The Kolmogorov Complexity of Liouville Numbers*

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Abstract

We consider for a real number α the Kolmogorov complexities of its expansions with respect to different bases. In the paper it is shown that, for usual and self-delimiting Kolmogorov complexity, the complexity of the prefixes of their expansions with respect to different bases r and b are related in a way which depends only on the relative information of one base with respect to the other.

More precisely, we show that the complexity of the length $l \cdot \log_r b$ prefix of the base r expansion of α is the same (up to an additive constant) as the $\log_r b$-fold complexity of the length l prefix of the base b expansion of α.

Then we use this fact to derive complexity theoretic proofs for the base independence of the randomness of real numbers and for some properties of Liouville numbers.

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Contents

Kolmogorov Complexity is mainly attributed to finite strings over a finite alphabet. As a function or, more coarsely, as a limit it measures the complexity of infinite strings.

Real numbers are described by their (infinite) r-ary expansions. Thus, choosing the base r, we may attribute Kolmogorov complexity also to real numbers, however, relative to the chosen base. Consequently, it might happen that the Kolmogorov complexity of a real number depends on the chosen base r.

Particular cases, where a property of a real number depends an the base r are disjunctiveness and Borel normality. An infinite r-ary expansion ξ of the real number $\nu_r(\xi) := 0.\xi$ is called disjunctive provided every finite r-ary string appears as an infix of ξ. Borel normality is defined in a similar way, taking into account also the relative frequencies of the infixes. For more detailed information see, e. g. , [Ca94, He96]. It was already shown in [Cs59, Sc60] that Borel normality and disjunctiveness are not invariant under changes of the base r. On the other hand, it was shown in [CJ94], and in another context in [HW98], that the property of randomness of an infinite expansion of a real number is invariant under base change. Besides that it was claimed in [CH94] that the Kolmogorov complexity (as a limit) does not depend on the chosen base r.

In this note we investigate in more detail the Kolmogorov complexities, $K_r(\xi/n)$ of expansions ξ of a real number with respect to different bases r. We show that, if a real number is expanded in the scales of r and b, respectively, then complexity of the length $l \cdot \log_r b$ prefix of the base r expansion of is the same (up to an additive constant) as the $\log_r b$-fold complexity of the length l prefix of the base b expansion.

This result provides a third proof of the fact that randomness is base invariant for real numbers. Next we investigate the complexity of Liouville numbers, a kind of real numbers famous for an elegant and constructive proof of the existence of transcendental real numbers. Finally, utilizing our complexity theoretic arguments, we calculate the Hausdorff dimension of the set of Liouville numbers and investigate disjunctive Liouville numbers.

1 Notation and Preliminaries

By $\mathbb{N} = \{0,1,2,\ldots\}$ we denote the set of natural numbers. In order to treat the Kolmogorov complexities for arbitrary alphabets we let $X_r := \{0,\ldots,r-1\}$ be our alphabet of cardinality $\mathrm{card} X_r = r$, $r \in \mathbb{N}, r \geq 2$. By X_r^* we denote the set of finite strings (words) on X_r, including the *empty* word e. We consider also the space X_r^ω of infinite sequences (ω-words) over X_r. For $w \in X_r^*$ and $\eta \in X_r^* \cup X_r^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious

way to subsets $W \subseteq X_r^*$ and $F \subseteq X_r^* \cup X_r^\omega$.
By $w \sqsubseteq \eta$ we denote the prefix relation, that is, $w \sqsubseteq \eta$ if and only if there is an η' such that $w \cdot \eta' = \eta$.
For $\eta \in X_r^* \cup X_r^\omega$ we denote by $\nu_r(\eta) := 0.\eta$ the real number with (finite or infinite) base r expansion η.
We will consider the self-delimiting as well as the non self-delimiting complexity (cf. [Ca94, LV93]). To this end we fix for every $r \in \mathbb{N}$ a universal algorithm $U_r : X_r^* \to X_r^*$ and a universal self-delimiting algorithm $C_r : X_r^* \to X_r^*$, the domain of the latter is a prefix-free subset of X_r^*. Moreover we fix a recursive standard bijection between $\mathbb{N}$ and X_r^*, $r\text{-}\mathsf{string} : \mathbb{N} \to X_r^*$. For the sake of convenience we agree that $r\text{-}\mathsf{string}(n)$ is the nth string in the quasilexicographical order of X_r^*. Then $|r\text{-}\mathsf{string}(n)| = \lfloor \log_r(n(r-1)+1) \rfloor \leq 1 + \log_r \max\{n, 1\}$.
The *Kolmogorov complexity* of a word $w \in X_r^*$ is defined as $K_r(w) := \inf\{|\pi| : \pi \in X_r^* \wedge U_r(\pi) = w\}$. Accordingly, the *self-delimiting Kolmogorov complexity* of a word $w \in X_r^*$ is $H_r(w) := \inf\{|\pi| : \pi \in X_r^* \wedge C_r(\pi) = w\}$.
In order to prove our results we need the following slight modifications of Theorem 5.1.b.ii in [Ca94] and Theorem 3.5 in [CC96]. We call a function $f : M \to M'$ of *bounded ambiguity* provided there is a $k \in \mathbb{N}$ such that for every $m \in M'$ the preimage $f^{-1}(m)$ has no more than k elements, and we call a function $h : \mathbb{N} \to \mathbb{N}$ *semi-computable from above* if the set $M_h := \{(n, j) : h(n) \leq j\}$ is recursively enumerable.

Theorem 1 *1. Let $f : \mathbb{N} \to X_r^*$ be a recursive function of bounded ambiguity. Then $\sum_{n \in \mathbb{N}} r^{-H_r(f(n))} < \infty$.*

2. If $g : \mathbb{N} \to X_r^*$ is recursive and $h : \mathbb{N} \to \mathbb{N}$ is semi-computable from above such that $\sum_{n \in \mathbb{N}} r^{-h(n)} < \infty$ then

$$\exists c (c \in \mathbb{N} \wedge \forall n (n \in \mathbb{N} \to H_r(g(n)) \leq h(n) + c)) \ .$$

Proof. 1. It is well-known that the self-delimiting complexity satisfies the inequality $\sum_{w \in X_r^*} r^{-H_r(w)} < \infty$ (see [Ca94, LV93]). Let $\operatorname{card} f^{-1}(w) \leq k$ for every $w \in X_r^*$. Then

$$\sum_{n \in \mathbb{N}} r^{-H_r(f(n))} = \sum_{w \in X_r^*} \operatorname{card} f^{-1}(w) \cdot r^{-H(w)} \leq k \cdot \sum_{w \in X_r^*} r^{-H_r(w)} < \infty \ .$$

2. If $\sum_{n \in \mathbb{N}} r^{-h(n)} < \infty$ then also $\sum\limits_{n \in \mathbb{N}} \sum\limits_{j \geq h(n)} r^{-j} = \frac{r}{r-1} \cdot \sum\limits_{n \in \mathbb{N}} r^{-h(n)} < \infty$. Consequently, there is an $m \in \mathbb{N}$ such that $\sum_{n \in \mathbb{N}} \sum_{j \geq h(n)} r^{-h(n)-m} \leq 1$.
Let $f_h : \mathbb{N} \to X_r^* \times \mathbb{N}$ be a recursive function enumerating the recursively enumerable set $M_h := \{(r\text{-}\mathsf{string}(n), j) : j \geq h(n) + m\}$. Above we derived the inequality

$\sum_{(r\text{-string}(n),j)\in M_h} r^{-j} \leq 1$. Thus, according to the Kraft-Chaitin Theorem (Theorem 4.17 in [Ca94]) there is a mapping $C : X_r^* \to X_r^*$ with prefix-free domain such that $C(w_{n,j}) = r\text{-string}(n)$ for some word $w_{n,j} \in X_r^j$ whenever $(r\text{-string}(n), j) \in M_h$.
Then $C' := g \circ r\text{-string}^{-1} \circ C : X_r^* \to X_r^*$ is a partial recursive function with the same prefix-free domain as C and $C'(w_{n,j}) = g(n)$ for all $n, j \in \mathbb{N}$. Since $H_r(g(n)) \leq H_{C'}(g(n)) + c$ where $H_{C'}(w) := \inf\{|\pi| : \pi \in X_r^* \wedge C'(\pi) = w\}$, we have $H_r(g(n)) \leq h(n) + m + c$. ❑

The next theorem relates the complexities K_r and H_r to their counterparts for alphabets of different size $\mathrm{card}\, X_b = b$, K_b and H_b, respectively. To this end we denote by $\mathsf{trans}_{b,r} := b\text{-string} \circ r\text{-string}^{-1} : X_r^* \to X_b^*$ the standard bijection between r-ary and b-ary words.

Theorem 2 *Let $f : \mathbb{N} \to X_b^*$ be a recursive function of bounded ambiguity, and let $g : \mathbb{N} \to X_r^*$ be a recursive function. Then there is a constant $c > 0$ such that for all $n \in \mathbb{N}$ the following inequalities hold true*

$$\begin{array}{rcl} K_r(g(n)) & \leq & \log_r b \cdot K_b(f(n)) + c \text{ and} \\ H_r(g(n)) & \leq & \log_r b \cdot H_b(f(n)) + c \ . \end{array}$$

Proof. Let $\mathrm{card}\, f^{-1}(w) \leq k$ for all $w \in X_b^*$. We define a function $\phi : X_r^* \to X_r^*$ in the following way:
If $|\pi| \leq k$ let $\phi(\pi) := e$ (the empty word). Otherwise split the input $\pi \in X_r^*$ in two parts $\pi_1 \cdot \pi_2$ such that $|\pi_1| = k$.
Set $m := (|r\text{-string}^{-1}(\pi_1)| \pmod k) \in \{1, \ldots, k\}$.
Then translate π_2 via the standard bijection $\mathsf{trans}_{b,r} : X_r^* \to X_b^*$ into a program $\sigma := \mathsf{trans}_{b,r}(\pi_2) \in X_b^*$. Compute $U_b(\sigma)$ for a universal computer w.r.t. X_b^*. If $U_b(\sigma)$ is defined then take from the set $\{i : f(i) = U_b(\sigma)\}$ the m-th element, n (say), and compute $g(n)$.
Thus, if $f(n) = U_b(\sigma)$ then $\mathrm{card}\, f^{-1}(f(n)) \leq k$ and there is a prefix π_1 such that we have $\phi(\pi) = g(n)$ for $\pi := \pi_1 \cdot \mathsf{trans}_{r,b}(\sigma)$. Finally observe that $K_\phi(g(n)) \leq |\pi| \leq k + \lceil |\sigma| \cdot \log_r b \rceil$.
In the case of self-delimiting complexity, the assertion follows from the previous theorem, because $\sum_{n\in\mathbb{N}} r^{-\log_r b \cdot H_b(f(n))} < \infty$. ❑

2 Base independence

In this section we consider expansions of real numbers with respect to different bases. It is well known that the mappings converting real numbers from scale r

to scale b are not continuous functions mapping the r-ary expansion $\xi \in X_r^\omega$ of a real number $\alpha \in [0,1]$ to a b-ary expansion $\Phi(\xi) \in X_b^\omega$ of the same number. For instance, in the case $r = 3$ and $b = 2$ for $\alpha = \frac{1}{2}$, that is, $\xi = 111\ldots \in \{0,1,2\}^\omega$ we do not know the first bit of the ω-word $\Phi(\xi) \in \{0,1\}^\omega$ until we know the whole infinite ω-word ξ. For a more detailed account see [We92].

Despite this fact, we can show that the Kolmogorov complexities of the expansions of the same real number α do not differ too much. To this end we denote by $K_r(\xi/l)$ $(H_r(\xi/l))$ the (self-delimiting) Kolmogorov complexity of the prefix of length l of the ω-word $\xi \in X_r^\omega$, that is, $K_r(\xi/l) := K_r(w)$ $(H_r(\xi/l) := H_r(w))$ where $w \sqsubset \xi$ and $|w| = l$.

The aim of this section is to prove the following theorem.

Theorem 3 *Let $\alpha \in [0,1]$ be a real number, and let $\xi \in X_r^\omega$ and $\beta \in X_b^\omega$ be its base r and base b expansions, respectively.*

Then there is a constant c such that for every $l \in \mathbb{N}$ the following equations hold true:

$$\begin{aligned} |K_r(\xi/\lfloor l \cdot \log_r b \rfloor) - \log_r b \cdot K_b(\beta/l)| &\leq c \text{ , and} \\ |H_r(\xi/\lfloor l \cdot \log_r b \rfloor) - \log_r b \cdot H_b(\beta/l)| &\leq c \text{ .} \end{aligned}$$

In order to prove Theorem 3, it suffices to show the inequalities

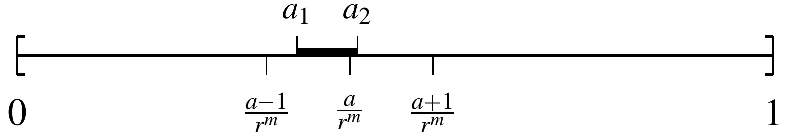
$$K_r(\xi/\lfloor l \cdot \log_r b \rfloor) \leq \log_r b \cdot K_b(\beta/l) + c \text{ , and} \tag{1}$$

$$H_r(\xi/\lfloor l \cdot \log_r b \rfloor) \leq \log_r b \cdot H_b(\beta/l) + c \text{ .} \tag{2}$$

To this end we derive the following facts establishing some connections between the prefixes of an r-ary expansion and a b-ary expansion of the same real number.

Fact 4 *Let $0 \leq a_1 < a_2 \leq 1$ for some real numbers $a_1, a_2 \in \mathbb{R}$ and let $r \in \mathbb{N}$, $r \geq 2$. Then there is at least one $a \in \mathbb{N}$ such that the interval $[a_1, a_2]$ is contained in the interval $\left[\frac{a-1}{r^m}, \frac{a+1}{r^m}\right]$ where $m := \lfloor \log_r \frac{1}{a_2 - a_1} \rfloor$.*

This fact is illustrated in the following picture.



Remark. Observe that for $a_2 - a_1 \leq r^{-m}$ it is not always possible to cover the interval $[a_1, a_2]$ by a single r-ary interval $\left[\frac{a}{r^m}, \frac{a+1}{r^m}\right]$. Fact 4 shows that, however, it is possible to cover $[a_1, a_2]$ by two adjacent r-ary intervals.

We note still that for $0 \leq a < r^m$ every real $\alpha \in \left[\frac{a}{r^m}, \frac{a+1}{r^m}\right]$ has an r-ary expansion which starts with the same prefix $w(a,m)$ of length m, that is, has an expansion between $w(a,m) \cdot 0^\omega$ and $w(a,m) \cdot (r-1)^\omega$. Here $w(a,m)$ is obtained by writing the integer $a \in \mathbb{N}$ in r-ary notation and filling with leading zeros up to the length m provided $a < r^m$.

The following fact summarizes our considerations about the containment of real intervals in r-ary intervals. To this end let $\mathsf{Cov}_r(a_1, a_2; a)$ denote the above illustrated fact that $[a_1, a_2] \subseteq \left[\frac{a-1}{r^m}, \frac{a+1}{r^m}\right]$ where $m := \lfloor \log_r \frac{1}{a_2 - a_1} \rfloor$.

Fact 5 *The relation*

$$R_r := \{(a_1, a_2, a) : a_1, a_2 \in \mathbb{Q} \cap [0,1] \wedge a \in \mathbb{N} \wedge \mathsf{Cov}_r(a_1, a_2; a)\}$$

is recursive and contains for every pair $a_1, a_2 \in \mathbb{Q} \cap [0,1]$ *such that* $a_1 < a_2$ *at least one triple* (a_1, a_2, a) *where* $a \in \mathbb{N}$.

As a consequence of Fact 5 we obtain that the functions h_r, $r \geq 2$ defined by

$$\begin{aligned} h_r &: (\mathbb{Q} \cap [0,1])^2 \to \mathbb{N} \text{ where} \\ h_r(a_1, a_2) &= \begin{cases} \mu a(a \in \mathbb{N} \wedge \mathsf{Cov}_r(a_1, a_2; a)), & \text{if } a_1 < a_2 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{3}$$

are computable and satisfy the following properties.

Property 6 *Let* $0 \leq a_1 < a_2 \leq 1$ *and* $m := \lfloor \log_r \frac{1}{a_2 - a_1} \rfloor$. *Then*

$$h_r(a_1, a_2) \leq r^m \quad \text{and} \tag{4}$$

$$[a_1, a_2] \subseteq \left[\frac{h_r(a_1, a_2) - 1}{r^m}, \frac{h_r(a_1, a_2) + 1}{r^m}\right]. \tag{5}$$

Proof of Theorem 3. Let $v = \beta(1) \ldots \beta(l)$, that is, $|v| = l$. Then $0 \leq \nu_b(\beta) - \nu_b(v) \leq b^{-|v|}$.

According to Fact 4 and Property 6 the numbers

$$\begin{aligned} a(v) &:= h_r\left(\nu_b(v), \nu_b(v) + b^{-|v|}\right) \text{ and} \\ m(v) &:= \lfloor \log_r b^{|v|} \rfloor \end{aligned}$$

satisfy $\nu_b(\beta) \in \left[\frac{a(v)-1}{r^{m(v)}}, \frac{a(v)+1}{r^{m(v)}}\right]$. Thus there is an r-ary expansion of $\nu_b(\beta)$ starting with $w(a(v)-1, m(v))$ or with $w(a(v), m(v))$.

Summarizing the preceding discussion, we obtained recursive functions $h_-, h_+ : X_b^* \to X_r^*$ such that $h_-(v) := w(a(v)-1, m(v))$ and $h_+(v) := w(a(v), m(v))$.[1]

The proof is now finished by applying Theorem 2 in the following way:

$$\left.\begin{array}{rcc} w(a(v),m(v)) & \xleftarrow{g} & 2n \\ w(a(v)-1,m(v)) & \xleftarrow{g} & 2n+1 \end{array}\right\} \xrightarrow{f} \quad v = b\text{-}\mathsf{string}(n)\ ,$$

that is, we associate with every word $v \in X_b^*$ two natural numbers $2n, 2n+1$ via

$$f(2n) := f(2n+1) := b\text{-}\mathsf{string}(n)\ ,$$

and, on the other hand, the function g maps the natural numbers $2n$ and $2n+1$ to the words $w(a(v)-1, m(v)) \in X_r^*$ and $w(a(v), m(v)) \in X_r^*$, respectively:

$$\begin{array}{rcl} g(2n) & := & h_-(b\text{-}\mathsf{string}(n)) \text{ and} \\ g(2n+1) & := & h_+(b\text{-}\mathsf{string}(n))\ . \end{array}$$

It is obvious that f is of bounded ambiguity, so Eqs. (1) and (2) follow from Theorem 2. ❑

3 The complexity of real numbers

In this section we consider the Kolmogorov complexity of real numbers with certain properties: the first class is the mentioned in the introduction class of random real numbers, and the second is the class of Liouville numbers, well-known as constructive examples of transcendental numbers.

To this end we introduce the lower and upper limit of the relative complexity of an ω-word $\xi \in X_r^*$.

$$\underline{\kappa}(\xi) := \liminf_{n\to\infty} \frac{K_r(\xi/n)}{n} \text{ and } \kappa(\xi) := \limsup_{n\to\infty} \frac{K_r(\xi/n)}{n} \tag{6}$$

[1]The choice between the two functions h_-, h_+ provides the missing information which prevented us, in the general case, from a continuous conversion between b-ary and r-ary expansions of real numbers. Observe, that the information we need to accomplish the choice between $w(a(v)-1, m(v))$ and $w(a(v), m(v))$ is only *one* bit.

Since $|H_r(\xi/n) - K_r(\xi/n)| \le o(n)$ it is of no importance whether we use the usual or self-delimiting complexity.

From Theorem 3 above we conclude that for a real number $\alpha \in [0,1]$ we can define its lower and upper limit of complexity in the same way as in Eq. (6):

$$\underline{\kappa}(\nu_r(\xi)) := \underline{\kappa}(\xi) \text{ and } \kappa(\nu_r(\xi)) := \kappa(\xi) \ .$$

3.1 Random reals

It was widely believed that the notion of randomness of a real number α is independent of the base of the expansion in which α is represented. Sound proofs of this fact were given only recently by different means [CJ94, HW98]. Here we give a third proof relying on the following definition of random sequences by self-delimiting Kolmogorov complexity (cf. [Ca94, Ch87, LV93]).

Definition 1 *An* ω*-word* $\xi \in X_r^\omega$ *is called random provided*

$$\lim_{l \to \infty} H_r(\xi/l) - l = \infty \ .$$

Now from Theorem 3 the proof of the independence result is immediate.

Lemma 7 *Let* $\alpha \in [0,1]$ *be a real number which is random in the scale of r. Then for* $b \in \mathbb{N},\ b \ge 2$ *the* ω*-word* $\beta \in X_b^\omega$ *with* $\nu_b(\beta) = \alpha$ *is random.*

3.2 The Kolmogorov complexity of Liouville numbers

The real numbers we deal with in this section are named after Liouville who invented them to demonstrate the existence of transcendental numbers. They are characterized by the fact that they have, although in a nonconstructive way, very tight rational approximations.

Definition 2 *A real number* $\alpha \in \mathbb{R}$ *is called a Liouville number provided*

1. α *is irrational.*
2. $\forall n (n \in \mathbb{N} \to \exists p, q (p, q \in \mathbb{N} \wedge q > 1 \wedge |\alpha - \frac{p}{q}| < \frac{1}{q^n}))$.

It should be noted that every Liouville number is transcendental (see [Ox71]).[2]

We obtain our first result.

[2]Moreover, since $n \ge l \cdot (\log_2 k + 1)$ and $|\alpha - \frac{p}{q}| < \frac{1}{q^n}$ imply $|(\alpha + \frac{m}{k}) - (\frac{p}{q} + \frac{m}{k})| < \frac{1}{(q \cdot k)^l}$, the sum of a Liouville number and a rational number is again a Liouville number, whereas, as we shall see below, every real is the sum of at most two Liouville numbers.

Lemma 8 *If* $\alpha \in [0,1]$ *is a Liouville number then* $\underline{\kappa}(\alpha) = 0$.

Proof. We show that for the binary expansion $\eta \in \{0,1\}^\omega$ of α for every $n \in \mathbb{N}$ there is an $l \geq n$ such that
$$\frac{K_2(\eta/l)}{l} \leq \frac{\log_2 n}{n}$$
.

Let $|\alpha - \frac{p}{q}| < \frac{1}{q^n}$, where $0 \leq p \leq q$. We use the function h_2 defined in Eq. (3). Since $\alpha \in \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n}\right)$, we obtain for $a := h_2(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n})$ the restriction $a < r^M$ where $m := \log_2 \frac{q^n}{2} = n \cdot \log_2 q - 1$. As in the discussion following Fact 4 we define words $w(a-1,m)$ and $w(a,m)$ of length $m = \lfloor n \cdot \log_2 q \rfloor - 1$, one of them being a prefix of η.

Both words $w(a-1,m)$ and $w(a,m)$ can be specified by the numbers n, p, q. Utilizing a prefix-free binary encoding $\mathsf{code} : \mathbb{N} \to \{0,1\}^*$ of the natural numbers, where $|\mathsf{code}(n)| \leq 2 \cdot \log_2 n$ for $n \geq 4$, we obtain programs of the form

$$\pi_{n,p,q}(i) := i \cdot \mathsf{code}(n) \cdot \mathsf{code}(p) \cdot \mathsf{code}(q),\ i \in \{0,1\}\ ,$$

and a computable function $\psi : \{0,1\}^* \to \{0,1\}^*$ such that

$$\psi(\pi_{n,p,q}(0)) = w(a-1,m) \text{ and } \psi(\pi_{n,p,q}(1)) = w(a,m)\ .$$

Consequently, $K_\psi(w(a-1,m)), K_\psi(w(a,m)) \leq 1 + 2\log_2 n + 2\log_2 p + 2\log_2 q \leq 1 + 2\log_2 n + 4\log_2 q$, and hence $K_2(w(a-1,m)), K_2(w(a,m)) \leq c + 2\log_2 n + 4\log_2 q$ for all triples (n,p,q) such that $|\alpha - \frac{p}{q}| < \frac{1}{q^n}$ and $n, q \geq 4$. Now, observe that in view of Definition 2 the values of the denominator q grow with the value of the exponent of precision n. Thus

$$\frac{K_2(\eta/\lfloor n \cdot \log_2 q - 1 \rfloor)}{\lfloor n \cdot \log_2 q - 1 \rfloor} \leq \frac{\log_2 n}{n}$$

if n (and hence q) is large enough. ❏

In connection with Definition 1 we obtain the following.

Corollary 9 *No Liouville number is a random real.*

Though Liouville numbers are not random, we show that the upper limit of complexity reaches its maximum value $\kappa(\alpha) = 1$ also for certain Liouville numbers α. We consider the following set constructed in a similar way as the one in Example 3.18 of [St93].

Example 10 Define

$$F := X_r \cdot \prod_{i\in\mathbb{N}} X_r^{2i\cdot(2i)!} \cdot 0^{(2i+1)\cdot(2i+1)!} .$$

It is interesting to note that the set of finite prefixes of F, $\mathbf{A}(F) := \{w : w \in X_r^* \wedge \exists\xi(\xi \in F \wedge w \sqsubset \xi)\}$, is recursive.

If we consider ω-words $\beta = 0 \cdot \prod_{i\in\mathbb{N}} w_i \cdot 0^{(2i+1)\cdot(2i+1)!}$ where $|w_i| = 2i\cdot(2i)!$ and $K_r(w_i) \geq |w_i| - c$ for some $c \in \mathbb{N}$ then Daley's [Da74] diagonalization argument shows $\kappa(\beta) = 1$. Since the set $\{w : w \in X_r^* \wedge K_r(w) \geq |w| - 2\}$ contains at least two elements, F contains uncountably many ω-words β having $\kappa(\beta) = 1$.

The following consideration verifies that the set of numbers $\{\nu_r(\xi) : \xi \in F\} \setminus \mathbb{Q}$ consists entirely of Liouville numbers:

Let $\xi \in F$, $n \in \mathbb{N}$ and consider the prefix $w \sqsubset \xi$ of length $(2n+1)! = 1 + \sum_{i=0}^{2n} i\cdot i!$. Then $\nu_r(w) = p \cdot r^{-(2n+1)!}$ for some $p \in \mathbb{N}$ and $w \cdot 0^{(2n+1)\cdot(2n+1)!} \sqsubset \xi$.

Consequently, $0 \leq \nu_r(\xi) - \nu_r(w) = \nu_r(\xi) - \dfrac{p}{r^{(2n+1)!}} < \dfrac{1}{r^{(2n+2)!-1}} \leq \dfrac{1}{\left(r^{(2n+1)!}\right)^n}$.

Thus, either $\nu_r(\xi)$ is rational or a Liouville number. ❏

Remark. In the same way one proves that $F' := \{0\} \cdot \prod_{i\in\mathbb{N}} 0^{2i\cdot(2i)!} \cdot X_r^{(2i+1)\cdot(2i+1)!}$ contains only rational or Liouville numbers. It is readily seen that every number $\alpha = \nu_r(\zeta) \in [0,1]$ can be represented as the sum $\nu_r(\zeta) = \nu_r(\xi) + \nu_r(\xi')$ where $\xi \in F$ and $\xi' \in F'$ are the letter-by-letter projections of ζ onto F or F', respectively. The numbers $\nu_r(\xi)$ and $\nu_r(\xi')$ in the above sum are rational or Liouville numbers, thus according to Footnote 2 $\nu_r(\zeta)$ is a Liouville number or the sum of two Liouville numbers. ❏

In the subsequent parts, we use the results obtained so far to give a complexity-theoretic proof of Theorem 2.4 in [Ox71] and to prove the existence of disjunctive Liouville numbers.

3.3 The Hausdorff dimension of Liouville numbers

First we consider the Hausdorff dimension of the set of Liouville numbers, $L \subseteq [0,1]$. It was mentioned in [MS94] that the Hausdorff dimension of a subset $M \subseteq [0,1]$ coincides with the one of $\{\xi : \xi \in X_r^\omega \wedge \nu_r(\xi) \in M\}$. The latter can be defined as follows (see, e.g., [St93, St98]).

Definition 3 The *Hausdorff dimension* of a set $F \subseteq X_r^\omega$, $\dim F$, is the smallest real number $\alpha \geq 0$ such that for all $\gamma > \alpha$ it holds

$$\forall \varepsilon(\varepsilon > 0 \rightarrow \exists W(W \subseteq X_r^* \wedge F \subseteq W \cdot X_r^\omega \wedge \sum_{w \in W} \left(r^{-\gamma}\right)^{|w|} < \varepsilon))\ .$$

From the definition it is evident that Hausdorff Dimension is monotone with respect to set inclusion and that $\dim\{\xi\} = 0$. We mention still that Hausdorff Dimension is also countably stable.

$$\dim \bigcup_{i \in \mathbb{N}} F_i = \sup_{i \in \mathbb{N}} \dim F_i \tag{7}$$

Thus every countable subset $F \subseteq X_r^\omega$ has $\dim F = 0$. For further properties of the Hausdorff dimension see, e.g., [Fa90].

We are going to give a complexity-theoretic proof of the fact (cf. [Ox71, Theorem 2.4]) that the set of Liouville numbers L is an uncountable set of Hausdorff dimension $\dim L = 0$.

In the papers [Ry86, St93, St98] close connections between Hausdorff dimension and Kolmogorov complexity are derived. We need here the following one (see [Ry86, Theorem 2] or [St93, Corollary 3.14]).

Lemma 11 *For every $F \subseteq X_r^\omega$ the following bound is true.*

$$\dim F \leq \sup\{\underline{\kappa}(\xi) : \xi \in F\}$$

Now Lemmas 8 and 11 yield the announced result.

Corollary 12 *The set of Liouville numbers $L \subseteq [0,1]$ has Hausdorff dimension* $\dim L = 0$.

3.4 Disjunctive Liouville numbers

In this last part we turn to disjunctive ω-words. As it was mentioned above, an ω-word $\xi \in X_r^\omega$ is called *disjunctive* provided every word $w \in X_r^*$ appears as an infix of ξ, that is, $\forall w(w \in X_r^* \rightarrow \exists v(v \sqsubset \xi \wedge v \cdot w \sqsubset \xi))$.

Proposition 8 of [JT88] proves that for every $r \geq 2$ there are uncountably many disjunctive $\xi \in X_r^\omega$ such that $\nu_r(\xi)$ is a Liouville number. The paper [He96] presents examples of Liouville numbers whose expansions are disjunctive with respect to one base, but not to with respect to all bases (e.g. $\sum_{i=1}^{\infty} r^{-i!-i}$ which is not disjunctive in the scale of r). We prove the existence of Liouville numbers disjunctive with respect to all bases.

Lemma 13 *There are uncountably many Liouville numbers* α *such that for every* $r \in \mathbb{N}$, $r \geq 2$ *the* ω*-word* $\xi \in X_r^\omega$ *with* $\nu_r(\xi) = \alpha$ *is disjunctive.*

Proof. Eq. (5.3) of [St93] shows that an ω-word $\xi \in X_r^\omega$ with $\kappa(\xi) = 1$ is disjunctive. In fact, if $\xi \in X_r^\omega$ does not contain a word $w \in X_r^*$ of length $|w| = l$ as infix then $\kappa(\xi) \leq l^{-1} \cdot \log_r(r^l - 1) < 1$. Now Example 10 yields the existence of Liouville numbers which are disjunctive in every scale r. ❏

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