How Much Can You Win When Your Adversary is Handicapped?*

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Abstract

We consider infinite games where a gambler plays a coin-tossing game against an adversary. The gambler puts stakes on heads or tails, and the adversary tosses a fair coin, but has to choose his outcome according to a previously given law known to the gambler. In other words, the adversary is not allowed to play all infinite heads-tails-sequences, but only a certain subset F of them.

We present an algorithm for the player which, depending on the structure of the set F, guarantees an optimal exponent of increase of the player's capital, independently on which one of the allowed heads-tails-sequences the adversary chooses.

Using the known upper bound on the exponent provided by the maximum Kolmogorov complexity of sequences in F we show the optimality of our result.

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Contents

It is well-known that random sequences do not admit successful gambling strategies. Here we consider a game where a player bets at fixed odds, but with unlimited amount, on the tosses of a coin. We further agree on the fact that the player must have no debt. It was explained in [Sc71, vL87, LV93] that in such a game a player playing according a computable gambling strategy cannot have unlimited gain if the tosses of the coin follow a random zero-one-sequence.

On the other hand, it is quite obvious that, if the zero-one-sequence follows partially a certain computable law, the player may have an unlimited gain. A simple example is a zero-one-sequence which repeats each value twice. Here the player may double his capital every second step just by betting all his remaining capital according to the previous outcome.

In this paper we investigate the exponent of the increase of the player's capital, λ, under the following assumptions on the game.

1. The player plays a computable gambling strategy, more precisely, he computes his bets from a complete history in a deterministic way.

2. The tosses of the coin follow a zero-one-sequence which belongs to a certain previously fixed set $F \subseteq \{0,1\}^\omega$.

3. The player can bet arbitrary nonnegative amounts not exceeding his capital, in particular, he must not have debts.

It is shown that under these and some additional computability assumptions on the set of admitted zero-one-sequences F there is always a strategy which guarantees the player an exponent λ which depends only upon the size of the constraint F.

Moreover, we show that our result is the best possible in two respects.

1. Regardless which constraint $F \subseteq \{0,1\}^\omega$ we consider, there is always a zero-one-sequence $\xi \in F$ such that $\lambda(\xi)$ cannot be better than the upper bound given by the size of F.

2. Our computability assumption on F guaranteeing the optimal exponent λ is a best one. It cannot be extended to admit larger classes of constraints.

The results of this papers relate several different areas of mathematics and theoretical computer science.

In the first section we give some necessary notation, and we present our notion of game. For these games we derive a description of gambling strategy via computable martingales. In Section 2 we derive an upper bound on the exponent of the increase of the player's capital in terms of Kolmogorov complexity.

The subsequent section introduces an appropriate size measure for sets of zero-one-sequences. It turns out that the Hausdorff dimension, known from fractal geometry, fulfills our requirements of being closely related to Kolmogorov complexity on the one hand and to gambling strategies on the other hand. In the fourth section we discuss the computability requirements which we have to put on our constraints $F \subseteq \{0,1\}^\omega$. Here we state also our main result.

Most of the results presented here are proved in [St98]. For the necessary background in computability, random sequences and Kolmogorov complexity we refer the reader to [Sc71], [LV93] and [Ca94]. For the definition of Hausdorff dimension and their properties see e.g. [Ed90, Fa90].

1 Notation and Definitions

By $\mathbb{N} = \{0,1,2,\ldots\}$ we denote the set of natural numbers. We consider the space $\{0,1\}^\omega$ of infinite zero-one-sequences (ω-words). By $\{0,1\}^*$ we denote the set of finite strings (words) on $\{0,1\}$, including the *empty* word e. For $w \in \{0,1\}^*$ and $b \in \{0,1\}^* \cup \{0,1\}^\omega$ let $w \cdot b$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq \{0,1\}^*$ and $B \subseteq \{0,1\}^* \cup \{0,1\}^\omega$.

Furthermore $|w|$ is the *length* of the word w. By b/n we denote the *length n prefix* of a string $b \in \{0,1\}^*$, $|b| \geq n$, or $b \in \{0,1\}^\omega$, and $\mathbf{A}(b) := \{b/n : n \in \mathbb{N} \wedge n \leq |b|\}$ and $\mathbf{A}(B) := \bigcup_{b \in B} \mathbf{A}(b)$ are the sets of all finite prefixes of $b \in \{0,1\}^* \cup \{0,1\}^\omega$ and $B \subseteq \{0,1\}^* \cup \{0,1\}^\omega$, respectively.

The set of all binary words $\{0,1\}^*$ may be also viewed at as the rooted infinite binary tree, where the empty word e is the root and $w0, w1$ are the successors of the node $w \in \{0,1\}^*$. Then $\{0,1\}^\omega$ is in a natural correspondence with the infinite paths through $\{0,1\}^*$ starting at the root, as any path $\xi \in \{0,1\}^\omega$ is uniquely specified by its finite initial paths $w \in \mathbf{A}(\xi)$.

This much notation suffices to describe our game.

Tree game on the binary tree $\{0,1\}^*$ given a set $F \subseteq \{0,1\}^\omega$ of admitted zero-one-sequences

$$\begin{array}{llcll} \texttt{Start}: & w & := & e & \text{[root node = empty word]} \\ & \mathcal{V}(e) & := & 1 & \text{[initial capital]} \end{array} \tag{1}$$

`For` $w := e$ `to` $\xi \in F$ `do`

player bets: $\mathcal{W}_0(w), \mathcal{W}_1(w) \in [0,1]$ where $\mathcal{W}_0(w) + \mathcal{W}_1(w) \leq 1$

adversary chooses $x \in \{0,1\}$ according to $\xi \in F$ and pays $2 \cdot \mathcal{W}_x(w) \cdot \mathcal{V}(w)$

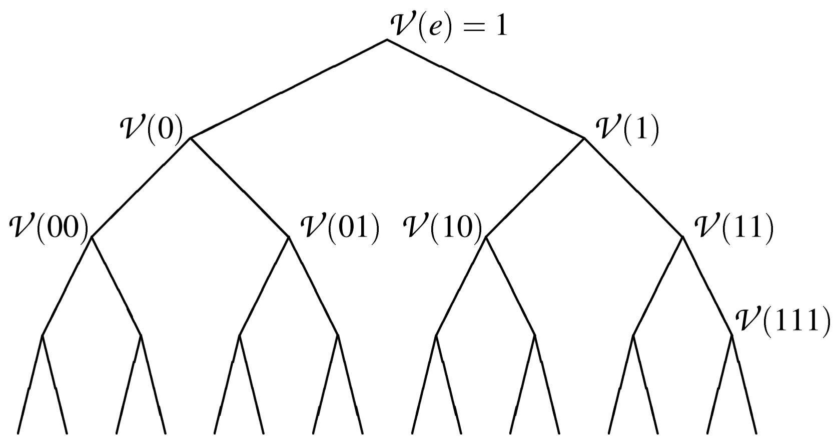
player's capital

$$\mathcal{V}(wx) := \mathcal{V}(w) \cdot (1 + \mathcal{W}_x(w) - \mathcal{W}_{\neg x}(w)) \tag{2}$$

$w := wx$

`Endfor`

We assume that $\mathcal{W}_0 : \{0,1\}^* \to \mathbb{R}$ and $\mathcal{W}_1 : \{0,1\}^* \to \mathbb{R}$ are computable functions. From the Equations (1) and (2) in the above description of our game we can compute in advance the player's capital $\mathcal{V}(w)$ in node w of the binary tree $\{0,1\}^*$ as illustrated in the picture below:



Here one easily observes that the capital function $\mathcal{V}$ has the following property

$$\mathcal{V}(w) = \frac{1}{2} \cdot (\mathcal{V}(w0) + \mathcal{V}(w1)) \ . \tag{3}$$

Conversely, if we have a function $\mathcal{V} : \{0,1\}^* \to \mathbb{R}$ satisfying (1) and (3) then defining

$$\mathcal{W}_x(w) := \begin{cases} \frac{\mathcal{V}(wx)}{\mathcal{V}(w)} & \text{, if } \mathcal{V}(w) > 0 \text{ and} \\ 0 & \text{, otherwise} \end{cases} \tag{4}$$

yields a gambling strategy $(\mathcal{W}_0, \mathcal{W}_1)$ which realizes the capital $\mathcal{V}(w)$ in the node w of the binary tree.

Thus, in the sequel, it suffices to consider (computable) capital functions satisfying (1) and (3). Those functions are also called (*computable*) *martingales* (cf. [Sc71, vL87, LV93]).

We conclude this section with two examples presenting gambling strategies for given constraints F_1 and F_2.

Example 1 As mentioned above in the introduction let our constrained satisfy $F_1 := \{00, 11\}^\omega$, that is, the adversary repeats its choice once.

A reasonable betting strategy for the player to maximize the growth of his capital would be given by

$$\mathcal{W}_x(w) := \begin{cases} 1, & \text{if } |w| \text{ is odd and } w \in \{0,1\}^* \cdot x \\ 0, & \text{otherwise,} \end{cases}$$

that is, to put every second step all of the capital on letter x if x was previously chosen by the adversary.

One easily calculates that

$$\mathcal{V}_1(w) = \begin{cases} 2^{\lfloor |w|/2 \rfloor}, & \text{if } w \in \mathbf{A}(F_1), \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

So, asymptotically we have $\log_2 \mathcal{V}_1(w) \approx \frac{|w|}{2}$ for $|w| \to \infty$ and $w \in \mathbf{A}(F_1)$.

Intuitively, it is clear that the player cannot do much better, for at every odd step the adversary might flip a coin to draw his outcome randomly, and, as it is well known (cf. [Sc71, vL87, LV93]), one cannot win against a random sequence. As we shall prove below, the asymptotic gain of the betting above strategy is optimal. ❑

The next example is a little bit more involved.

Example 2 Let $F_2 := \{0,1\}^* \cdot 0^\omega$, that is a typical zero-one-sequence in $F2$ has the form

$$\underbrace{x_1 \ldots x_m}_{\substack{m \in \mathbb{N} \\ \textit{arbitrary}}} \cdot \underbrace{0000000\cdots}_{\textit{ad infinitum}} .$$

A reasonable betting strategy for maximizing the player's capital would be to put larger and larger parts of the capital on 0, because the adversary's ultimate behaviour is to draw only zeros. Observe here that, albeit the player is not allowed to make loans, he is allowed to retain arbitrarily small positive amounts.

Thus we might choose

$$\mathcal{W}_x(w) := \begin{cases} 1 - 2^{-(|w|+1)} & \text{, if } x = 0 \text{, and} \\ 0 & \text{, otherwise.} \end{cases}$$

If $w \in v \cdot 0^*$ then

$$\begin{aligned} \mathcal{V}_2(w) &\geq \prod_{i=1}^{|v|} 2^{-i} \cdot \prod_{i=|v|+1}^{|w|} 2 \cdot (1 - 2^{-i}) \\ &\geq 2^{|w| - \frac{|v| \cdot (|v|+3)}{2}} \cdot \prod_{i=1}^{\infty} (1 - 2^{-i}) . \end{aligned}$$

Using the fact that $\prod_{i=1}^{\infty}(1-2^{-i}) > 0$ we obtain that $\mathcal{V}_2(w) \geq c_\xi \cdot 2^{|w|}$ as $w \to \xi$ for every $\xi \in F_2$. Here $c_\xi > 0$ is a constant depending on v when $\xi = v \cdot 0^\omega$. ❑

2 Upper Bounds by Kolmogorov Complexity

In this section we derive an upper bound on the exponent of the increase of the player's capital for arbitrary (even non-computable) constraints $F \subseteq \{0,1\}^\omega$. Moreover, we show that, in general, there is no computable gambling strategy which reaches this upper bound.

Before we proceed to the results, we make precise what we mean by the exponent of the increase of the player's capital function $\mathcal{V}$, $\lambda_{\mathcal{V}}$.

Definition 1 Let $\mathcal{V} : \{0,1\}^* \to [0,\infty)$ be a capital function. Then

$$\lambda_{\mathcal{V}}(\xi) := \limsup_{n \to \infty} \frac{\log_2 \mathcal{V}(\xi/n)}{n} \tag{5}$$

is the *exponent of the increase* of $\mathcal{V}$ on the zero-one-sequence ξ.

The task of the subsequent sections is, given a constraint $F \subseteq \{0,1\}^\omega$, to maximize the value of $\inf_{\xi \in F} \lambda_{\mathcal{V}}(\xi)$ for computable $\mathcal{V} : \{0,1\}^* \to [0,\infty)$.

First we give the announced upper bound on $\lambda_{\mathcal{V}}(\xi)$.

To this end we introduce the *Kolmogorov complexity* of finite and infinite strings.

For a given algorithm (computable partial function) $\mathfrak{A} : \{0,1\}^* \to \{0,1\}^*$ we define the complexity of a word $w \in \{0,1\}^*$ as the length of the shortest program $\pi \in \{0,1\}^*$ for which $\mathfrak{A}$ prints w:

$$K_{\mathfrak{A}}(w) := \inf\{|\pi| : \mathfrak{A}(\pi) = w\} \ . \tag{6}$$

Then it holds

Theorem 1 (Solomonoff, Kolmogorov) *There is an optimal algorithm $\mathfrak{U}$ such that for all algorithms $\mathfrak{A}$ we have*

$$\forall w \in \{0,1\}^* : \ K_{\mathfrak{U}}(w) \leq K_{\mathfrak{A}}(w) + c_{\mathfrak{A},\mathfrak{U}} \ ,$$

for an appropriately chosen constant $c_{\mathfrak{A},\mathfrak{U}}$.

In what follows, when considering the Kolmogorov complexity of a finite string $w \in \{0,1\}^*$, we shall refer to a fixed optimal algorithm $\mathfrak{U}$.

For infinite strings here we shall consider the following notion of Kolmogorov complexity.

Definition 2 The *lower Kolmogorov complexity* of an infinite string $\xi \in \{0,1\}^\omega$ is the value

$$\underline{\kappa}(\xi) := \liminf_{n \to \infty} \frac{K_{\mathfrak{U}}(\xi/n)}{n} \ .$$

Utilizing Levin's universal semicomputable semimeasure (cf. [ZL70] or [LV93]) it was shown in [Ry93] that the exponent $\lambda_{\mathcal{V}}(\xi)$ is bounded from above by $1 - \underline{\kappa}(\xi)$ provided the gambler plays according to a computable strategy.

Lemma 2 (Upper bound by Kolmogorov complexity) *Let $\mathcal{V}$ be a computable capital function. Then*

$$\lambda_{\mathcal{V}}(\xi) \leq 1 - \underline{\kappa}(\xi) \tag{7}$$

for every $\xi \in \{0,1\}^\omega$.

This upper bound, though being helpful as we shall see in the next sections, is inaccessible. From the well-known noncomputability of Levin's universal semicomputable semimeasure one obtains readily the following.

Theorem 3 *There is no computable capital function* $\mathcal{V}_{\mathrm{opt}} : \{0,1\}^* \to \mathbb{R}$ *such that* $\forall \xi \in \{0,1\}^\omega : \lambda_{\mathcal{V}_{\mathrm{opt}}}(\xi) \geq \lambda_{\mathcal{V}}(\xi)$ *for all other computable capital functions* $\mathcal{V} : \{0,1\}^* \to \mathbb{R}$.

As a corollary we obtain further.

Corollary 4 *There is no computable capital function* $\mathcal{V} : \{0,1\}^* \to \mathbb{R}$ *such that* $\lambda_{\mathcal{V}}(\xi) = 1 - \underline{\kappa}(\xi)$ *for all* $\xi \in \{0,1\}^\omega$.

3 An Appropriate Size Measure

So far we did not agree upon the notion of size of a set $F \subseteq \{0,1\}^\omega$. The upper bound on $\lambda_{\mathcal{V}}$ by Lemma 2, in view of $\inf_{\xi \in F} \lambda_{\mathcal{V}}(\xi) \leq 1 - \sup_{\xi \in F} \underline{\kappa}(\xi)$, suggests to choose a value like $\sup_{\xi \in F} \underline{\kappa}(\xi)$ as the size of F. This proposal, however, seems to be a bit too artificial.

Take for example a random zero-one-sequence ζ. Those sequences have $\underline{\kappa}(\zeta) = 1$. Thus the size of a singleton $\{\zeta\}$ would be 1 in contrast to the size of the uncountable set F_1 of Example 1 which has a size of $\frac{1}{2}$ only.

Consequently, we follow an different line, mentioning that several papers investigated the relationship between the Kolmogorov complexity of infinite strings and size measures known from information theory and fractal geometry. It turned out in [Ry86], [St93] and [St98] that the Hausdorff dimension of subsets $F \subset \{0,1\}^\omega$ is closely related to $\sup_{\xi \in F} \underline{\kappa}(\xi)$.

Definition 3 The *Hausdorff dimension* of a set $F \subseteq \{0,1\}^\omega$, $\dim F$, is the smallest real number $\alpha \geq 0$ such that for all $\gamma > \alpha$ it holds

$$\forall \varepsilon > 0 : \exists W \subseteq \{0,1\}^* : F \subseteq W \cdot \{0,1\}^\omega \wedge \sum_{w \in W} \left(2^{-\gamma}\right)^{|w|} < \varepsilon \,.$$

From Definition 3 it is evident that Hausdorff dimension is monotone with respect to set inclusion and that $\dim\{\xi\} = 0$. We observe that Hausdorff dimension fulfills the following stronger property. Let $(F_i)_{i \in \mathbb{N}}$ be a countable family of subsets of $\{0,1\}^\omega$. Then

$$\dim \bigcup_{i \in \mathbb{N}} F_i = \sup_{i \in \mathbb{N}} \dim F_i \,. \tag{8}$$

Eq. (8) implies that $\dim E = 0$ for every at most countable set $E \subseteq \{0,1\}^\omega$.

As a first result we mention a lower bound to Kolmogorov complexity by Hausdorff dimension which states that sets of large dimension contain complex sequences.

Theorem 5 ([Ry86])

$$\dim F \leq \sup\{\underline{\kappa}(\xi) : \xi \in F\}$$

Consider a $\xi \in F$ satisfying $\underline{\kappa}(\xi) \geq \dim F - \varepsilon$. Then Lemma 2 proves $\lambda_{\mathcal{V}}(\xi) \leq 1 - \underline{\kappa}(\xi) = \dim F - \varepsilon$. Thus we obtain the following worst case behaviour of capital functions.

Lemma 6 *Let $F \subseteq \{0,1\}^\omega$ and let $\mathcal{V} : \{0,1\}^* \to \mathbb{R}$ be a computable capital function. For all $\varepsilon > 0$ there is a $\xi \in F$ such that $\lambda_{\mathcal{V}}(\xi) \leq 1 - \dim F + \varepsilon$.*

As announced previously, according to Lemma 6 the adversary has always the possibility to limit the exponent of growth of the player's capital function $\lambda_{\mathcal{V}}$ close to the value $1 - \dim F$ (or even below) when F is his constraint.

Example 1 (continued) Consider again $F_1 := \{00, 11\}^\omega$. One can easily show that $\dim F_1 = \frac{1}{2}$. Thus according to Lemma 6 the asymptotic growth of the capital function $\log_2 \mathcal{V}_1(w) \approx \frac{|w|}{2}$ is optimal. ❑

4 Which constraints are reasonable

It follows from the behaviour of computable capital functions that $1 - \dim F$ is a worst case upper bound to the exponent of their growth. In this section we investigate which subsets $F \subseteq \{0,1\}^\omega$ allow for computable betting strategies $(\mathcal{W}_0, \mathcal{W}_1)$ such that the player achieves a guaranteed exponent of the corresponding capital function $\lambda_{\mathcal{V}}$ which is close to the bound $1 - \dim F$ regardless which infinite sequence $\xi \in F$ the adversary plays.

First we derive an example where the constraint $E \subseteq \{0,1\}^\omega$ is in some sense effectively presented, but nevertheless there is a large gap between $1 - \dim E$ and $\lambda_{\mathcal{V}}(\zeta)$ for at least one $\zeta \in E$ for all computable capital functions $\mathcal{V}$.

Example 3 ([St98, Lemma 6]) *There is a countable subset $E \subseteq \{0,1\}^\omega$ such that $\mathbf{A}(E)$ is recursively enumerable*[1] *and contains a random zero-one-sequence ζ. Since $\underline{\kappa}(\zeta) = 1$, as ζ is random, and since $\dim E = 0$, as E is countable, we have $0 = \lambda_{\mathcal{V}}(\zeta) = 1 - \underline{\kappa}(\zeta) < 1 - \dim E = 1$ for every computable capital function $\mathcal{V}$.*❑

Remark. A more subtle consideration of the proof of Lemma 6 of [St98] shows that E contains exactly one random zero-one-sequence ζ, and $E \setminus \{\zeta\} \subseteq \{0,1\}^* \cdot 0^\omega$. Thus E might be seen as an effective presentation of the

[1] A subset $W \subseteq \{0,1\}^*$ is called *recursively enumerable* provided $W = \emptyset$ or there is a computable function $f : \mathbb{N} \to \{0,1\}^*$ such that $f(\mathbb{N}) = W$.

random zero-one-sequence ζ, although infinite random sequences seem to be objects which cannot be presented effectively.

Our Example 3 leads to the conclusion that we have to restrict the range of computability of the constraints.

Definition 4 (Σ_2-definable sets) A subset $F \subseteq \{0,1\}^\omega$ is referred to as Σ_2-*definable* provided there is a computable function $f_F : \mathbb{N} \times \{0,1\}^* \to \{0,1\}$ such that

$$\xi \in F \quad \longleftrightarrow \quad \exists i \in \mathbb{N} : \forall n \in \mathbb{N} : f_F(i, \xi/n) = 0 \; .$$

Remark. The set E of Example 3 can be defined in a similar way: There is a computable function $g_E : \mathbb{N} \times \{0,1\}^* \to \{0,1\}$ such that

$$\xi \in E \;\longleftrightarrow\; \forall n \in \mathbb{N} : \exists i \in \mathbb{N} : g_E(i, \xi/n) = 0 \; .$$

Observe, however, that the order of the quantifiers is reversed and, besides that, here the outer quantifier $\forall n$ is related to the sequence ξ.

Now we can derive our main result.

Theorem 7 (Main Theorem) *If $F \subseteq \{0,1\}^\omega$ is Σ_2-definable, then for every $\gamma > \dim F$ there is a computable capital function $\mathcal{V}$ such that*

$$\forall \xi \in F : \lambda_{\mathcal{V}}(\xi) > 1 - \gamma \; .$$

If, moreover, $\dim F$ is a computable real number[2] *then there is a computable capital function $\mathcal{V}$ such that*

$$\forall \xi \in F : \lambda_{\mathcal{V}}(\xi) \geq 1 - \dim F \; .$$

For a proof see [St98].

Example 2 (continued) The set $F_2 = \{0,1\}^* \cdot 0^\omega$ is countable. Consequently, $\dim F_2 = 0$. The capital function $\mathcal{V}_2$ introduced above satisfies $\lambda_{\mathcal{V}_2}(\xi) = 1$ whenever $\xi \in F_2$.

It should be mentioned that, in contrast to $\lambda_{\mathcal{V}_1}(\xi) = 0$ for $\xi \in F_1$ where $\mathcal{V}_1$ is the capital function of Example 1, for $\mathcal{V}_2$ we may have even $\lambda_{\mathcal{V}_2}(\xi) = 1$ for some $\xi \notin F_2$ provided the infinitely many ones in ξ are distributed sparsely. ❑

Finally, combining the results of Theorems 5 and 7 and Lemma 2, we obtain an exact bound on the maximum lower Kolmogorov complexity for Σ_2-definable subsets of $\{0,1\}^\omega$.

[2] A number $\gamma \in \mathbb{R}$ is *computable* provided there is a computable function $f_\gamma : \mathbb{N} \to \mathbb{Q}$ such that $|f_\gamma(n) - \gamma| < 2^{-n}$ for all $n \in \mathbb{N}$.

Theorem 8 (Exact bound for Σ_2-definable sets)

$$\dim F = \sup\{\underline{\kappa}(\xi) : \xi \in F\}$$

if $F \subseteq \{0,1\}^\omega$ *is* Σ_2*-definable.*

Proof. One inequality is Theorem 5. For the converse inequality, observe that Theorem 7 and Lemma 2 prove that $\gamma > \underline{\kappa}(\xi)$ whenever $\gamma > \dim F$, $\xi \in F$ and $F \subseteq \{0,1\}^\omega$ is Σ_2-definable. Thus, $\dim F \geq \sup_{\xi \in F} \underline{\kappa}(\xi)$. ❑

Concluding Remark

Our Theorems 7 and 8 in connection with previous results of Ryabko ([Ry86, Ry93]) and this author ([St93, St98]) give evidence that there is a strong coincidence between the concepts of Kolmogorov complexity, gambling strategies and Hausdorff dimension for a class of recursive (computable) sets of infinite zero-one-sequences. The results of the last section show a borderline in the Arithmetical hierarchy[3] up to which this coincidence holds true, and our Example 3 gives evidence that it does not extend much further in the Arithmetical Hierarchy.

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[3] For the Arithmetical hierarchy of ω-languages see e.g. [St97].

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