Logical Characterization of Petri Net ω-Languages

Hideki YAMASAKI
Department of Mathematics, Hitotsubashi University
Kunitachi, Tokyo 186-8601 JAPAN *
Electronic Mail: yamasaki@math.hit-u.ac.jp

Abstract

In this paper, we study some classes of Petri net definable ω-languages. We consider several types of accepting conditions on Petri nets, corresponding to those considered on finite automata in the theory of ω-regular languages. Then, we establish a neat correspondence between the classes of Petri net definable ω-languages and the classes of finite automata definable ω-languages. Moreover, each class of Petri net definable ω-languages is characterized by logical formulas.

*This paper was written during my visit to Institut für Informatik, Martin-Luther-Universität Halle-Wittenberg, 1999.

Contents

1 Introduction **3**

2 Preliminary **4**

3 Inclusion relations **7**

4 Normal form of Petri nets **10**

5 Characterizations by logical formulas **12**

6 Decomposition of Petri nets **15**

7 Concluding remarks **16**

1 Introduction

The study of ω-languages was started by Büchi to solve the decision problem of the monadic second-order theory S1S over natural numbers. He proved that an ω-language L is defined by a formula in S1S if and only if it is accepted by a nondeterministic finite automaton, so called a Büchi automaton [1].

In the study of ω-languages accepted by finite automata, many types of accepting conditions has been considered [5, 6, 7, 11]. It is shown that there exists a neat correspondence between accepting conditions for automata and logical formulas in the theory S1S [10].

In this paper, we study the ω-languages accepted by Petri nets under several types of accepting conditions and their relation to logical formulas in the theory K over natural numbers, which is obtained from S1S by augmenting the relation $\preceq$ between sets of natural numbers. $V \preceq W$ means that there exists a one to one function $f : W \to V$ such that $f(x) \leq x$ for any $x \in W$, that is, any $x \in W$ has its 'origin' $f(x)$ in V.

Petri nets studied in this paper are λ-free labeled Petri nets. Let N be a Petri net with a λ-free labeling function e, and $\uparrow F$ be the set of markings greater than or equal to a marking in the designated finite set F of accepting markings. Then we mainly study the ω-languages accepted by N defined as follows:

$$L_{True}(N) = \{e(\alpha) \,|\, \alpha \text{ is an infinite firing sequence of } N\}$$

$$L_{ran\sqcap}(N) = \{e(\alpha) \,|\, \begin{array}{l}\alpha \text{ is an infinite firing sequence of } N,\\ \text{and passes a marking in } \uparrow F\ \}\end{array}$$

$$L_{inf\sqcap}(N) = \{e(\alpha) \,|\, \begin{array}{l}\alpha \text{ is an infinite firing sequence of } N,\\ \text{and passes markings in } \uparrow F \text{ infinitely often }\}.\end{array}$$

Then we can show that for each $\gamma \in \{True, ran\sqcap, inf\sqcap\}$, the class $\mathbf{P}_\gamma$ of ω-languages of the form $L_\gamma(N)$, neatly corresponds to the class defined by finite automata with the similar accepting condition. Moreover, these classes are characterized by logical formulas. That is, for each $\gamma \in \{True, ran\sqcap, inf\sqcap\}$, we can specify the set of formulas in prenex normal form in K which exactly define the ω-languages in $\mathbf{P}_\gamma$.

The study of this direction was initiated by Parigot and Pelz [3, 4]. The accepting conditions they considered are directly defined by the designated finite set F of accepting markings, instead of $\uparrow F$ in the above definitions. In that case, for each type of accepting conditions, we have the class of logical formulas that can represent the ω-languages accepted by Petri nets with those accepting conditions, but the inverse does not hold except for so called Muller type acceptance.

We give preliminary definitions and results in section 2. We establish the inclusion relations and some closure properties for the classes of Petri net ω-languages, in section 3. We give normal form theorem for Petri net in section 4. The logical characterization is given in section 5. Finally, in section 6, we give a kind of decomposition theorem for Petri nets.

2 Preliminary

The set of integers $\{0, 1, -1, 2, -2, \dots\}$ is denoted by $\mathbf{Z}$, and the set of nonnegative integers is denoted by $\mathbf{N}$. For sets X and Y, Y^X denotes the set $\{f \mid f : X \to Y\}$ of all functions from X to Y. For a finite set $X = \{x_1, x_2, \dots, x_n\}$, a function $f \in \mathbf{Z}^X$ is identified with the n-dimensional vector $\langle f(x_1), f(x_2), ..., f(x_n)\rangle$ over $\mathbf{Z}$. Then for functions $f, g \in \mathbf{Z}^X$ and $z \in \mathbf{Z}$, the addition $f + g$, the scalar product zf, and the partial ordering $f \leq g$ are defined componentwise as usual.

Let Σ be a finite alphabet. We call a mapping $\alpha \in \Sigma^{\mathbf{N}}$ an ω-*word* over Σ, and write $\alpha = a_0a_1a_2\cdots$ where $a_n = \alpha(n)$ for each n. The set of all ω-words over Σ is denoted by Σ^ω, and that of all finite words over Σ is denoted by Σ^* as usual. The empty word is denoted by λ.

If $u = \alpha(0)\dots\alpha(n)$ for some n, then u is called a *prefix* of α. We define $pref(\alpha) = \{v \in \Sigma^* \mid v \text{ is a prefix of } \alpha\}$ for $\alpha \in \Sigma^\omega$, and $pref(L) = \bigcup_{\alpha\in L} pref(\alpha)$ for $L \subseteq \Sigma^\omega$.

For any alphabets Σ and Δ, a function $f : \Sigma \to \Delta$ is naturally extended to a morphism $f : \Sigma^* \to \Delta^*$ or $f : \Sigma^\omega \to \Delta^\omega$, by $f(a_0a_1\cdots a_n) = f(a_0)f(a_1)\cdots f(a_n)$ or $f(a_0a_1\cdots) = f(a_0)f(a_1)\cdots$, respectively. In this case, we call a morphism f a *projection*. Especially, we will use in the sequel the projection $pr_i : \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n \to \Sigma_i$ defined by $pr_i(\langle a_1, a_2, \dots, a_n\rangle) = a_i$ $(i = 1, \dots, n)$.

For $\alpha \in \Sigma^\omega$, we also define $ran(\alpha) = \{a \mid a = \alpha(n) \text{ for some } n\}$, and $inf(\alpha) = \{a \mid a = \alpha(n) \text{ for infinitely many } n\}$. That is, $ran(\alpha)$ is the set of letters appearing in α, and $inf(\alpha)$ is the set of letters appearing infinitely many times in α.

For $K \subseteq \Sigma^*$ and $L \subseteq \Sigma^\omega$, we define $KL = \{u\alpha \mid u \in K \text{ and } \alpha \in L\}$ and $K^\omega = \{v_1v_2\dots \mid v_1, v_2, \dots \in K - \{\lambda\}\}$, where $u\alpha$ is the ω-word obtained by concatenating u before α, and $v_1v_2\dots$ is the ω-word obtained by concatenating $v_1, v_2, \dots$ one after another.

We can consider Σ^ω as a metric space with the distance d defined by:

$$d(\alpha, \beta) = \begin{cases} 0, & \text{if } \alpha = \beta \\ 2^{-Min\{n \mid \alpha(n)\neq\beta(n)\}}, & \text{if } \alpha \neq \beta. \end{cases}$$

Then $L \subseteq \Sigma^\omega$ is open if and only if $L = K\Sigma^\omega$ for some $K \subseteq \Sigma^*$, and L is closed if and only if $L = \{\alpha \mid pref(\alpha) \subseteq pref(L)\}$.

In this paper, when we mention a *net* or a *Petri net* N, we mean a marked λ-free labeled Petri net $N = (P, T, A, e, m_0, F)$, where P is a finite set of *places*, T a finite set of *transitions*, $A : T \to \mathbf{N}^P \times \mathbf{N}^P$, $e : T \to \Sigma$ a λ-free *labelling function*, $m_0 \in \mathbf{N}^P$ an *initial marking*, and $F \subseteq \mathbf{N}^P$ a finite set of *accepting markings*.

A *marking* m of a Petri net N is a function in $\mathbf{N}^P$, that is, an assignment of tokens to places. We say that the place p has $m(p)$ tokens at the marking m. For each transition t, $A(t) = \langle {}^\bullet A(t), A(t)^\bullet \rangle$ assigns a pair of functions ${}^\bullet A(t)$ and $A(t)^\bullet$ in $\mathbf{N}^P$ called the *input* and *output vector* of t, respectively.

Example 1 *Let* $N = (\{p\}, \{s, t\}, A, e, \langle 0 \rangle, \{\langle 2 \rangle\})$, *where* $A(s) = \langle \langle 0 \rangle, \langle 1 \rangle \rangle$, $A(t) = \langle \langle 1 \rangle, \langle 0 \rangle \rangle$, $e(s) = a$ *and* $e(t) = b$. *Then the Petri net* N *is illustrated as follows:*

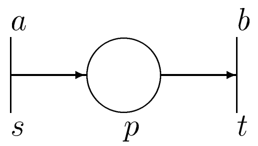

Figure 1: An example of Petri net.

In this diagram, circles and bars represent places and transitions, respectively. The tokens are usually represented by dots in circles.

A transition t is *fireable* in a marking m if ${}^\bullet A(t) \leq m$, and if so, t may *be fired* at m resulting in the marking

$$m' = m - {}^\bullet A(t) + A(t)^\bullet.$$

In this case, we write $m[t\rangle$ or $m[t\rangle m'$. Intuitively, the firing of t removes ${}^\bullet A(t)(p)$ tokens from the place p, and distributes $A(t)^\bullet(p)$ tokens to p.

These definitions and notations are extended to finite or infinite sequences of transitions: $m[t_0 t_1 \cdots t_n\rangle m'$ or $m[t_0 t_1 \cdots t_n\rangle$ if $m[t_0\rangle m_1[t_1\rangle m_2 \cdots m_n[t_n\rangle m'$, and $m[\alpha\rangle$ if $m[\alpha(0)\rangle m_1[\alpha(1)\rangle m_2 \cdots$.

We define the infinite behaviour of a Petri net N as the projective image of infinite firing sequences by the λ-free labeling function e. More precisely, for a

Petri net $N = (P, T, A, e, m_0, F)$ and $\alpha \in T^\omega$, we define $N(\alpha) = m_0 m_1 m_2 \cdots$ if $m_0[\alpha(0)\rangle m_1[\alpha(1)\rangle m_2 \cdots$. Let

$$\uparrow F = \{m' \mid m' \geq m \text{ for some } m \in F\}.$$

Then we consider the following five types of ω-languages accepted by N:

$$L_{True}(N) = \{e(\alpha) \mid m_0[\alpha\rangle\},$$

$$L_{ran\sqcap}(N) = \{e(\alpha) \mid m_0[\alpha\rangle \text{ and } ran(N(\alpha)) \cap \uparrow F \neq \phi\},$$
$$L_{ran\subseteq}(N) = \{e(\alpha) \mid m_0[\alpha\rangle \text{ and } ran(N(\alpha)) \subseteq \uparrow F\},$$
$$L_{inf\sqcap}(N) = \{e(\alpha) \mid m_0[\alpha\rangle \text{ and } inf(N(\alpha)) \cap \uparrow F \neq \phi\}, \text{ and}$$
$$L_{inf\subseteq}(N) = \{e(\alpha) \mid m_0[\alpha\rangle \text{ and } inf(N(\alpha)) \subseteq \uparrow F\}.$$

Here we use the symbol $\sqcap$ to denote the condition that the intersection with $\uparrow F$ is nonempty. In this paper we use γ to denote one of the accepting conditions, $True, ran\sqcap, ran\subseteq, inf\sqcap$ and $inf\subseteq$, considered above. We define $\mathbf{P}_\gamma = \{L_\gamma(N) \mid N \text{ is a Petri net}\}$ for each accepting condition γ.

Example 2 *For the Petri net N over $\Sigma = \{a, b\}$ in the previous example,*

$$L_{True}(N) = \{\alpha \mid \#_a(u) \geq \#_b(u) \text{ for any prefix } u \text{ of } \alpha\},$$

$$L_{ran\sqcap}(N) = L_0(N) - (ab)^\omega,$$
$$L_{ran\subseteq}(N) = \phi,$$
$$L_{inf\sqcap}(N) = L_0(N) - D(ab)^\omega, \text{ and}$$
$$L_{inf\subseteq}(N) = \{u \mid \#_a(u) = \#_b(u) + 2\}L_0(N) \cap L_0(N),$$

where $\#_a(u)$ is the number of occurrence of the letter a in the string u,and $D = \{u \in \Sigma^* \mid \#_a(u) = \#_b(u)\}$ is the Dyck set.

Let $M = (Q, \Sigma, \delta, s, F)$ be a (nondeterministic, partial) *finite automaton* with the finite set Q of *states*, the *input alphabet* Σ, the *transition relation* $\delta \subseteq Q \times \Sigma \times Q$, the *initial state* $s \in Q$, and the set $F \subseteq Q$ of *accepting states*. Any $\alpha = \langle q_0, a_0, p_0\rangle\langle q_1, a_1, p_1\rangle\langle q_2, a_2, p_2\rangle \ldots \in \delta^\omega$ (or $u = \langle q_0, a_0, p_0\rangle\langle q_1, a_1, p_1\rangle \ldots \langle q_n, a_n, p_n\rangle \in \delta^*$) is called a run of M, if $q_0 = s$ and $p_i = q_{i+1}$ for any i (or $i < n$, respectively). Borrowing the notation for Petri nets, we write $s[\alpha\rangle$ (or $s[u\rangle p_n$) in this case.

Then we can also define the following five types of ω-languages accepted by a finite automaton M:

$$L_{True}(M) = \{pr_2(\alpha) \mid s[\alpha\rangle\},$$

$$L_{ran\sqcap}(M) = \{pr_2(\alpha) \mid s[\alpha\rangle \text{ and } ran(pr_1(\alpha)) \cap F \neq \phi\},$$
$$L_{ran\subseteq}(M) = \{pr_2(\alpha) \mid s[\alpha\rangle \text{ and } ran(pr_1(\alpha)) \subseteq F\},$$
$$L_{inf\sqcap}(M) = \{pr_2(\alpha) \mid s[\alpha\rangle \text{ and } inf(pr_1(\alpha)) \cap F \neq \phi\}, \text{ and}$$
$$L_{inf\subseteq}(M) = \{pr_2(\alpha) \mid s[\alpha\rangle \text{ and } inf(pr_1(\alpha)) \subseteq F\}.$$

We define $\mathbf{E}_\gamma = \{L_\gamma(M) \mid M \text{ is a finite automaton}\}$ for each accepting condition γ. Note that a finite automaton M can be naturally considered to be a Petri net N with the following interpretation: a place of N is a state of M, N has only one token which indicates the current state, and each transition of M is represented by a transition of N which causes the corresponding move of the token. Thus, $\mathbf{E}_\gamma \subset \mathbf{P}_\gamma$ for any γ, since the proper inclusion is clear from the above example.

3 Inclusion relations

In the case of ω-languages accepted by (nondeterministic) finite automata, it is known that $\mathbf{E}_{True} = \mathbf{E}_{ran\subseteq} \subset \mathbf{E}_{ran\sqcap} = \mathbf{E}_{inf\subseteq} \subset \mathbf{E}_{inf\sqcap}$ [5, 7, 11]. We show that the similar results hold for the classes $\mathbf{P}_\gamma$ of Petri net ω-languages.

In this section, we introduce a new type of accepting condition for Petri nets which is described by a language over transitions. Let $N = (P, T, A, e, m_0, \phi)$ and $R \subseteq T^\omega$. We define

$$L(N, R) = \{e(\alpha) \mid m_0[\alpha\rangle \text{ and } \alpha \in R\}.$$

Moreover, in the proof of the following theorems, we use the following notations to simplify the description. For $f \in \mathbf{Z}^X$ and $g \in \mathbf{Z}^Y$ $f \oplus g$ denotes the function in $\mathbf{Z}^{X \cup Y}$, defined by

$$f \oplus g(x) = \begin{cases} f(x) + g(x), & \text{if } x \in X \cap Y \\ f(x), & \text{if } x \in X \\ g(x), & \text{if } x \in Y. \end{cases}$$

For $n \in \mathbf{N}$ and a set X, $\mathbf{n}^X$ denote the constant function in $\mathbf{N}^X$ such that $\mathbf{n}^X(x) = n$ for any $x \in X$. If X is a singleton $\{x\}$, then we write $\mathbf{n}^x$ instead of $\mathbf{n}^{\{x\}}$. Thus, for example, for $p_0 \in P$,

$$\mathbf{0}^P \oplus \mathbf{1}^{p_0}(p) = \begin{cases} 1, & \text{if } p = p_0 \\ 0, & \text{if } p \neq p_0. \end{cases}$$

Theorem 1 *For any accepting condition γ, $\mathbf{P}_\gamma = \{L(N, R) \mid N$ is a Petri net and $R \in \mathbf{E}_\gamma\}$.*

Proof. Let $N = (P, T, A, e, m_0, \phi)$ and $M = (Q, T, \delta, s, F)$ be a finite automaton such that $L_\gamma(M) = R$. We construct a Petri net which accepts the ω-language $L(N, R)$ in the sense of γ. Define the Petri net

$$N \times M = (P \cup Q, \delta, A', e', m_0 \oplus \mathbf{0}^Q \oplus \mathbf{1}^s, \{\mathbf{0}^P \oplus \mathbf{0}^Q \oplus \mathbf{1}^q \mid q \in F\}),$$

where

$$A'(\langle q, t, q' \rangle) = \langle {}^\bullet A(t) \oplus \mathbf{0}^Q \oplus \mathbf{1}^q, A(t)^\bullet \oplus \mathbf{0}^Q \oplus \mathbf{1}^{q'} \rangle,$$

and

$$e'(\langle q, t, q' \rangle) = e(t)$$

for any $\langle q, t, q' \rangle$.

The Petri net $N \times M$ simulates N and M simultaneously. More precisely, for any $w \in \delta^*$, $m \oplus \mathbf{0}^P \oplus \mathbf{1}^q[w\rangle m' \oplus \mathbf{0}^P \oplus \mathbf{1}^{q'}$ in $N \times M$ if and only if $m[pr_2(w)\rangle m'$ in N and $q[w\rangle q'$ in M. Hence it is easy to see that $L(N, R) = L(N, L_\gamma(M)) = L_\gamma(N \times M)$.

Assume $L = L_\gamma(N)$ with $N = (P, T, A, e, m_0, F)$. For each $t \in T$ and $m \in F$, we add a new transition t_m to N, such that $m_1[t_m\rangle m_2$ if and only if $m_1 \geq m$ and $m_1[t\rangle m_2$. Since $m_1 \geq m \in F$ means $m_1 \in \uparrow F$, t_m works same as t, and can check whether the current marking is in $\uparrow F$ or not.

More precisely, we construct the Petri net

$$N' = (P, T \cup T_F, A', e', m_0, \phi),$$

where

$$T_F = \{t_m \mid t \in T \text{ and } m \in F\},$$

$$A'(t) = A(t) \text{ and } e'(t) = e'(t_m) = e(t),$$

for each $t \in T$ and $m \in F$. Moreover,

$$A'(t_m) = \langle {}^\bullet A(t) \sqcup m, ({}^\bullet A(t) \sqcup m) - {}^\bullet A(t) + A(t)^\bullet \rangle,$$

where ${}^\bullet A(t) \sqcup m$ is the least upper bound of the vectors ${}^\bullet A(t)$ and m.

Then it is clear that $L_{True}(N) = L(N', T^\omega)$, $T^\omega \in \mathbf{E}_{True}$, $L_{ran\sqcap}(N) = L(N', T^*T_FT^\omega)$, $T^*T_FT^\omega \in \mathbf{E}_{ran\sqcap}$, $L_{ran\subseteq}(N) = L(N', T_F^\omega)$, $T_F^\omega \in \mathbf{E}_{ran\subseteq}$, $L_{inf\sqcap}(N) = L(N', (T^*T_F)^\omega)$, $(T^*T_F)^\omega \in \mathbf{E}_{inf\sqcap}$, $L_{inf\subseteq}(N) = L(N', T^*T_F^\omega)$ and $T^*T_F^\omega \in \mathbf{E}_{inf\subseteq}$. **Q.E.D.**

Remark. The above theorem and its proof indicate that the class $\mathbf{P}_\gamma$ can be defined by Petri nets with designated set of transitions, or equivalently, with finite control whose accepting condition γ is given on the finite control in the sense of finite automata. Thus, the classes $\mathbf{P}_\gamma$ coincide to the classes $\mathbf{K}_\omega^i$ and $\mathbf{B}_\omega^i$ defined by Valk [8, 9]. The following inclusion relations on the classes $\mathbf{P}_\gamma$ are proved in [8, 9], but we give the proofs for the self-completeness of this paper. See also [2].

Corollary 2 $\mathbf{P}_{True} = \mathbf{P}_{ran\subseteq} \subseteq \mathbf{P}_{ran\sqcap} = \mathbf{P}_{inf\subseteq} \subseteq \mathbf{P}_{inf\sqcap}$

Proof. Since $\mathbf{E}_{True} = \mathbf{E}_{ran\subseteq} \subset \mathbf{E}_{ran\sqcap} = \mathbf{E}_{inf\subseteq} \subset \mathbf{E}_{inf\sqcap}$, it is clear from Theorem 1. **Q.E.D.**

In the sequel, we consider the classes $\mathbf{P}_\gamma$ for $\gamma \in \{True, ran\sqcap, inf\sqcap\}$. So we again use γ to denote one of the accepting conditions $True, ran\sqcap$ and $inf\ \sqcap$. To show the strict inclusions between these classes, we prove the following topological properties of the classes $\mathbf{P}_{True}$ and $\mathbf{P}_{ran\sqcap}$.

Lemma 3 *For any Petri net N, $L_{True}(N)$ is a closed set, and $L_{ran\sqcap}(N)$ is a denumerable union of closed sets.*

Proof. Let $N = (P, T, A, e, m_0, F)$. Since the set $T(N) = \{\alpha \mid m_0[\alpha\rangle\}$ is a closed set, $L_{True}(N) = e(T(N))$ is closed.

Let $N_m = (P, T, A, e, m, F)$ for $m \in \mathbf{N}^P$. Then,

$$L_{ran\sqcap}(N) = \bigcup\{e(w)L_{True}(N_m) \mid m_0[w\rangle m \in \uparrow F\},$$

which is a denumerable union of closed sets. **Q.E.D.**

The next theorem follows from the topological characterizations of ω-regular languages, and we can find the following examples in [5, 7].

$$a^*b^\omega \in \mathbf{P}_{ran\sqcap} - \mathbf{P}_{True}$$

$$(a^*b)^\omega \in \mathbf{P}_{inf\sqcap} - \mathbf{P}_{ran\sqcap}$$

Theorem 4 $\mathbf{P}_{True} = \mathbf{P}_{ran\subseteq} \subset \mathbf{P}_{ran\sqcap} = \mathbf{P}_{inf\subseteq} \subset \mathbf{P}_{inf\sqcap}$.

Here, we show some closure properties.

Theorem 5 *The classes $\mathbf{P}_\gamma$ of Petri net ω-languages are closed under union, intersection, projections and inverse projections.*

Proof. The closure under projections and inverse projections are clear from the definition.

Let $N_j = (P_j, T_j, A_j, e_j, m_j, \phi)$ for $j = 1, 2$. We may assume that $P_1 \cap P_2 = \phi$ and $T_1 \cap T_2 = \phi$. To show the closure under union, we define a Petri net N which can simulate N_1 and N_2 independently as follows:

$$N = (P_1 \cup P_2, T_1 \cup T_2, A, e, m_1 \oplus m_2, \phi),$$

where

$$A(t_1) = \langle {}^\bullet A_1(t_1) \oplus \mathbf{0}^{P_2}, A_1(t_1)^\bullet \oplus \mathbf{0}^{P_2}\rangle,$$
$$A(t_2) = \langle \mathbf{0}^{P_1} \oplus {}^\bullet A_2(t_2), \mathbf{0}^{P_1} \oplus A_2(t_2)^\bullet\rangle,$$
$$e(t_1) = e_1(t_1),\ \text{and}\ e(t_2) = e_2(t_2)$$

for any $t_1 \in T_1$ and $t_2 \in T_2$. Then it is clear that

$$L(N_1, R_1) \cup L(N_2, R_2) = L(N, R_1 \cup R_2).$$

Moreover, since $\mathbf{E}_\gamma$ is closed under union, $R_1 \cup R_2$ is in the same class with R_1 and R_2.

To show the closure under intersection, we define a Petri net $N_1 \times N_2$ which can simulate N_1 and N_2 simultaneously, as follows:

$$N_1 \times N_2 = (P_1 \cup P_2, T_1 \times T_2, A, e, m_1 \oplus m_2, \phi),$$

where

$$A(\langle t_1, t_2\rangle) = \langle {}^\bullet A_1(t_1) \oplus {}^\bullet A_2(t_2), A_1(t_1)^\bullet \oplus A_2(t_2)^\bullet\rangle,$$
$$e(\langle t_1, t_2\rangle) = e_1(t_1),$$

for any $t_1 \in T_1$ and $t_2 \in T_2$.

Then it is easy to see that for any $\langle u, v\rangle \in (T_1 \times T_2)^*$, $m_1 \oplus m_2[\langle u, v\rangle\rangle m_1' \oplus m_2'$ if and only if $m_1[u\rangle m_1'$ and $m_2[v\rangle m_2'$.

For any $R_j \subseteq T_j^\omega$ $(j = 1, 2)$ in $\mathbf{E}_\gamma$, let

$$R = \{\alpha \in (T_1 \times T_2)^\omega \mid pr_1(\alpha) \in R_1 \text{ and } pr_2(\alpha) \in R_2 \text{ and } e_1(pr_1(\alpha)) = e_2(pr_2(\alpha))\}.$$

Then it is clear that

$$L(N_1, R_1) \cap L(N_2, R_2) = L(N, R).$$

Moreover, since $R = pr_1^{-1}(R_1) \cap pr_2^{-1}(R_2) \cap \{\langle t_1, t_2\rangle \mid e_1(t_1) = e_2(t_2)\}^\omega$, R is in $\mathbf{E}_\gamma$. **Q.E.D.**

4 Normal form of Petri nets

In this section, we define a normal form of Petri nets and show that any Petri net can be transformed into a normal form Petri net.

We say that a Petri net $N = (P, T, A, e, m_0, \phi)$ is in *normal form* if
1) $m_0 = \mathbf{0}^P$,
2) for any $p \in P$, and $t \in T$, ${}^\bullet A(t)(p) \leq 1$ and $A(t)^\bullet(p) \leq 1$, that is, each place p gets or loses at most one token at once.

Theorem 6 *For any Petri net $N = (P, T, A, e, m_0, \phi)$ and $R \subseteq T^\omega$, we can construct a Petri net $N' = (P', T', A', m_0', e', \phi)$ in normal form and $R' \subseteq T'^\omega$ such that $L(N, R) = L(N', R')$. Moreover R and R' are in the same class $\mathbf{E}_\gamma$.*

Proof. First we show that any Petri net $N = (P, T, A, e, m_0, \phi)$ can be transformed into a Petri net $N' = (P', T', A', \mathbf{0}^P, e', \phi)$ which satisfies the condition 1). Let $P' = P$ and $T' = T \cup \{t' \mid m_0[t\rangle\}$. We define

$$A'(t) = A(t),$$

$$A'(t') = \langle \mathbf{0}^P, (m_0 - {}^\bullet A(t) + A(t)^\bullet)\rangle,$$
$$e'(t) = e'(t') = e(t),$$

and

$$R' = \{t'v \mid tv \in R \text{ and } m_0[t\rangle\}.$$

Then Petri net N' satisfies 1). It is clear from the construction that $L(N, R) = L(N', R')$, and R and R' are in the same class $\mathbf{E}_\gamma$.

Now, we assume that a Petri Net $N = (P, T, A, e, \mathbf{0}^P, \phi)$ satisfies the condition 1). We show that we can decrease the number of places $q \in P$ such that $Max\{{}^\bullet A(t)(q), A(t)^\bullet(q) \mid t \in T\} = n > 1$. Repeating the process, we can transform N into a Petri net satisfying conditions 1) and 2).

In the process, we replace the place q of N by n new places $q_1, q_2, \cdots, q_n$. Moreover, for each transition t of N, let

$$D_i^t \ (1 \leq i \leq k_t) \text{ and } E_j^t \ (1 \leq j \leq l_t)$$

be the enumeration of the subsets of $\{q_1, q_2, \cdots, q_n\}$ with ${}^\bullet A(t)(q)$ and $A(t)^\bullet(q)$ elements, respectively. Then we replace the transition t by $k_t \times l_t$ transitions $t_{i,j}$ $(1 \leq i \leq k_t, 1 \leq j \leq l_t)$ and we give input- and output-vectors and labels for $t_{i,j}$ as follows:

$$^\bullet A(t_{i,j})(p) = \begin{cases} {}^\bullet A(t)(p), & \text{if } p \neq q \\ 1, & \text{if } p \in D_i^t \\ 0, & \text{if } p \notin D_i^t \end{cases},$$

$$A(t_{i,j})^\bullet(p) = \begin{cases} A(t)^\bullet(p), & \text{if } p \neq q \\ 1, & \text{if } p \in E_j^t \\ 0, & \text{if } p \notin E_j^t \end{cases}$$

and $e(t_{i,j}) = e(t)$.

Note that on the Petri net obtained by this process, the tokens in q on N are distributed to the places $q_1, q_2, \cdots, q_n$, and the effect of the firing of t

for q on N is accomplished by firing one of the transitions $t_{i,j}$ for the group of places $\{q_1, \ldots, q_n\}$.

Then it is easy to see that $L(N, R) = L(N', \psi^{-1}(R))$, where N' is the result of the above process, and ψ is the projection defined by $\psi(t_{i,j}) = t$ and $\psi(s) = s$ for $s \neq t$. Since $\mathbf{E}_\gamma$ is closed under the inverse projections, R and $\psi^{-1}(R)$ are in the same class. **Q.E.D.**

5 Characterizations by logical formulas

We define the monadic second-order theory K over an alphabet Σ for natural numbers, following to the one given in [4]. K has two sorts of variables, number variables $x, y, \ldots$ ranging over $\mathbf{N}$, and set variables $X, Y, \ldots$ ranging over the power set of $\mathbf{N}$. K also has set constants C_a for each $a \in \Sigma$.

The terms of K are expressions of form n or $x + n$, where x is a number variable and n is a constant in $\mathbf{N}$. The atomic formulas of K are expressions of form $u \leq t$, $t \in W$ or $V \preceq W$, where u, t are terms and V, W are set variables or C_a for some $a \in \Sigma$. Here, $\leq$ and $\in$ are usual 'less than or equal to' and 'belongs to' relations, and $V \preceq W$ is true if and only if there exists a one to one function $f : W \to V$ such that $f(x) \leq x$ for any $x \in W$.

The formulas of K, called K-formulas, are defined as usual: $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$, $\forall x\varphi$, $\exists x\varphi$, $\boldsymbol{\forall} X\varphi$, and $\boldsymbol{\exists} X\varphi$ are formulas for any formulas or atomic formulas φ, ψ, number variable x and set variable X. We use boldface quantifier symbols $\boldsymbol{\forall}$ and $\boldsymbol{\exists}$ for set variables to distinguish from those for number variables.

Note that the K-formulas not containing the symbol $\preceq$ is the S1S formulas studied in Büchi [1].

We say that an ω-word $\alpha \in \Sigma^\omega$ satisfies K-sentence (i.e., formulas without free variables) ψ, if ψ is true under the interpretation $C_a = \{n \mid \alpha(n) = a\}$. Then, K-sentence ψ define the set $L(\psi)$ of all ω-words satisfying ψ. For a set of K-formulas Δ, we define that $\mathbf{L}(\Delta) = \{L(\psi) \mid \psi \in \Delta\}$, the class of ω-languages defined by the sentences in Δ.

For a regular expression R over quantifier symbols $\{\forall, \exists, \boldsymbol{\forall}, \boldsymbol{\exists}\}$, $[R]$ denotes the set of S1S formulas of the prenex normal form

$$\Xi_1\xi_1\Xi_2\xi_2 \cdots \Xi_n\xi_n \ \psi(\xi_1, \xi_2, \cdots, \xi_n),$$

where $\Xi_1\Xi_2 \cdots \Xi_n$ is a string represented by R, and ψ is a quantifier-free formula.

On the relation between S1S formulas and ω-regular languages, we have shown the following theorem [10].

Theorem 7 *$\mathbf{E}_{True} = \mathbf{L}([\exists^*\forall])$, $\mathbf{E}_{ran\sqcap} = \mathbf{L}([\exists^*\exists\forall])$ and $\mathbf{E}_{inf\sqcap} = \mathbf{L}([\exists^*\forall\exists])$.*

Let Δ be a set of S1S formulas and $\overline{\Delta}$ be the smallest set of K-formulas constructed from the formulas in Δ and formulas $V \preceq W$ using logical symbols $\wedge$, $\vee$, $\exists$ and $\exists$.

Now, we show the main theorem.

Theorem 8 *If $\mathbf{L}(\Delta) = \mathbf{E}_\gamma$ then $\mathbf{L}(\overline{\Delta}) = \mathbf{P}_\gamma$ for any accepting condition γ.*

Proof. In the proof, we say that $\alpha \in (\{0,1\}^{n+k} \times \Sigma)^\omega$ satisfies the formula $\psi(X_1, \ldots, X_n, x_1, \ldots, x_k)$, if $pr_{n+k+1}(\alpha) \in \Sigma^\omega$ satisfies $\psi(C_1, \ldots, C_n, d_1, \ldots, d_k)$, where $C_i = \{j \mid pr_i(\alpha)(j) = 1\}$ for $i = 1, \ldots, n$ and $pr_{n+i}(\alpha)(j) = 1$ if and only if $j = d_i$ for $i = 1, \ldots, k$.

Let $\mathbf{L}(\Delta) = \mathbf{E}_\gamma$. First we note that a Petri net can simulate a nondeterministic finite automaton with trivial correspondence between transitions. Thus if $\psi \in \Delta$ then $L(\psi) \in \mathbf{P}_\gamma$. Moreover the ω-language ($\subseteq (\{0,1\}^2)^\omega$) defined by the formula $X_1 \preceq X_2$ is accepted by the following Petri net without the accepting condition, that is, $L(X_1 \preceq X_2) \in \mathbf{P}_{True} \subseteq \mathbf{P}_\gamma$.

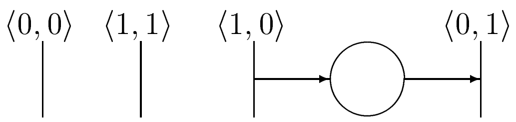

Figure 2: A petri net accepting $L(X_1 \preceq X_2)$.

Since each class of Petri net ω-languages is closed under union, intersection, projections and inverse projections, we have shown that $\mathbf{L}(\overline{\Delta}) \subseteq \mathbf{P}_\gamma$.

Let $N = (P, T, A, e, \mathbf{0}^P, \phi)$ be a Petri net in normal form. Note that in normal form Petri net, each place p can get or lose at most one token at once. To describe the infinite behaviour of the net N, we use the following set variables.

- X_t for each $t \in T$, to represent when t fires
- E_p for each $p \in P$, to represent when p gets a new token
- S_p for each $p \in P$, to represent when p loses one token.

Let

$$\psi_1(x) = \bigvee_{t \in T} ((x \in X_t) \wedge (x \in C_{e(t)}) \wedge (\bigwedge_{t' \neq t} \neg(x \in X_{t'})))$$

which means that there exists a unique transition t that fires at time x. Let

$$\psi_2(x) = \bigwedge_{p \in P} ((x \in S_p) \leftrightarrow \bigvee_{\bullet A(t)(p)=1} (x \in X_t))$$

which means that a place p loses a token at time x if and only if a transition t with ${}^{\bullet}A(t)(p) = 1$ fires at the same time x. Let

$$\psi_3(x) = \bigwedge_{p \in P} ((x+1 \in E_p) \leftrightarrow \bigvee_{A(t)^{\bullet}(p)=1} (x \in X_t))$$

which means that a place p gets a token at time $x+1$ if and only if a transition t with $A(t)^{\bullet}(p) = 1$ fires at time x. Let

$$\psi_4 = \bigwedge_{p \in P} \neg(0 \in E_p)$$

which represents the condition for the initial marking. Let

$$\psi_5 = \bigwedge_{p \in P} (E_p \preceq S_p)$$

which means that each place p can lose only tokens which p got previously.

Finally, let φ be an S1S formula defining the ω-regular language $R \subseteq T^{\omega}$ using X_t as the set constant for $t \in T$ and $\Phi = \forall x(\psi_1(x) \wedge \psi_2(x) \wedge \psi_3(x) \wedge \psi_4) \wedge \varphi$.

Then, $L(N, R)$ is defined by

$$\exists_{t \in T} X_t \exists_{p \in P} E_p \exists_{p \in P} S_p (\Phi \wedge \psi_5).$$

Note that all $\psi_1, \ldots, \psi_4$ are S1S-formulas with no quantifiers. Thus, $L(\Phi)$ belongs to the same class $\mathbf{E}_{\gamma}$ with $L(\varphi) = R$. Therefore, if $\varphi \in \Delta$ with $\mathbf{L}(\Delta) = \mathbf{E}_{\gamma}$, then there exists a formula in Δ which is equivalent to Φ. It means that $L(N, R) \in \mathbf{L}(\overline{\Delta})$. Thus $\mathbf{P}_{\gamma} \subseteq \mathbf{L}(\overline{\Delta})$. **Q.E.D.**

For a language R over quantifier symbols $\{\forall, \exists, \forall, \exists\}$, let $[R]_K$ denote the set of K-sentences of the prenex normal form

$$\Xi_1 \xi_1 \Xi_2 \xi_2 \cdots \Xi_n \xi_n \ \psi(\xi_1, \xi_2, \cdots, \xi_n),$$

where $\Xi_1 \Xi_2 \cdots \Xi_n$ is a string in R, and ψ is a quantifier-free formula constructed from the atomic formulas $V \preceq W$ and S1S-formulas using $\wedge$ and $\vee$.

Corollary 9 $\mathbf{P}_{True} = \mathbf{L}([\exists^* \forall]_K)$, $\mathbf{P}_{ran\sqcap} = \mathbf{L}([\exists^* \exists \forall]_K)$ *and* $\mathbf{P}_{inf\sqcap} = \mathbf{L}([\exists^* \forall \exists]_K)$.

6 Decomposition of Petri nets

Let $N = (P, T, A, e, m_0, \phi)$ be a normal form Petri net over Σ, and R be an ω-regular language over the transitions T of N. Then, intuitively speaking, the machine recognizing $L(N, R)$ can be decomposed into the form $M_{e^{-1}} \circ N \circ M_R$, where $M_{e^{-1}}$ is a nondeterministic finite transducer which converts an input ω-word $\alpha \in \Sigma^\omega$ into an ω-word $\beta \in e^{-1}(\alpha) \subseteq T^\omega$ of transitions nondeterministically, N checks whether β is fireable in N or not, and M_R is a (deterministic) finite automaton accepting R. We use $M \circ N$ to denote the usual notion of composition of functions induced from machines, that is, the output of M is used as the input of N, one by one.

Moreover, the K-formula expressing $L(N, R)$ in the last section suggests the decomposition of a Petri net into a finite automaton and some one-place loop-free Petri nets in Figure 2.

In this section we show that any Petri net N is decomposed into the form $M_{e^{-1}} \circ N_1 \circ \cdots \circ N_k \circ M_R$, where each N_i is a one-place loop-free Petri net with the identity labeling function. Here, we consider a Petri net $N = (P, T, A, id, m_0, \phi)$ with the identity labeling function id, as a transducer from T^ω to T^ω, which checks whether a given input ω-word is fireable or not. That is, for any $\alpha \in T^\omega$, $N(\alpha) = \alpha$ if α is fireable in N, and $N(\alpha)$ is undefined otherwise.

Lemma 10 *For any normal form Petri net $N = (P, T, A, id, \mathbf{0}, \phi)$, there exist one-place normal form Petri nets $N_1, \ldots, N_k$ with the same set T of transitions, such that $N = N_1 \circ \cdots \circ N_k$.*

Proof. For each place $p \in P$, let $N_p = (\{p\}, T, A_p, id, \langle 0 \rangle, \phi)$, where $A_p(t) = \langle \langle {}^\bullet A(t)(p) \rangle, \langle A(t)^\bullet(p) \rangle \rangle$ is the restriction of $A(t)$ to $\{p\}$ for each $t \in T$. Then, it is easy to see that

$$N = N_1 \circ \cdots \circ N_k.$$

Q.E.D.

Next, utilizing finite control, we delete all loop transitions t with $A(t) = \langle \langle 1 \rangle, \langle 1 \rangle \rangle$ from one-place normal form Petri net N.

Lemma 11 *For any one-place normal form Petri net $N_p = (\{p\}, T, A, \langle 0 \rangle, id, \phi)$, there exist a one-place normal form loop-free Petri net $N'_p = (\{p\}, T', A', \langle 0 \rangle, e', \phi)$, and a regular ω-language $R'_p \subseteq T'^\omega$, such that $L_{True}(N_p) = L(N'_p, R'_p)$.*

Proof. To simplify the proof, we assume that $N_p = (\{p\}, \{a, b, c, d\}, A, \langle 0\rangle, id, \phi)$, with $A(a) = \langle\langle 0\rangle, \langle 1\rangle\rangle$, $A(b) = \langle\langle 1\rangle, \langle 0\rangle\rangle$, $A(c) = \langle\langle 1\rangle, \langle 1\rangle\rangle$, and $A(d) = \langle\langle 0\rangle, \langle 0\rangle\rangle$.

Let $N'_p = (\{p\}, \{a, a', b, b', c, c', d, d'\}, A', \langle 0\rangle, e', \phi)$, with $A'(a) = A'(a') = \langle\langle 0\rangle, \langle 1\rangle\rangle$, $A'(b) = A'(c) = \langle\langle 1\rangle, \langle 0\rangle\rangle$, $A'(b') = A'(c') = A'(d) = A'(d') = \langle\langle 0\rangle, \langle 0\rangle\rangle$, $e'(a) = e'(a') = a$, $e'(b) = e'(b') = b$, $e'(c) = e'(c') = c$, and $e'(d) = e'(d') = d$. Let R'_p be the ω-regular languages defined by the automaton $M = (\{0, 1\}, \{a, a', b, b', c, c', d, d'\}, \delta, 0, \phi)$ with $\delta = \{(0, a, 0), (1, a', 1), (0, d, 0), (1, d', 1), (0, c, 1), (1, c', 1), (0, b, 0), (1, b', 0)\}$.

Then it is easy to see that for any $u \in \{a, b, c, d\}^*$, $\langle 0\rangle[u\rangle\langle k\rangle$ in N if and only if for some $v \in \delta^*$ and $i \in \{0, 1\}$ $\langle 0\rangle[pr_2(v)\rangle\langle(k - i)\rangle$ in N', $0[v\rangle i$ in M and $e'(pr_2(v)) = u$.

Thus, $T(N_p) = L(N'_p, R'_p)$. Note that the set T' of new transitions are defined independent from the structure of the original net N. The symbols a' and d' are only introduced to hold this property. **Q.E.D.**

Theorem 12 *For any Petri net N over Σ and ω-regular language R over the set of transitions of N, there exist a set of transitions T', a projection $h : T' \to \Sigma$, an ω-regular language $R' \subseteq T'^{\omega}$, and normal form one-place loop-free Petri nets $N_1, \ldots, N_k$ with the set T' of transitions such that (N, R) is decomposed into the form $M_{h^{-1}} \circ N_1 \circ \cdots \circ N_k \circ M_{R'}$.*

Proof. We may assume that N is a normal form Petri net and (N, R) is decomposed into the form $M_{e^{-1}} \circ N \circ M_R$. First, from lemma 10 we decompose N into the form $N = N_1 \circ \cdots \circ N_k$. Next, from lemma 11 each N_i is decomposed into a one-place normal form loop-free Petri net N'_i with the set T' of transitions and an ω-regular language R_i over T'. Let $g : T' \to T$ be a natural injection. Then, it is easy to see that $(N, R) = M_{(g \circ e)^{-1}} \circ N'_1 \circ \cdots \circ N'_k \circ M_{R'}$, where $R' = R_1 \cap \cdots \cap R_k \cap g^{-1}(R)$. **Q.E.D.**

7 Concluding remarks

In this paper, we define several classes of Petri net ω-languages and study them in relation with the classes of ω-regular languages. We give the logical characterization of the classes of Petri net ω-languages, and the decomposition of a Petri net into finite automata and one-place normal form Petri nets without loops.

We have considered only λ-free Petri nets. It is natural to allow the transitions labeled with the empty word λ. But then, some infinite firing sequence may generate the word of finite length, and we can not always bound the markings reachable with λ-transitions from the given marking. These facts may bring some difficulties into the study of Petri net ω-languages in contrast with the case of regular ω-languages.

Acknowledgment. I would like to thank Ludwig Staiger for inviting me to the Martin-Luther-Universtät Halle-Wittenberg and many valuable discussions for improving this paper.

References

[1] J.R. Büchi, On a decision method in restricted second-order arithmetic, *Logic, Methodology and Philosophy of Science* (Stanford Univ. Press, 1960) 1–11.

[2] J. Engelfriet and H.J. Hoogeboom, X-automata on ω-words, *T.C.S.* 110 (1993) 1, 1–51.

[3] M. Parigot and E. Pelz, A logical formalism for the study of the finite behaviour of Petri nets, *Advances on Petri nets 1985, LNCS* 222, 346–361.

[4] E. Pelz, ω-languages of Petri nets and logical sentences, *Proceedings of Application and Theory of Petri Nets* 1986.

[5] L. Staiger and K. Wagner, Automatentheoretische und automatenfreie Charakterisierungen topologischer Klassen regulärer Folgenmengen, *E.I.K.* 10 (1974) 379–392.

[6] T. Moriya and H. Yamasaki, Accepting conditions for automata on ω-words, *T.C.S.* 61 (1988), 137-147.

[7] M. Takahashi and H. Yamasaki, A note on ω-regular languages, *T.C.S.* 23 (1983) 217–225.

[8] R. Valk, Infinite behaviour of Petri nets, *T.C.S* 25 (1983) 311–341.

[9] R. Valk, Infinite behaviour and fairness in Petri nets, *LNCS 188* Springer-Verlag (1985) 83–100.

[10] H. Yamasaki, M. Takahashi and K. Kobayashi, Characterization of ω-regular languages by monadic second-order formulas, *T.C.S.* 46 (1986) 91–99.

[11] K. Wagner, On ω-regular sets, *Inform. and Contr.* 43 (1979) 123–177.