost periodic and of uniformly ergodic functions. In Section 2 we give a short proof of an abstract Tauberian theorem. This theorem relates the asymptotic behaviour of bounded, measurable functions to their spectrum. In Sections 3 and 4 we apply the Tauberian theorem to bounded solutions of the evolutionary integral equations (0.5) and (0.6), respectively. We show, in fact, that it is possible to calculate the spectrum of a solution from spectral properties of the underlying equation. Section 5 contains a special case of a kernel A which is of particular interest in linear viscoelasticity, for which the results from Sections 3 and 4 apply, and which leads to situations which are not accessible by existing methods.

1 Preliminaries

Let us shortly present the notations which we will use throughout the paper. By X, Y and Z we denote Banach spaces. We will always assume that Y is densely embedded into X. The letter J will stand either for the line $\mathbb{R}$ or for the half line $\mathbb{R}_+$.

The Fourier transform on $L^1(\mathbb{R};X)$ and its inverse will be denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$, respectively. They are given by the following formulae

$$\begin{aligned}\mathcal{F}h(s) &= \int_{\mathbb{R}} e^{-its}h(t)\,dt,\\ \mathcal{F}^{-1}h(s) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{its}h(t)\,dt \quad (h\in L^1(\mathbb{R};X), s\in\mathbb{R}).\end{aligned}$$

We will always regard $L^1_{loc}(\mathbb{R}_+;X)$ as a subspace of $L^1_{loc}(\mathbb{R};X)$ by extending the functions by 0 on $\mathbb{R}_-$. Therefore, the Fourier transforms are equally defined on $L^1(\mathbb{R}_+;X)$.
A function $h\in L^1_{loc}(\mathbb{R}_+;X)$ is called Laplace transformable, if for some $\omega\in\mathbb{R}$

one has $\int_0^\infty e^{-t\omega}\|h(t)\|\ dt < \infty$. In that case $\hat{h}(z) := \int_0^\infty e^{-tz}h(t)\ dt$ exists whenever $\mathrm{Re} z > \omega$ and it defines a holomorphic function in that half plane, called the Laplace transform of h.

For further notations concerning Stieltjes integrals and Laplace-Stieltjes or Fourier-Stieltjes transforms, we refer to [18].

By $AP(\mathbb{R};X)$ and $AP(\mathbb{R}_+;X)$ we denote the spaces of vector-valued *almost periodic (a.p.) functions* on the line and on the half line, respectively. When studying functions on the half line, it is also useful to consider the space $AAP(\mathbb{R}_+;X) = AP(\mathbb{R}_+;X) \oplus C_0(\mathbb{R}_+;X)$ of *asymptotically almost periodic (a.a.p.) functions.* For the theory of a.p. and a.a.p. functions we refer to [15].

For functions $h \in AP(J;X)$ the Mean Ergodic Theorem [15] says, that for all $\eta \in \mathbb{R}$ the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\eta t}h(t+s)\ dt \quad \text{exists uniformly in } s \in J. \tag{1.1}$$

Conversely, we say that a function $h \in BUC(J;X)$ is *uniformly ergodic in* $\eta \in \mathbb{R}$, whenever relation (1.1) holds for this η. If a function h is uniformly ergodic in all $\eta \in \mathbb{R}$, then h is also called *totally uniformly ergodic* ([2], [3], [12]). It is easy to see that every $h \in AAP(\mathbb{R}_+;X)$ is totally uniformly ergodic. For any function h which is uniformly ergodic in η we call the limit in (1.1) with $s = 0$ the *mean of* h *in* η.

For $G \in L^1(\mathbb{R};B(X,Z))$ and $h \in L^\infty(\mathbb{R};X)$ we define the convolution $G * h$ by

$$G * h(t) := \int_{\mathbb{R}} G(t-\tau)h(\tau)\ d\tau \quad (t \in \mathbb{R}).$$

Putting $G = \phi \cdot I$ for $\phi \in L^1(\mathbb{R})$, the convolution is therefore also defined for scalar functions ϕ and vector valued functions h.

2 An abstract Tauberian theorem

In this section we present a general Tauberian theorem for bounded, vector-valued functions defined either on the line or on the half line. It is remarkable, that the proof will make no difference between these two cases. In this Tauberian theorem we use the notion of a *spectrum* (with respect to a homogeneous subspace $\mathcal{E}$ of $BUC(J;X)$) of bounded, measurable functions, which is very useful for evolutionary integral equations, as we will show in the following sections. The Tauberian theorem itself can also be found in [6, Theorem 4.2.6], but we present here a more compact proof which relies on an idea in [12].

Definition 2.1 *Let $\mathcal{E}$ be a closed subspace of $BUC(J;X)$.*
(i) The space $\mathcal{E}$ is called translation-biinvariant, *if for all $s \in J$ one has $\mathcal{E} = \{h \in BUC(J;X) : h_s \in \mathcal{E}\}$, where h_s is the translate of h: $h_s(t) = h(s+t)$ $(s,t \in J)$.*
(ii) We say that $\mathcal{E}$ is operator-invariant, *if $G \circ h \in \mathcal{E}$ for all operators $G \in B(X)$ and all $f \in \mathcal{E}$.*
(iii) If $\mathcal{E}$ is translation-biinvariant and operator-invariant, then we call $\mathcal{E}$ homogeneous *(see also [18, Def. 11.1]).*

Examples of homogeneous spaces are $AP(\mathbb{R};X)$, the space $W(J;X)$ of *weakly almost periodic functions in the sense of Eberlein*, the space $WAP(J;X)$ of all *weakly (asymptotically) almost periodic functions in the sense of Amerio and Prouse*, $C_0(\mathbb{R}_+;X)$ or $AAP(\mathbb{R}_+;X)$. Let $\Lambda \subset \mathbb{R}$ be closed. Then $E_\Lambda(J;X) := \{f \in BUC(J;X) : f$ is uniformly ergodic in each $\eta \in \Lambda\}$ is also a homogeneous space. We will see more examples later.

Definition 2.2 ([6], [7]) *Let $\mathcal{E}$ be a translation-biinvariant subspace of $BUC(J;X)$ and let $h \in L^\infty(J;X)$. We define the spectrum $\mathrm{Sp}_{\mathcal{E}}(h)$ of h with respect to $\mathcal{E}$ by*

$$\begin{aligned} \eta \notin \mathrm{Sp}_{\mathcal{E}}(h) \quad :\Leftrightarrow \quad & \exists \varepsilon > 0 \text{ such that for all } \phi \in L^1(\mathbb{R}) \text{ with} \\ & \mathrm{supp}\mathcal{F}\phi \subset (\eta - \varepsilon, \eta + \varepsilon) \text{ one has } (h * \phi)|_J \in \mathcal{E}. \end{aligned}$$

For the special case of $AP(\mathbb{R};X)$ this definition coincides with that in [14, p.166] (at least in the scalar case $X = \mathbb{C}$). There, the points $\eta \notin \mathrm{Sp}_{AP}(h)$ are called the points of almost periodicity. Notice, that for $\mathcal{E} = \{0\}$ and $J = \mathbb{R}$ one obtains the classical Beurling spectrum. The notion of the Beurling spectrum makes no real sense for functions having their support in the half line: one has $\mathrm{Sp}_{\{0\}}(h) = \mathbb{R}$, whenever h is different from 0. For functions defined on the half-line the basic space is $C_0(\mathbb{R}_+;X)$. In [12, Proposition 1.3] one can find a useful condition for points η not to belong to $\mathrm{Sp}_{C_0}(h)$.

Proposition 2.3 *Let $h \in L^\infty(\mathbb{R}_+;X)$ and assume that for some $\eta \in \mathbb{R}$ there exist $\varepsilon > 0$ and $g \in L^1((\eta-\varepsilon, \eta+\varepsilon);X)$ such that*

$$\lim_{\xi \to 0} \hat{h}(\xi + i\cdot)|_{(\eta-\varepsilon,\eta+\varepsilon)} = g \quad \text{in } \mathcal{D}'((\eta - \varepsilon, \eta + \varepsilon);X).$$

Then $\eta \notin \mathrm{Sp}_{C_0}(h)$.

Using the notion of the spectrum $\mathrm{Sp}_{\mathcal{E}}(h)$ it is possible to define more homogeneous subspaces of $BUC(\mathbb{R};X)$. If we put for a closed set $\Lambda \subseteq \mathbb{R}$

$$\mathcal{H}_\Lambda := \left\{ f \in BUC(\mathbb{R};X) : \mathrm{Sp}_{\{0\}}(f) \subseteq \Lambda \right\}, \tag{2.1}$$

then $\mathcal{H}_\Lambda$ is homogeneous.

Lemma 2.4 ([6], Theorem 4.1.4) *Let $\mathcal{E}$ be a translation-biinvariant subspace of $BUC(J;X)$ and let h be bounded and uniformly continuous. Then $\mathrm{Sp}_{\mathcal{E}}(h)$ is empty if and only if $h \in \mathcal{E}$.*

Let us turn to a weaker condition for h to belong to $\mathcal{E}$, namely countability of the spectrum $\mathrm{Sp}_{\mathcal{E}}(h)$. This result is due to Basit ([6, Theorem 4.2.6]). We present here a simpler proof.

Theorem 2.5 *Let $\mathcal{E}$ be a translation-biinvariant subspace of $BUC(J;X)$ that contains $AP(J;X)$. Let $h \in L^\infty(J;X)$ be such that $\mathrm{Sp}_{\mathcal{E}}(h)$ is countable. If h is uniformly ergodic in each $\eta \in \mathrm{Sp}_{\mathcal{E}}(h)$, then $\mathrm{Sp}_{\mathcal{E}}(h)$ is empty. If, in addition, h is uniformly continuous, then $h \in \mathcal{E}$.*

Proof: Consider the adjoint $\kappa' : Z' \to L^\infty(\mathbb{R})$ of the following bounded operator

$$\begin{array}{rcl} \kappa : L^1(\mathbb{R}) & \to & BUC(J;X)/\mathcal{E} =: Z \\ \phi & \mapsto & (h * \phi)|_J + \mathcal{E}. \end{array}$$

Choose $\phi \in \mathcal{D}(\mathbb{R})$ such that $\mathrm{supp}\phi \cap \mathrm{Sp}_{\mathcal{E}}(h) = \emptyset$. Then we have $(h * \mathcal{F}^{-1}\phi)|_J \in \mathcal{E}$ so that for every $z' \in Z'$ one obtains

$$\langle \mathcal{F}^{-1}(\kappa' z'), \phi \rangle = \langle z', (h * \mathcal{F}^{-1}\phi)|_J + \mathcal{E} \rangle = 0. \tag{2.2}$$

So the support of the distribution $\mathcal{F}^{-1}(\kappa' z')$ is contained in $\mathrm{Sp}_{\mathcal{E}}(h)$. Hence, since $\mathrm{Sp}_{\mathcal{E}}(h)$ is supposed to be countable, $\kappa' z' * \psi$ is almost periodic for all $\psi \in L^1(\mathbb{R})$ by Loomis's Theorem (see [14, Ex. 7, p.169] or [15, Section 6.4, Theorem 4]).

Let $\eta \in \mathrm{Sp}_{\mathcal{E}}(h)$. We obtain

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{-i\eta t} (\kappa' z' * \psi)(-t)\, dt &= \lim_{T\to\infty} \langle \kappa' z' * \psi, e^{i\eta \cdot} \frac{1}{T} 1_{[-T,0]} \rangle \\ &= \lim_{T\to\infty} \langle \kappa' z', (e^{i\eta \cdot} \frac{1}{T} 1_{[-T,0]}) * \check{\psi} \rangle \\ &= \lim_{T\to\infty} \langle z', (h * (e^{i\eta \cdot} \frac{1}{T} 1_{[-T,0]}) * \check{\psi})|_J + \mathcal{E} \rangle. \end{aligned}$$

The assumption that h be uniformly ergodic in each $\eta \in \mathrm{Sp}_{\mathcal{E}}(h)$ implies that

$$(2.3) \qquad \lim_{T\to\infty} (h * (e^{i\eta\cdot}\frac{1}{T}1_{[-T,0]}))|_J = \left(\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-i\eta t}h(t)\,dt\right) e^{i\eta\cdot} \in AP(J;X).$$

Since $\mathcal{E}$ contains $AP(J;X)$, we obtain

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-i\eta t}(\kappa' z' * \psi)(-t)\,dt = 0,$$

and the Uniqueness Theorem for almost periodic functions [15] yields, that $\kappa' z' * \psi = 0$ for every $z' \in Z'$ and every $\psi \in L^1(\mathbb{R})$. Thus, $\kappa = 0$, or, in other words, $(h * \phi)|_J \in \mathcal{E}$ for all $\phi \in L^1(\mathbb{R})$. This implies $\mathrm{Sp}_{\mathcal{E}}(h) = \emptyset$.
If h is uniformly continuous, then Lemma 2.4 implies the last result. $\square$

Remark 2.6 In Theorem 2.5 we assumed that the space $\mathcal{E}$ contains $AP(J;X)$. Let us point out that, by equation (2.3), it suffices that $\mathcal{E}$ contains just

$$(2.4) \qquad \mathcal{H} := \overline{\mathrm{span}}\left\{ e^{i\eta\cdot}x_\eta : \eta \in \mathrm{Sp}_{\mathcal{E}}(h),\ x_\eta := \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-i\eta t}h(t)\,dt \right\}.$$

If all the means x_η of h are 0, then actually $\mathcal{H} = \{0\}$ and $\mathcal{E}$ can be chosen arbitrarily in Theorem 2.5.

3 Evolutionary integral equations on the line

In this section we want to study the asymptotic behaviour of bounded solutions of the following evolutionary integral equation on the line, which was already mentioned in the Introduction:

$$(3.1) \qquad \dot{u}(t) = \int_0^\infty A_0(\tau)\dot{u}(t-\tau)\,d\tau + \int_0^\infty dA_1(\tau)\,u(t-\tau) + f(t), \quad t \in \mathbb{R}.$$

We assume that $A_0 \in L^1(\mathbb{R}_+;\mathcal{B}(Y,X))$, $A_1 \in BV(\mathbb{R}_+;\mathcal{B}(Y,X))$ and $f \in BC(\mathbb{R};X)$.

Definition 3.1 *(i) A function $u \in BC(\mathbb{R};X)$ is called a* strong solution *of (3.1) if $u \in BC(\mathbb{R};Y)$, $u - A_0 * u \in BC^1(\mathbb{R};X)$ and $(u - A_0 * u)^{\cdot} = dA_1 * u + f$.*
(ii) A function $u \in BC(\mathbb{R};X)$ is called a (mild) solution *of (3.1) if there are $f_n \in BC(\mathbb{R};X)$ which converge uniformly to f and strong solutions $u_n \in BC(\mathbb{R};X)$ of (3.1) (with f replaced by f_n), such that $(u_n)_{n\in\mathbb{N}}$ converges to u in $BC(\mathbb{R};X)$.*

Lemma 3.2 *Let $u \in BC(\mathbb{R};X)$ be a (mild) solution of (3.1). Then u is uniformly continuous.*

Proof: If u is a strong solution of (3.1), then the equation $(u - A_0 * u)^{\cdot} = dA_1 * u + f$ yields, that $u - A_0 * u$ is actually Lipschitz continuous. Since the convolution $A_0 * u$ is always uniformly continuous, every strong solution is uniformly continuous. As mild solutions are uniform limits of strong solutions by definition, the claim follows. $\square$

Assume that u is a bounded solution of (3.1) and assume that f belongs to a given homogeneous subspace $\mathcal{E}$ of $BUC(\mathbb{R};X)$. When does u also belong to $\mathcal{E}$?

In order to study this question we introduce the spectrum Λ_0 of equation (3.1) (see [18, p. 289]). We put

(3.2) $\Lambda_0 := \left\{\eta \in \mathbb{R} : i\eta - i\eta\widehat{A_0}(i\eta) - \widehat{dA_1}(i\eta) \text{ is not invertible in } B(Y,X)\right\}.$

The following fundamental proposition shows the relation between Λ_0 and $\mathrm{Sp}_{\mathcal{E}}(u)$.

Proposition 3.3 *Let $u \in BUC(\mathbb{R};X)$ be a (mild) solution of (3.1) and let $\phi \in \mathcal{D}(\mathbb{R})$ be such that $\mathrm{supp}\phi \cap \Lambda_0 = \emptyset$. Let $\mathcal{E}$ be a homogeneous subspace of $BUC(\mathbb{R};X)$. If $f \in L^\infty(\mathbb{R};X)$ satisfies $\mathrm{Sp}_{\mathcal{E}}(f) = \emptyset$, then $u * \mathcal{F}^{-1}\phi \in \mathcal{E}$.*

Proof: Let $\phi \in \mathcal{D}(\mathbb{R})$ be such that $\mathrm{supp}\phi \cap \Lambda_0 = \emptyset$. When u is a solution of (3.1), then $u * \mathcal{F}^{-1}\phi$ is a solution of the same equation where only the inhomogeneity f is replaced by $f * \mathcal{F}^{-1}\phi$. It follows from [18, Theorem 11.1], that $u * \mathcal{F}^{-1}\phi = G_\phi * f$ for some function $G_\phi \in L^1(\mathbb{R};\mathcal{B}(X,Y))$. By the Cohen-Hewitt Factorisation Theorem [13, Thm. 32.22], we find $\psi \in L^1(\mathbb{R})$ and $\widetilde{G} \in L^1(\mathbb{R};\mathcal{B}(Y,X))$ such that $G_\phi = \widetilde{G} * \psi$. The assumption $\mathrm{Sp}_{\mathcal{E}}(f) = \emptyset$ and a partition of the unity argument yield $f * \psi \in \mathcal{E}$. For $s \in \mathbb{R}$ we obtain

$$\begin{aligned}(G_\phi * f)(s) &= (\widetilde{G} * (f * \psi))(s)\\ &= \int_{\mathbb{R}} \widetilde{G}(s-t)(f*\psi)(t)\, dt\\ &= \left(\int_{\mathbb{R}} \widetilde{G}_{-t} \circ (f * \psi)\, dt\right)(s),\end{aligned}$$

with $\widetilde{G}.\circ (f * \psi) \in L^1(\mathbb{R};\mathcal{E})$ by the properties of $\widetilde{G}$ and $\mathcal{E}$. Thus, $G_\phi * f \in \mathcal{E}$, so that $u * \mathcal{F}^{-1}\phi \in \mathcal{E}$. $\square$

The following main theorem of this section can be found in [3] for the semigroup case. Our proof of part (i) via Theorem 2.5 differs from that one given in [3], which does not seem to work for Volterra equations.

Theorem 3.4 *Let $u \in BUC(\mathbb{R};X)$ be a (mild) solution of (3.1). Suppose that Λ_0 is countable and that $f \in L^\infty(\mathbb{R};X)$ satisfies $\mathrm{Sp}_{AP}(f) = \emptyset$. Suppose that, in addition, one of the following assumptions holds:*
(i) The solution u is uniformly ergodic in each $\eta \in \Lambda_0$.
(ii) The space X contains no isomorphic copy of c_0.
(iii) The solution u has relatively weakly compact range in X.
(iv) The spectra Λ_0 and $\mathrm{Sp}_{\{0\}}(f)$ are discrete.
Then $u \in AP(\mathbb{R};X)$.

Proof: (i) Proposition 3.3 yields, that $\mathrm{Sp}_{AP}(u)$ is contained in Λ_0 which is countable. The claim follows from Theorem 2.5.

The proof of part (ii) of Theorem 3.4 makes use of a vector-valued version of a Theorem of Bohr in the theory of almost periodic functions, due to Kadets, which is only valid in Banach spaces not containing an isomorphic copy of c_0. Actually, the proof for parts (ii) and (iii) is contained in [15, Theorem 4, p.92].

(iv) Let $\phi \in \mathcal{D}(\mathbb{R})$ be such that $\mathrm{supp}\phi \cap (\Lambda_0 \cup \mathrm{Sp}_{\{0\}}(f)) = \emptyset$, and choose $\phi_1 \in \mathcal{D}(\mathbb{R})$ such that $\phi_1 \equiv 1$ on $\mathrm{supp}\phi$ and $\mathrm{supp}\phi_1 \cap (\Lambda_0 \cup \mathrm{Sp}_{\{0\}}(f)) = \emptyset$. Then $u * \mathcal{F}^{-1}\phi = u * \mathcal{F}^{-1}\phi * \mathcal{F}^{-1}\phi_1$. By the proof of Proposition 3.3 we have $u * \mathcal{F}^{-1}\phi = \widetilde{G} * f * \psi * \mathcal{F}^{-1}\phi_1$. By assumption, we obtain $f * \mathcal{F}^{-1}\phi_1 = 0$, so that $u * \mathcal{F}^{-1}\phi = 0$. Thus $\mathrm{Sp}_{\{0\}}(u) \subset \Lambda_0 \cup \mathrm{Sp}_{\{0\}}(f)$ is discrete. Recalling that $\mathrm{Sp}_{\{0\}}(u)$ coincides with the Beurling spectrum, the claim follows from [5, Theorem 3.5].□

Remark 3.5 Proposition 3.3 and Theorem 2.5 show that in Theorem 3.4 (i) we can replace the space $AP(\mathbb{R};X)$ by any homogeneous subspace $\mathcal{E}$ of $BUC(\mathbb{R};X)$, that contains the space $\mathcal{H}$ as it was defined in equation (2.4). This will be expressed in the following corollary.

Corollary 3.6 *Let u be a bounded solution of (3.1), $\mathcal{E}$ a homogeneous subspace of $BUC(\mathbb{R};X)$ and $f \in L^\infty(\mathbb{R};X)$ be such that $\mathrm{Sp}_{\mathcal{E}}(f) = \emptyset$. Suppose that Λ_0 is countable and that u is uniformly ergodic in each $\eta \in \Lambda_0$. Define $\mathcal{H}$ as in equation (2.4). Then $u \in \overline{\mathcal{E} + \mathcal{H}}$.*
In particular, if $\mathcal{E} = \mathcal{H}_\Lambda$ for some closed $\Lambda \subseteq \mathbb{R}$, then $u \in \mathcal{H}_{\Lambda \cup \Lambda_0}$.

4 Evolutionary integral equations on the half line

In this section we will consider the evolutionary integral equation on the half line

$$\dot{v}(t) = \left(\int_0^t A(t-\tau)v(\tau)\,d\tau\right)^{\cdot} + g(t), \quad v(0) = x, \qquad t \in \mathbb{R}_+, \tag{4.1}$$

where $A \in L^1_{loc}(\mathbb{R}_+; B(Y,X))$ and $g \in L^1_{loc}(\mathbb{R}_+; X)$.

Definition 4.1 *(i) A function $v \in C(\mathbb{R}_+; X)$ is called a* strong solution *of (4.1), if $v \in L^\infty_{loc}(\mathbb{R}_+; Y)$ and for all $t \in \mathbb{R}_+$*

$$v(t) - x = \int_0^t A(t-\tau)v(\tau)\, d\tau + \int_0^t g(s)\, ds. \tag{4.2}$$

(ii) A function $v \in C(\mathbb{R}_+; X)$ is called a (mild) solution, *if there are functions $g_n \in C(\mathbb{R}_+; X)$ and strong solutions $v_n \in C(\mathbb{R}_+; X)$ of (4.1) with g replaced by g_n such that $g_n \to g$ and $v_n \to v$ in $C(\mathbb{R}_+; X)$.*

Throughout this section we want to make the following restriction on the kernel A:

$$(\mathcal{A}) \qquad \begin{aligned} & A = \textstyle\sum_{i=0}^N A_i \text{ for some } N \in \mathbb{N} \text{ with} \\ & A_0 \in L^1(\mathbb{R}_+; B(Y,X)) \\ & A_j \in W^{j-1,1}_{loc}(\mathbb{R}_+; B(Y,X)) \quad (1 \le j \le N) \\ & A_j^{(j-1)} \in BV(\mathbb{R}_+; B(Y,X)) \quad (1 \le j \le N). \end{aligned}$$

Clearly, if $A \in L^1_{loc}(\mathbb{R}_+; B(Y,X)$ satisfies the condition $(\mathcal{A})$, then its Laplace transform $\hat{A}$ exists on the right half plane $\mathbb{C}_+$ and has a continuous extension to $i\mathbb{R}\backslash\{0\}$. Assumption $(\mathcal{A})$ is fairly reasonable from the applications point of view; cf. Section 5.

The problem whether bounded solutions of equation (4.1) are uniformly continuous follows not as easily as in the line case. In fact, there are bounded solutions of well-posed Cauchy problems (corresponding to constant kernels in our case) which are not uniformly continuous (see [11, Example 6]). At least, we can give a positive answer for strong solutions under some reasonable conditions.

Proposition 4.2 *Let $g \in L^\infty(\mathbb{R}_+; X)$, let $v \in BC(\mathbb{R}_+; X) \cap L^\infty(\mathbb{R}_+; Y)$ be a strong solution of (4.1) and assume that A satisfies $(\mathcal{A})$. Then either of the following conditions implies the uniform continuity of v:*
(i) The function $t \to \int_0^t g(s)\, ds$ is bounded.
(ii) $g \in W^{N-1,1}_{loc}(\mathbb{R}_+; X)$ and $g^{(N-1)} \in L^\infty(\mathbb{R}_+; X)$.
(iii) $N \le 1$ in assumption $(\mathcal{A})$.

Proof: Let $v \in BC(\mathbb{R}_+; X) \cap L^\infty(\mathbb{R}_+; Y)$ be a strong solution, so that it actually satisfies equation (4.2).

If $A = \sum_{j=0}^{N} A_j$ as in condition $(\mathcal{A})$ for some $N \geq 1$, then the identities

$$\widehat{A_j}(\lambda) = \sum_{k=1}^{j-1} A_j^{(k-1)}(0)\frac{1}{\lambda^k} + d\widehat{A_j^{(j-1)}}(\lambda)\frac{1}{\lambda^j} \qquad (1 \leq j \leq N)$$

yield the representation

$$\hat{A}(\lambda) = \widehat{A_0}(\lambda) + \sum_{k=1}^{N-1}\left(\sum_{j=k+1}^{N-1} A_j^{k-1}(0)\right)\frac{1}{\lambda^k} + \sum_{k=1}^{N} d\widehat{A_k^{(k-1)}}(\lambda)\frac{1}{\lambda^k},$$

which can in turn be written as $\hat{A}(\lambda) = \widehat{B_0}(\lambda) + \sum_{k=1}^{N} \widehat{dB_k}(\lambda)\frac{1}{\lambda^k}$, where $B_0 \in L^1(\mathbb{R}_+; B(Y,X))$ and $B_j \in BV(\mathbb{R}_+; B(Y,X))$. This means, that in condition $(\mathcal{A})$ we may assume w.l.o.g. $A_j^{(k)}(0) = 0$ $(0 \leq k \leq j-1)$.

Next write

$$\begin{aligned}
\widehat{dB_j}(\lambda)\frac{1}{\lambda^j} &= \left(1+\frac{1}{\lambda}\right)^j \frac{\widehat{dB_j}(\lambda)}{(1+\lambda)^j} \\
&= \sum_{k=0}^{j}\binom{j}{k}\frac{1}{\lambda^k}\frac{\widehat{dB_j}(\lambda)}{(1+\lambda)^j} \\
&= \frac{\widehat{dB_j}(\lambda)}{(1+\lambda)^j} + \frac{1}{\lambda^{j+1}}\sum_{k=1}^{j}\binom{j}{k}\widehat{dB_j}(\lambda)\frac{\lambda^{j+1-k}}{(1+\lambda)^j}
\end{aligned}$$

to see that the terms with B_j $(1 \leq j \leq N-1)$ can be eliminated successively. Therefore we may assume also $A_j = 0$ $(1 \leq j \leq N-1)$ in condition $(\mathcal{A})$.
Thus we obtain $A_N * v(t) = v(t) - A_0 * v(t) - x - \int_0^t g(s)\, ds$. Notice that the integrability of A_0 implies that $A_0 * v$ is uniformly continuous.
If (i) holds, then $A_N * v$ is bounded and by assumption it is N times differentiable with bounded N-th derivative $dA_N^{(N-1)} * v$. By interpolation, $(A_N * v)'$ is bounded, so that $A_N * v$ is globally Lipschitz continuous. Since $t \to \int_0^t g(s)\, ds$ is Lipschitz continuous, too, the claim follows.
If (ii) holds, then $t \to A_N * v(t) + \int_0^t g(s)\, ds$ is bounded and by assumption it is N times differentiable with bounded N-th derivative $dA_N^{(N-1)} * v + g^{(N-1)}$. By interpolation, $(A_N * v)' + g$ is bounded, so that $v - A_0 * v - x$ is globally Lipschitz continuous. The claim follows.
Notice that (ii) implies (iii). □

In analogy to the definition of Λ_0 in Section 3 we define a *spectrum* Σ_0 *of equation (4.1)*:

$$\Sigma_0 := \mathbb{R}\backslash\{\eta \in \mathbb{R} \ : \ \exists\varepsilon > 0 \text{ such that } K(\lambda) := (\lambda - \lambda\hat{A}(\lambda))^{-1} \in B(X,Y)$$

exists for $\lambda \in B(i\eta, \varepsilon) \cap \mathbb{C}_+$ and K has a
continuous extension to $B(i\eta, \varepsilon) \cap \overline{\mathbb{C}_+}\}$.

Assume that the kernel A satisfies assumption $(\mathcal{A})$ for some $N \geq 1$. Then it makes sense to define a *modified spectrum* Σ_0' which differs from Σ_0 at most in the point 0. We put $\Sigma_0' := \Sigma_0$ whenever $d\widehat{A_N^{(N-1)}}(0)$ is invertible in $B(Y, X)$ and $\Sigma_0' := \Sigma_0 \cup \{0\}$ otherwise. It follows that $0 \notin \Sigma_0'$ if and only if $\lambda \mapsto \lambda^{-N} K(\lambda)$ has a continuous extension to $B(0, \varepsilon) \cap \overline{\mathbb{C}_+}$ for some $\varepsilon > 0$.

We want to remark that Σ_0, Σ_0' and Λ_0 coincide if $N \leq 1$ in assumption $(\mathcal{A})$. Moreover, the spectrum Σ_0 corresponds (up to a factor i) to the imaginary spectrum of the generator in the case of C_0-semigroups, i.e. $A(t) \equiv A$. Compare our definition of the spectrum with [18, p.297].

Proposition 4.3 *Assume that A satisfies the assumption $(\mathcal{A})$. Let $\phi \in \mathcal{D}(\mathbb{R})$ be such that* $\operatorname{supp}\phi \cap \Sigma_0' = \emptyset$. *Let $H = K(i\cdot)\phi$ (where H is defined to be 0 outside the support of ϕ). Then H is the Fourier transform of a function $G_\phi \in L^1(\mathbb{R}; B(X, Y))$.*

Proof: We remark first, that $G_\phi = \mathcal{F}^{-1}H$ extends to an entire function. We have to show, that G_ϕ is integrable.
Let A be as in assumption $(\mathcal{A})$ and assume w.l.o.g. $N \geq 1$ in the sequel. For $m \in \mathbb{N} \cup \{\infty\}$ we define the functions $A_{0m} := A_0 \cdot 1_{[0,m)}$ and $A_{jm} \in W^{j-1,1}_{loc}(\mathbb{R}_+; B(Y, X))$ such that $A_{jm}^{(j-1)}(t) = A_j^{(j-1)}(t)$ for $t \in [0, m)$, $A_{jm}^{(j-1)}(t) = A_j^{(j-1)}(m)$ for $t \in [m, \infty)$, $1 \leq j \leq N-1$. We define further $A_m = \sum_{j=0}^N A_{jm}$. Note that we have $A_{j\infty} = A_j$ and therefore $A_\infty = A$.
Next we let

$$L_m(\lambda) := \lambda^N - \lambda^N \widehat{A_{0m}}(\lambda) - \sum_{j=1}^N \lambda^{N-j} d\widehat{A_{jm}^{(j-1)}}(\lambda), \lambda \in \overline{\mathbb{C}_+}.$$

Then L_m is continuous on $\overline{\mathbb{C}_+}$ and it extends even to an entire function if $m < \infty$. Moreover, we obtain

$$(4.3) \qquad L_m(\lambda) \overset{m\to\infty}{\longrightarrow} L_\infty(\lambda) \quad \text{uniformly on compact subsets of } \overline{\mathbb{C}_+}.$$

The equation

$$(4.4) \qquad \lambda - \lambda\widehat{A_m}(\lambda) = \left(\lambda - \lambda\hat{A}(\lambda)\right) \left[I - K(\lambda)\frac{1}{\lambda^{N-1}}\left(L_\infty(\lambda) - L_m(\lambda)\right)\right]$$

shows that there exists a neighbourhood U of $i\mathrm{supp}\phi$ in $\overline{\mathbb{C}_+}$ such that $K_m(\lambda) := (\lambda - \lambda\widehat{A_m}(\lambda))^{-1}$ exists for all $\lambda \in U \cap \mathbb{C}_+$, at least for large m.
By equation (4.4), we obtain

$$K_m(\lambda) = \sum_{k=0}^{\infty} \left(K(\lambda) \frac{1}{\lambda^{N-1}} \left(L_\infty(\lambda) - L_m(\lambda) \right) \right)^k K(\lambda),$$

from where we deduce that K_m and $\lambda \to \frac{1}{\lambda^{N-1}} K_m(\lambda) = L_m(\lambda)^{-1}$ have a continuous extension to U. Since L_m is an entire function, we find that $L_m(\cdot)^{-1}$ and K_m are actually analytic on U.
The equation (4.4) implies in addition

$$H(\eta) = K_m(i\eta)\phi(\eta) - H(\eta) \left(L_\infty(i\eta) - L_m(i\eta) \right) L_m(i\eta)^{-1} \phi_1(\eta) \tag{4.5}$$

for some $\phi_1 \in \mathcal{D}(\mathbb{R})$ with $\phi_1 = 1$ on $\mathrm{supp}\phi$ and $\mathrm{supp}\phi_1 \subset -iU \cap \mathbb{R}$.
Since $K_m(i\cdot)\phi \in C_c^2(\mathbb{R}; B(X,Y))$, this function is the Fourier transform of some function $G_m \in L^1(\mathbb{R}; B(X,Y))$.
Since $\eta \to (i\eta)^{N-j} L_m(i\eta)^{-1}\phi_1(\eta) \in C_c^2(\mathbb{R}; B(X,Y))$ as well (for $0 \leq j \leq N$), these functions are Fourier transforms of functions $G_{mj} \in L^1(\mathbb{R}; B(X,Y))$. When we put

$$R_m := \left(A_0 1_{(m,\infty)} \right) * G_{m0} + \sum_{j=1}^{N} d \left(A_j 1_{(m,\infty)} \right) * G_{mj},$$

then $R_m \in L^1(\mathbb{R}; B(X))$ and by equation (4.5) we obtain

$$G_\phi = G_m + G_\phi * R_m. \tag{4.6}$$

When m is large enough, then $I - \mathcal{F}R_m(\eta) = I - (L_m(i\eta) - L_\infty(i\eta)) L_m(i\eta)^{-1}\phi_1(\eta)$ is invertible for all $\eta \in \mathbb{R}$. By the Paley-Wiener Lemma [18, Theorem 0.6], there exists a resolvent $S_m \in L^1(\mathbb{R}; B(X))$ for equation (4.6), so that

$$G_\phi = G_m + G_m * S_m.$$

Thus, $G_\phi \in L^1(\mathbb{R}; B(X,Y))$. □

Proposition 4.4 *Let $v \in BC(\mathbb{R}_+; X)$ be a (mild) solution of (4.1) and assume that $A \in L^1_{loc}(\mathbb{R}_+; B(Y,X))$ satisfies condition $(\mathcal{A})$. Let $\phi \in \mathcal{D}(\mathbb{R})$ be such that $\mathrm{supp}\phi \cap \Sigma_0' = \emptyset$. Let $\mathcal{E}$ be a homogeneous subspace of $BUC(\mathbb{R}_+; X)$. If $g \in L^\infty(\mathbb{R}_+; X)$ satisfies $\mathrm{Sp}_{\mathcal{E}}(g) = \emptyset$, then $(v * \mathcal{F}^{-1}\phi)|_{\mathbb{R}_+} \in \mathcal{E} \cap BUC(\mathbb{R}_+; Y)$.*

Proof: Let $\phi \in \mathcal{D}(\mathbb{R})$ be such that $\operatorname{supp}\phi \cap \Sigma_0 = \emptyset$ and let $t \in \mathbb{R}$. Notice, that the Laplace transform of a bounded (mild) solution v of (4.1) satisfies the equation

$$\left(\lambda - \lambda \hat{A}(\lambda)\right) \hat{v}(\lambda) = \hat{g}(\lambda) + x, \qquad \lambda \in \mathbb{C}_+.$$

So one calculates, using Plancherel's Theorem,

$$\begin{aligned}
v * \mathcal{F}^{-1}\phi(t) &= \int_0^\infty v(s)\mathcal{F}^{-1}\phi(t-s)\, ds \\
&= \lim_{\xi \to 0} \int_0^\infty e^{-\xi s} v(s) \mathcal{F}^{-1}\phi(t-s)\, ds \\
&= \lim_{\xi \to 0} \int_{\mathbb{R}} \hat{v}(\xi + i\eta) e^{i\eta t} \phi(\eta)\, d\eta \\
&= \lim_{\xi \to 0} \int_{\mathbb{R}} K(\xi + i\eta) \left(\hat{g}(\xi + i\eta) + x\right) e^{i\eta t}\phi(\eta)\, d\eta \\
&= \lim_{\xi \to 0} \int_{\mathbb{R}} K(\xi + i\eta) \left(\int_0^\infty e^{-(\xi + i\eta)s} g(s)\, ds \, + x \right) e^{i\eta t} \phi(\eta)\, d\eta \\
&= \lim_{\xi \to 0} \int_0^\infty \int_{\mathbb{R}} K(\xi + i\eta) e^{i\eta(t-s)} \phi(\eta)\, d\eta\, e^{-\xi s} g(s)\, ds + \\
&\qquad + \int_{\mathbb{R}} K(i\eta) x e^{i\eta t} \phi(\eta)\, d\eta.
\end{aligned}$$

By Proposition 4.3 (ii), the function $K(i\cdot)\phi \in C_c(\mathbb{R}; B(X,Y))$ is the Fourier transform of some $G_\phi \in L^1(\mathbb{R}; B(X,Y))$. The Dominated Convergence Theorem yields

$$v * \mathcal{F}^{-1}\phi(t) = \int_0^\infty G_\phi(t-s) g(s)\, ds + \int_{\mathbb{R}} K(i\eta) x e^{i\eta t}\phi(\eta)\, d\eta.$$

By the Cohen-Hewitt Factorisation Theorem [13, Theorem 32.22], we find $\psi \in L^1(\mathbb{R})$ and $\widetilde{G} \in L^1(\mathbb{R}; \mathcal{B}(X,Y))$ such that $G_\phi = \widetilde{G} * \psi$. The assumption and a partition of unity argument yield $(g * \psi)|_{\mathbb{R}_+} \in \mathcal{E}$. By [8, Lemma 7.1], the first term above $(\widetilde{G} * g * \psi)|_{\mathbb{R}_+}$ belongs to $\mathcal{E} \cap BUC(\mathbb{R}_+; Y)$. By the Lemma of Riemann-Lebesgue, the same is true for the second term, since $C_0(\mathbb{R}_+; Y) \subset \mathcal{E} \cap BUC(\mathbb{R}_+; Y)$. Hence, $(v * \mathcal{F}^{-1}\phi)|_{\mathbb{R}_+} \in \mathcal{E} \cap BUC(\mathbb{R}_+; Y)$. $\square$

Theorem 4.5 *Let v be a bounded and uniformly continuous solution of (4.1) and assume that A satisfies $(\mathcal{A})$. Let $\mathcal{E}$ be a homogeneous subspace of $BUC(\mathbb{R}_+; X)$ that contains $AAP(\mathbb{R}_+; X)$, and let $g \in L^\infty(\mathbb{R}_+; X)$ be such that $\mathrm{Sp}_{\mathcal{E}}(g) = \emptyset$. If Σ_0 is countable and if v is uniformly ergodic in each $\eta \in \Sigma_0'$, then $v \in \mathcal{E}$.*

Proof: Proposition 4.4 yields that $\mathrm{Sp}_{\mathcal{E}}(v)$ is contained in Σ_0', which is countable. Theorem 2.5 together with the assumption that v is uniformly continuous and uniformly ergodic in each $\eta \in \Sigma_0'$ implies $v \in \mathcal{E}$. $\square$

5 A SPECIAL CASE

In this section we want to study an important special case of (3.1) and (4.1), namely the scalar case $A(t) = a(t)A$, where A denotes a closed linear operator in X with spectrum $\sigma(A) \subset (-\infty, 0]$, and a is a *creep kernel.* The latter means that a is of the form

$$a(t) = a_0 + a_\infty t + \int_0^t a_1(s)\, ds, \qquad t \in \mathbb{R}_+, \tag{5.1}$$

where $a_0, a_\infty \geq 0$, and $a_1 \in L^1_{loc}(\mathbb{R}_+) \cap C_0(0,\infty)$ is *2-monotone*, i.e. nonnegative, nonincreasing and convex. Of course we assume also $a \not\equiv 0$, to avoid a trivial case. Note that $a \cdot A$ satisfies the condition $(\mathcal{A})$ from Section 4, with $N = 1$ in the case $a_\infty = 0$ and $a_1 \in L^1(\mathbb{R}_+)$, and $N = 2$ otherwise; see below. Because of its importance in various fields of mathematical physics like viscoelasticity or electrodynamics with memory, equations of scalar type with kernels of this form have been investigated intensively; cf. Prüss [18]. To be precise, we study the following equation on the half line

$$\dot v(t) = A(a * v)\dot{}(t) + g(t), \quad v(0) = x, \quad t \in \mathbb{R}_+, \tag{5.2}$$

and the corresponding equation on the line

$$\begin{aligned} \ddot u(t) \;&=\; Aa_0 \cdot \dot u(t) + Aa_\infty \cdot u(t) + Aa_2 * \dot u(t) + \\ &\qquad Ada_3 * u(t) + \dot f(t), \quad t \in \mathbb{R}. \end{aligned} \tag{5.3}$$

Note, that we have decomposed a_1 into a sum $a_2 + a_3$, where $a_2 \in L^1(\mathbb{R}_+)$ and $a_3 \in BV(\mathbb{R}_+)$, and we had to write the equation on the line as a second order equation so that the convolutions make (at least formally) sense for bounded functions u having bounded derivatives $\dot u$.

Let us point out that typical examples of operators A satisfying the above condition would be the Laplace operator Δ or the Stokes operator $P\Delta$ on an open domain $\Omega \subset \mathbb{R}^n$ with certain boundary conditions. Typical Banach spaces would be for example $X = L^p(\Omega)$ $(1 \leq p < \infty)$ and Y would of course be the domain of A equipped with the graph norm.

Note that even for these two examples the following difficulties arise:
First, if the function a_1 is only convex then existence of a resolvent family $(S(t))_{t\geq 0}$ (i.e. well-posedness) is even in one space dimension not clear, except for the case $X = L^2(\Omega)$. Second, if Ω is unbounded, then A is in general not

invertible. Hence, even if a resolvent family $(S(t))_{t\geq 0}$ exists, it will not be integrable. Third, special cases of functions a_1 lead to discrete countable spectra Σ_0 and Λ_0.

In order to study equation (5.3) we want to rewrite it as a first order equation. This is done by convolving equation (5.3) with the function ϕ defined by $\phi(t) = e^{-t}h_0(t)$, where h_0 is the Heaviside function. We then obtain

$$\begin{aligned} \dot{u}(t) &= (\phi + Aa_0 \cdot \phi + Aa_2 * \phi) * \dot{u}(t) + \\ &\qquad A(a_\infty \cdot \phi + da_3 * \phi) * u(t) + \tilde{f}(t), \quad t \in \mathbb{R}, \end{aligned} \tag{5.4}$$

where $\tilde{f} = f - f * \phi$, and this equation is of the form studied in Section 3. Notice that $\mathrm{Sp}_{\mathcal{E}}(f) = \emptyset$ for some translation-biinvariant subspace $\mathcal{E}$ of $BUC(\mathbb{R}; X)$ implies $\mathrm{Sp}_{\mathcal{E}}(\tilde{f}) = \emptyset$ as well. Since $\mathcal{F}\phi(\eta) \neq 0$ for all $\eta \in \mathbb{R}$ (i.e. ϕ belongs to the Wiener class mentioned in [18, p.286]), equations (5.3) and (5.4) are easily seen to be equivalent.

Proposition 5.1 *Let A and a be as above. Then:*
(a) The operator $\lambda - \lambda\hat{a}(\lambda)A$ is invertible for all $\lambda \in \mathbb{C}_+$.
(b) If Σ_0 and Λ_0 denote the spectra of equations (5.2) and (5.4), respectively, then $\Lambda_0\backslash\{0\} = \Sigma_0\backslash\{0\}$.
(c) If $\Sigma_0\backslash\{0\}$ is nonempty, then

$$a(t) = a_\infty t + \int_0^t \sum_{k\geq 1} \alpha_k (k\tau - s)_+ \, ds, \quad t \in \mathbb{R}_+, \tag{5.5}$$

where $(\alpha_k \cdot k)_{k\geq 1} \in l^1(\mathbb{N})^+$ and $\tau > 0$.
(d) If a is of the form (5.5), then $\Sigma_0 \subset \frac{2\pi}{k_0\tau}\mathbb{Z}$ where $k_0 \in \mathbb{N}$ denotes the smallest k such that $\alpha_k > 0$.
(e) $0 \notin \Sigma_0$ if and only if A is invertible, and then $K(0) \neq 0$ if and only if $a_\infty = 0$ and $a_1 \in L^1(\mathbb{R}_+)$.
(f) $0 \notin \Sigma_0'$ if and only if A is invertible and either $a_\infty > 0$, or $a_\infty = 0$ and $a_1 \in L^1(\mathbb{R}_+)$.

Proof: (a) We have to study invertibility of the operator $\lambda - \lambda\hat{a}(\lambda)A$ for $\lambda \in \mathbb{C}_+$. Because of (5.1) we have

$$\hat{a}(\lambda) = \frac{a_0}{\lambda} + \frac{a_\infty}{\lambda^2} + \frac{\widehat{a_1}(\lambda)}{\lambda}.$$

Next, convexity of a_1 implies that a_1 is of *positive type* (see [18, Section 3]), hence $\mathrm{Re}\widehat{a_1}(\lambda) > 0$ in the open right half-plane, unless $a_1 \equiv 0$. This shows $\hat{a}(\lambda) \notin (-\infty, 0]$ for $\mathrm{Re}\lambda > 0$, i.e. $\lambda - \lambda\hat{a}(\lambda)A$ is invertible for all $\lambda \in \mathbb{C}_+$.

(b) Since a_1 can be decomposed into $a_1 = a_2 + a_3$ with $a_2 \in L^1(\mathbb{R}_+)$ and $a_3 \in BV(\mathbb{R}_+)$, we obtain

$$\hat{a}(\lambda) = \frac{a_0 + \hat{a}_2(\lambda)}{\lambda} + \frac{a_\infty + \widehat{da_3}(\lambda)}{\lambda^2}.$$

This implies that $\hat{a}(\lambda)$ extends continuously to the imaginary axis minus the origin, so that, by (a), $\eta \in \Sigma_0\backslash\{0\}$ if and only if the operator $i\eta - i\eta\hat{a}(i\eta)A$ is not invertible.

Let $\eta \in \mathbb{R}\backslash\{0\}$. We then calculate, using the notation of Section 3,

$$i\eta - i\eta\widehat{A_0}(i\eta) - \widehat{dA_1}(i\eta) = \mathcal{F}\phi(\eta) i\eta(i\eta - i\eta\hat{a}(i\eta)A),$$

and we see that the left side is invertible if and only if $(i\eta - i\eta\hat{a}(i\eta)A)$ is invertible. This implies $\Lambda_0\backslash\{0\} = \Sigma_0\backslash\{0\}$.

(c) Assume that $\Sigma_0\backslash\{0\}$ is nonempty, and let $0 \neq \lambda_0 = i\eta_0 \in i\Sigma_0$, where w.l.o.g. $\eta_0 > 0$. Then $\hat{a}(\lambda_0) = \mathrm{Re}\hat{a}(\lambda_0) < 0$, which means

$$\begin{aligned} -\eta_0 \mathrm{Im}\hat{a}(i\eta_0) = a_0 + \mathrm{Re}\widehat{a_1}(i\eta_0) &= 0, \\ -\eta_0^2 \mathrm{Re}\hat{a}(i\eta_0) = a_\infty - \eta_0 \mathrm{Im}\widehat{a_1}(i\eta_0) &> 0. \end{aligned}$$

The first of these properties yield $a_0 = 0$ and

$$0 = \eta_0^2 \mathrm{Re}\widehat{a_1}(i\eta_0) = \int_0^\infty (1 - \cos(\eta_0 t))\ d\dot{a}_1(t),$$

which by convexity of a_1 yields $\mathrm{supp} d\dot{a}_1 \subset \frac{2\pi}{\eta_0}\mathbb{N}_0$, i.e.

$$d\dot{a}_1(t) = \sum_{k=0}^\infty \alpha_k \delta_0(t - \frac{2\pi k}{\eta_0}), \qquad t \geq 0,$$

δ_0 denoting Dirac's measure at the origin, and $\alpha_k \geq 0$. This in turn implies

$$a_1(t) = \sum_{k\geq 1} \alpha_k \left(\frac{2\pi k}{\eta_0} - t\right)_+, \quad t \in \mathbb{R}_+.$$

Since $a_1(t)$ has to be finite, we conclude $(\alpha_k \cdot k)_{k\geq 1} \in l^1(\mathbb{N})^+$, and the creep kernel a is of the form given in (5.5).

(d) Assume that a is of the form (5.5), and let $\eta \in \mathbb{R}\backslash\{0\}$. Then

$$\widehat{a_1}(i\eta) = -\eta^{-2} \sum_{k=1}^{\infty} \alpha_k \left(ik\tau\eta + e^{-ik\tau\eta} - 1 \right),$$

hence

$$\mathrm{Re}\widehat{a_1}(i\eta) = \eta^{-2} \sum_{k=1}^{\infty} \alpha_k \left(1 - \cos(k\tau\eta)\right).$$

Thus, $\mathrm{Re}\widehat{a_1}(i\eta) = 0$ (which is necessary for $\eta \in \Sigma_0$) implies $k\eta \in \frac{2\pi}{\tau}\mathbb{Z}$ for all $k \in \mathbb{N}$ with $\alpha_k > 0$. In particular, $\eta \in \frac{2\pi}{k_0\tau}\mathbb{Z}$ where $k_0 \in \mathbb{N}$ denotes the smallest k such that $\alpha_k > 0$.

(e) Obviously, the invertibility of A implies that 0 does not belong to Σ_0. So assume that $0 \notin \Sigma_0$. Since $\lim_{\lambda\to 0} \hat{a}(\lambda) = \infty$ (unless a is trivial), the boundedness of $K(\lambda) = \frac{1}{\lambda\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)} - A)^{-1}$ near the origin implies that A is invertible. On the other hand, $\lim_{\lambda\to 0} \lambda\hat{a}(\lambda) = \infty$ if and only if $a_\infty > 0$ or a_1 is not integrable, and this in turn is equivalent to $K(0) = 0$.

(f) If a is a scalar kernel as above, then $N \leq 2$ in assumption $(\mathcal{A})$. Hence, the claim follows from (e), the fact that $\widehat{dA_2^{(1)}}(0) = a_\infty A$ in case $a_\infty > 0$ or $\widehat{dA_1^{(0)}}(0) = (a_0 + \widehat{a_1}(0))A$ if $a_\infty = 0$ and $a_1 \in L^1(\mathbb{R}_+)$, and by the definition of Σ_0'. □

As in Definition 2.1 we denote for $s \in \mathbb{R}_+$ and $h \in L^1_{loc}(\mathbb{R}_+; X)$ by h_s the shifted function $h(s + \cdot)$. It is well known (see e.g. [4]) that uniform ergodicity of a function $h \in L^\infty(\mathbb{R}_+; X)$ in a point $\eta \in \mathbb{R}$ is equivalent to the condition that the limit $\lim_{\xi\to 0} \xi\widehat{h_s}(\xi + i\eta)$ exists uniformly in $s \in \mathbb{R}_+$. Thus, uniform ergodicity of a function h is also characterized by the behaviour of its Laplace transform $\hat{h}$ near points of the imaginary axis.

By Definition 4.1 we obtain for a (mild) solution v and $s \in \mathbb{R}_+$ in the case of the scalar evolutionary integral equation (5.2)

$$\begin{aligned}
v_s(t) &= x + A\int_0^{t+s} a(t+s-r)v(r)\,dr + \int_0^{t+s} g(r)\,dr \\
&= x + A\int_0^t a(t-s)v_s(r)\,dr + A\int_0^s a(t+s-r)v(r)\,dr + \\
&\quad + \int_0^{t+s} g(r)\,dr.
\end{aligned}$$

Taking Laplace transform on both sides, we find that for $\lambda \in \mathbb{C}_+$

$$\widehat{v_s}(\lambda) - A\hat{a}(\lambda)\widehat{v_s}(\lambda) = \frac{x}{\lambda} + A\int_0^s \widehat{a_{s-r}}(\lambda)v(r)\,dr +$$

$$\frac{1}{\lambda}\int_0^s g(r)\ dr + \frac{\widehat{g_s}(\lambda)}{\lambda}.$$

We calculate the Laplace transform of the shifted creep kernel:

$$\begin{aligned}
\widehat{a_s}(\lambda) &= \int_0^\infty e^{-\lambda t}\left(a_0 + a_\infty(t+s) + \int_0^{t+s} a_1(r)\ dr\right)\ dt \\
&= \frac{a_0}{\lambda} + \frac{a_\infty}{\lambda^2} + \frac{a_\infty s}{\lambda} + \frac{1}{\lambda}\int_0^s a_1(r)\ dr + \frac{\widehat{a_{1s}}(\lambda)}{\lambda} \\
&= \frac{a(s)}{\lambda} + \frac{a_\infty}{\lambda^2} + \frac{\widehat{a_{1s}}(\lambda)}{\lambda}.
\end{aligned}$$

Hence, we obtain for the Laplace transform of v_s the functional equation

$$\begin{aligned}
(\lambda - \lambda\hat{a}(\lambda)A)\widehat{v_s}(\lambda) &= v(s) + \widehat{g_s}(\lambda) + A\frac{a_\infty}{\lambda}\int_0^s v(r)\ dr + \\
&\quad + A\int_0^s \widehat{a_{1,s-r}}(\lambda)v(r)\ dr,
\end{aligned} \tag{5.6}$$

and since $K(\lambda) = (\lambda - \lambda\hat{a}(\lambda)A)^{-1}$ exists by Proposition 5.1 (a), we obtain for $\lambda \in \mathbb{C}_+$

$$\begin{aligned}
(\mathrm{Re}\lambda)\widehat{v_s}(\lambda) &= (\mathrm{Re}\lambda)K(\lambda)v(s) + (\mathrm{Re}\lambda)K(\lambda)\widehat{g_s}(\lambda) + \\
&\quad + \frac{\mathrm{Re}\lambda}{\lambda}K(\lambda)Aa_\infty\int_0^s v(r)\ dr + \\
&\quad + \frac{\mathrm{Re}\lambda}{\lambda\hat{a}(\lambda)}\left(\frac{1}{\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)} - A)^{-1} - I\right)\int_0^s \widehat{a_{1,s-r}}(\lambda)v(r)\ dr.
\end{aligned} \tag{5.7}$$

If we want to study the uniform ergodicity of a solution v, then we have to study the behaviour of the functions K, $\widehat{a_s}$ and $\widehat{g_s}$ near the imaginary axis.

We start with the function K. If A is a closed linear operator on a Banach space X, then we call a point $\lambda \in \mathbb{C}$ *simple for A* if
(i) there exists a closed cone Σ with vertex in λ and $\varepsilon > 0$ such that $\Sigma \cap B(\lambda,\varepsilon)\backslash\{\lambda\} \subset \varrho(A)$ and
(ii) for all closed cones Σ with vertex in λ such that $\Sigma \cap B(\lambda,\varepsilon)\backslash\{\lambda\} \subset \varrho(A)$ (for some $\varepsilon > 0$) the estimate $\sup_{\mu\in\Sigma\cap B(\lambda,\varepsilon)\backslash\{\lambda\}} \|(\mu-\lambda)R(\mu,A)\| < \infty$ holds.

It is clear from the definition that the set of all simple points for a closed linear operator A is a subset of $\overline{\varrho(A)}$. For simple points in $\overline{\varrho(A)}$ the proof of the following lemma is straightforward and we leave it to the reader (compare also with the Mean Ergodic Theorem for bounded C_0-semigroups in [22]).

Lemma 5.2 *Let A be a closed linear operator on a Banach space X, and let $\lambda \in \mathbb{C}$ be simple for A. Let Σ be a closed cone with vertex in λ such that*

$\Sigma \cap B(\lambda, \varepsilon)\backslash\{\lambda\} \subset \varrho(A)$. *Then the following assertions are equivalent:*
(i) The limit $\lim_{\substack{\mu\to\lambda \\ \mu\in\Sigma\backslash\{\lambda\}}} (\mu-\lambda)R(\mu, A) =: P_\lambda$ *exists strongly.*
(ii) $N(\lambda - A) \oplus \overline{R(\lambda - A)} = X$.
If (i) and (ii) hold, then P_λ *is the projection onto* $N(\lambda - A)$.

Notice that for operators on reflexive Banach spaces the conditions (i) and (ii) of Lemma 5.2 are always satisfied for all simple points $\lambda \in \mathbb{C}$. Lemma 5.2 is important for ergodicity of a bounded solution as we will see below.

Next, we study the inhomogeneity g in equation (5.2).

Lemma 5.3 *Let* $g \in L^\infty(\mathbb{R}_+; X)$ *and* $\eta \in \mathbb{R}$ *be such that the function* G *defined by* $G(t) := \int_0^t e^{-i\eta r} g(r)\, dr$ *has relatively compact range and is uniformly ergodic in* 0. *Then* $\lim_{\alpha\to 0} \widehat{g_s}(\alpha + i\eta) =: h(s)$ *exists uniformly in* $s \geq 0$ *and* h *has relatively compact range.*

Proof: Let $g \in L^\infty(\mathbb{R}_+; X)$ be such that $G(t) := \int_0^t e^{-i\eta r} g(r)\, dr$ has relatively compact range. Let $s \in \mathbb{R}_+$. Then one calculates for $\lambda \in \mathbb{C}_+$

$$\lambda \widehat{G_s}(\lambda) = G(s) + \widehat{g_s}(\lambda + i\eta).$$

Thus, the uniform ergodicity and the boundedness of G implies that $\lim_{\alpha\to 0} \widehat{g_s}(\alpha + i\eta) =: h(s)$ exists uniformly in $s \geq 0$. Since G has relatively compact range and since $s \mapsto \lim_{\lambda\to 0} \lambda \widehat{G_s}(\lambda)$ is constant, the function h has also relatively compact range. □

Remark 5.4 Let $g \in AAP(\mathbb{R}_+; X)$ be such that $g = g_0 + g_1$ with $g_0 \in C_0(\mathbb{R}_+; X)$ and $g_1 \in AP(\mathbb{R}_+; X)$. Assume that the space c_0 can not be isomorphically imbedded into the Banach space X and that either
(i) the Laplace transform $\hat{g}$ extends analytically near 0, or
(ii) $g_0 \in L^1(\mathbb{R}_+; X)$ and $\hat{g}$ extends continously to $\mathbb{C}_+ \cup \{0\}$.

Then the conditions of Lemma 5.3 are verified for $\eta = 0$.

In fact, if (i) or (ii) holds, then the Laplace transform $\widehat{g_1}$ extends analytically near the origin. Thus, the primitives G, G_0 and G_1 of g, g_0 and g_1, respectively, are bounded by [4, Remark 3.2]. By the vector-valued Bohr Theorem for almost periodic functions, G_1 is therefore also almost periodic. Moreover, the limit $\lim_{t\to\infty} G_0(t)$ exists by [4, Remark 3.3] in the case (i), or trivially in the case (ii). Hence, the function G is asymptotically almost periodic. In particular, it is totally uniformly ergodic and it has relatively compact range.

Theorem 5.5 *Let A be a closed linear operator on X such that $\sigma(A) \subset (-\infty, 0]$. Let $a \in L^1_{loc}(\mathbb{R}_+)$ be a creep kernel which is* not *of the form in (5.5).*

Let $\mathcal{E}$ be a translation-biinvariant subspace of $BUC(\mathbb{R}_+; X)$, let $g \in L^\infty(\mathbb{R}_+; X)$ be such that $\mathrm{Sp}_{\mathcal{E}}(g) = \emptyset$, and let $v \in BUC(\mathbb{R}_+; X)$ be a solution of (5.2). Assume further that either
(i) the operator A is invertible and $a_\infty > 0$, or
(ii) the operator A is invertible, $a_\infty = 0$ and $a_1 \in L^1(\mathbb{R}_+)$, or
(iii)(a) 0 is a simple point and $N(A) \oplus \overline{R(A)} = X$.
(b) $a_\infty = 0$ and $a_1 \in L^1(\mathbb{R}_+)$.
(c) The function $G(t) := \int_0^t g(r)\, dr$ is uniformly ergodic in 0 and it has relatively compact range.
(d) The space $\mathcal{E}$ contains the constant functions if $N(A) \neq \{0\}$.
(e) The solution v has relatively compact range.
Then $v \in \mathcal{E}$.

Proof: Since the creep kernel a is not of the form (5.5), we obtain $\Sigma_0 \subset \{0\}$ by Proposition 5.1 (c).

(i) If, in addition, the operator A is invertible and $a_\infty > 0$, then Σ_0' is empty by Proposition 5.1 (f). The assertion follows from Theorem 4.5.

(ii) If $a_\infty = 0$ and $a_1 \in L^1(\mathbb{R}_+)$, then the kernel a is of bounded variation and $N = 1$ in assumption $(\mathcal{A})$. Hence, the spectra Σ_0 and Σ_0' coincide by the remark following the definition of Σ_0'. If the operator A is invertible, then Σ_0' is empty by Proposition 5.1 (e), and the assertion follows again from Theorem 4.5.

(iii) In order to apply Theorem 4.5 in this case we have to show that the solution v is uniformly ergodic in 0. To this end, we use representation (5.7).

Since $a_\infty = 0$, the third term on the right hand side of equation (5.7) vanishes. Moreover, the assumptions on g together with Lemma 5.3 and on A together with Lemma 5.2 imply that the second term on the right hand side of equation (5.7) converges uniformly in $s \geq 0$ as λ tends to 0.

Next, we consider the fourth term on the right hand side of equation (5.7). We have

$$
\begin{aligned}
\left\| \lambda \int_0^s \widehat{a_{1,r}}(\lambda) v(s-r)\, dr \right\| &\leq \|v\|_\infty \lambda \int_0^s \int_0^\infty e^{-\lambda t} a_1(t+r)\, dt\, dr \\
&\leq \|v\|_\infty \lambda \int_0^\infty \int_0^\infty e^{-\lambda t} a(t+r)\, dr\, dt \\
&= \|v\|_\infty \lambda \int_0^\infty e^{-\lambda t} \int_t^\infty a_1(r)\, dr\, dt \\
&= \|v\|_\infty \int_0^\infty a_1(t)(1 - e^{-\lambda t})\, dt \;\to\; 0 \text{ as } \lambda \to 0
\end{aligned}
$$

by the Dominated Convergence Theorem. Since 0 is a simple point for A, i.e. $\|\frac{1}{\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)} - A)^{-1} - I\| \leq M$ for all $\lambda \in (0,1]$, the fourth term on the right hand side of (5.7) converges to 0 uniformly in $s \geq 0$.

The property $\lim_{\lambda\to 0} \lambda K(\lambda) = P_0$ strongly implies that the limit is uniform on relatively compact subsets of X. Hence $\lim_{\lambda\to 0} \lambda K(\lambda)v(s) = P_0 v(s)$ uniformly in $s \geq 0$. This completes the proof of uniform ergodicity of v, and thus $v \in \mathcal{E}$. $\square$

Theorem 5.6 *Let A be a closed linear operator on X such that $\sigma(A) \subset (-\infty, 0]$, and let $a \in L^1_{loc}(\mathbb{R}_+)$ be a creep kernel which is of the form (5.5). Let $\mathcal{E}$ be a translation-biinvariant subspace of $BUC(\mathbb{R}_+; X)$ containing $AP_{\Sigma_0}(\mathbb{R}_+; X)$, and let $g \in L^\infty(\mathbb{R}_+; X)$ be such that $\mathrm{Sp}_{\mathcal{E}}(g) = \emptyset$. Let $v \in BUC(\mathbb{R}_+; X)$ be a solution of (5.2). Assume that*
(a) all points $\lambda \in \sigma(A)$ are simple and $N(\lambda - A) \oplus \overline{R(\lambda - A)} = X$ for $\lambda \in \sigma(A)$.
(b) $a_\infty = 0$ and $(\alpha_k \cdot k^3)_{k\geq 0} \in l^1(\mathbb{N})^+$.
(c) For each $\eta \in \Sigma_0$ the function $G(t) := \int_0^t e^{-i\eta r} g(r)\, dr$ is uniformly ergodic in 0 and it has relatively compact range.
(d) The solution v has relatively compact range.

Then $v \in \mathcal{E}$.

Proof: By assumption (b), the kernel a is just the integral of the function a_1 which is given by $a_1(t) = \sum_{k\geq 1} \alpha_k (k\tau - t)_+$ $(t \in \mathbb{R}_+)$. Since, by assumption (b) again, the sum $\sum_{k\geq 1} \alpha_k \cdot k^3$ is finite, one easily finds that actually $a_1 \in L^1(\mathbb{R}_+; (1+t)\, dt)$. In particular, the kernel a satisfies $(\mathcal{A})$ with $N = 1$ so that, by Proposition 5.1 (f), $\Sigma_0 = \Sigma_0'$. Moreover, by Proposition 5.1 (d), the spectrum Σ_0 is discrete, hence countable. In order to apply Theorem 4.5, we have to show that the solution v is uniformly ergodic in every $\eta \in \Sigma_0$. We will proceed as in Theorem 5.5, and we use again representation (5.7).

First, if $0 \in \Sigma_0$, then the same arguments as in the proof of Theorem 5.5 show that v is uniformly ergodic in 0. So let $\eta \in \Sigma_0 \backslash \{0\}$.

From assumption (a) we obtain the existence of $\lim_{\xi\to 0} \xi K(\xi + i\eta) = c \cdot P$ strongly, hence also uniformly on relatively compact subsets of X. Since the solution v is supposed to have relatively compact range, the first term on the right hand side of equation (5.7) converges as $\xi = \mathrm{Re}\lambda \to 0$, uniformly in $s \geq 0$. The same holds for the second term by assumption (c) and by Proposition 5.3. The third term vanishes by the assumption $a_\infty = 0$.

Since $a_1 \in L^1(\mathbb{R}_+; (1+t)\, dt)$, the Laplace transforms $\widehat{a_{1,r}}$ extend continuously to the imaginary axis. We even obtain the estimate

$$\begin{aligned} \int_0^\infty |\widehat{a_{1,r}}(i\eta)| \ dr &\leq \int_0^\infty \int_0^\infty |a_1(t+r)| \ dt \ dr \\ &= \int_0^\infty t|a_1(t)| \ dt < \infty. \end{aligned}$$

Thus the integral $\int_0^s \widehat{a_{1,r}}(\lambda)v(s-r) \ dr$ is a bounded function in s for all λ in the closed right half plane $\overline{\mathbb{C}_+}$, and since v has relatively compact range, this convolution has also relatively compact range. Moreover, by the Dominated Convergence Theorem, we obtain

$$\lim_{\xi \to 0} \int_0^s \widehat{a_{1,r}}(\xi + i\eta)v(s-r) \ dr = \int_0^s \widehat{a_{1,r}}(i\eta)v(s-r) \ dr \text{ uniformly in } s \geq 0.$$

Since $\limsup_{\lambda \to i\eta} |\frac{1}{\lambda \hat{a}(\lambda)^2}| < \infty$, we finally obtain that the fourth term on the right hand side converges as $\xi = \mathrm{Re}\lambda \to 0$, uniformly in $s \geq 0$.

Hence, the solution v is uniformly ergodic in every $\eta \in \Sigma_0 = \Sigma_0'$, and $v \in \mathcal{E}$ by Theorem 4.5. $\square$

We conclude the article with a result for equation (5.4) on the line.

Theorem 5.7 *Let A be a closed linear operator on X such that $\sigma(A) \subset (-\infty, 0]$, and let $a \in L^1_{loc}(\mathbb{R}_+)$ be a creep kernel.*

Let $f \in L^\infty(\mathbb{R}; X)$ be such that $\mathrm{Sp}_{AP}(f) = \emptyset$. Assume that either the spectrum $\mathrm{Sp}_{\{0\}}(f)$ is discrete, or c_0 can not be isomorphically embedded into the Banach space X.

If $u \in BUC(\mathbb{R}; X)$ is a solution of (5.4), then $u \in AP(\mathbb{R}; X)$.

Proof: By Proposition 5.1, the spectrum Λ_0 of equation (5.4) is discrete. The assertion thus follows directly from Theorem 3.4. $\square$

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