THE EXAMINATION OF NONLINEAR STABILITY AND SOLVABILITY OF THE ALGEBRAIC EQUATIONS FOR THE IMPLICIT TAYLOR SERIES METHOD

HANS-EBERHARD SCHOLZ

ABSTRACT. Through the availability of efficient implementations of automatic differentiation, the numerical algorithms in which (higher) derivatives of functions are used become more interesting. Such methods are, for example, the the implicit Taylor series methods for solving stiff initial value problems for ordinary differential equations. By constructing relevant examples, we show that these methods (except the implicit Euler method) are not AN- or B-stable, and that for non-autonomous dissipative differential equations (except for the Euler method and the trapezoidal rule) the algebraic equations of A-stable Taylor series methods are not solvable for all stepsizes. For autonomous dissipative equations (except for the Euler method and the trapezoidal rule), the algebraic equations of A-stable methods do not always have a unique solution for all stepsizes.

Key words. Stiff initial value problems, implicit Taylor series methods, nonlinear stability, solvability of the algebraic equations.

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1. Introduction

In past years great strides were made in the development of efficient implementations of automatic differentiation (cf. e.g. [3], [2]). So it is sure worth implementing and judging anew numerical algorithms in which (higher) derivatives of functions are used, or to deduce new methods of this kind.

For numerically solving stiff initial value problems (IVPs) for systems of ordinary differential equations (ODEs),

$$y'(x) = f(x, y(x)), \quad y(x) \in \mathbb{R}^m \tag{1.1a}$$

$$y(x_0) = y_0 \tag{1.1b}$$

in [5] a class of methods which uses higher derivaties of the right-hand side function f was taken up again, and employing improved automatic differentiation techniques, a new implementation was suggested and examined. Those methods have been known since 1940 and can be termed implicit Taylor series methods or Hermite-Taylor series methods. They are also frequently called Obreschkoff or HOP methods. These methods may attain any given order of consistency, and they are A- or L-stable with an approriate choice of parameters. Special cases of these methods are for example the implicit Euler method and the trapezoidal rule. The reader is referred to [5] for a survey of the literature.

If stiff IVPs are solved numerically, the numerical method should, amongst others,

1. mirror the stability propertis of the exact solution as good as possible, so one can use the largest possible stepsizes if the size of the local error and the smoothness of the solution allow us to do so.

Such methods with sufficiently good stability propertis will usually be implicit. To be able to make use of the advantages of the stability, the algebraic equations of the method whose root is the next approximate of the ODE should

2. be solvable for large stepsizes.

It can not generally be expected that the equations are always solvable for large stepsizes. For dissipative systems of ODEs, however, the BDF-methods have a unique solution for all stepsizes and there are BN-stable implicit Runge-Kutta methods for which the equations are also solvable for all stepsizes.

Apart from the good properties of linear stability, I do not know of any results about non-linear stability for the implicit Taylor series methods. As far as I know there have also not yet been any investigations into the solvability of the algebraic equations for large stepsizes. The present paper is about these problems.

The paper is organized as follows: in section 2 we give a constructive proof for the existence of a unique solution of the algebraic equations of the implicit Taylor series methods in the case of sufficiently small stepsizes. In section 3 we formulate two theorems concerning non-autonomous ODEs: except for the implicit Euler method, implicit Taylor series methods are not AN-stable and except for imlicit Euler and for the trapezoidal rule, the algebraic equations of A-stable methods for dissipative systems are not always solvable for all stepsizes. Those theorems are proved by the construction of corresponding examples in section 4. In section 5 the corresponding theorems for autonomous equations are given and proved: except for the implicit Euler method, implicit Taylor series methods are not B-stable, and except for imlicit Euler and for the trapezoidal rule, the algebraic equations of A-stable methods do not always have an unique solution for all stepsizes.

2. The Taylor series method

Let y_n be a known approximation for the solution of the IVP (1.1) at the point x_n and we look for an approximate y_{n+1} for $y(x_{n+1})$ with $x_{n+1} = x_n + h_n$.

If f is sufficiently often differentiable and is then, as for example in [7], for functions $g\colon \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ the differential operator D defined by

$$(Dg)(x,y) = \frac{\partial g}{\partial x}(x,y) + \frac{\partial g}{\partial y}(x,y) \cdot f(x,y), \tag{2.1}$$

then for the solution of (1.1) we have

$$y^{(i)}(x) = (D^i y)(x, y(x)), \qquad i = 0, 1, 2, \ldots$$

and the here examined Taylor series method can be written as

$$\sum_{j=0}^{q} (-1)^j \left(D^j y\right)(x_{n+1}, y_{n+1}) \frac{h_n^j}{j!} \frac{\binom{q}{j}}{\binom{p+q}{j}} = \sum_{i=0}^{p} \left(D^i y\right)(x_n, y_n) \frac{h_n^i}{i!} \frac{\binom{p}{i}}{\binom{p+q}{i}}, \tag{2.2}$$

see e.g. [7] or [5].

For $q = 0$, (2.2) is the well known (explicit) Taylor series method. For $q > 0$, (2.2) is an implicit equation for determing y_{n+1}. In [7] the solvability of this equation is derived from the implicit function theorem. This theorem guarantees the existence of a unique solution of (2.2) for sufficiently small $|h_n|$. If we assume that the $(D^j y)(x,z)$ satisfy the Lipschitz conditions

$$\| \, (D^j y)(x,u) - (D^j y)(x,v) \, \| \leq L_j \, \| \, u - v \, \|, \qquad j = 1, \ldots, q \tag{2.3}$$

in a neighbourhood of (x_n, y_n), then one can, like in other methods, deduce an estimate for $|h_n|$:

From $q > 0$ it follows that (2.2) can be expressed as

$$y_{n+1} = \sum_{i=0}^{p} \, (D^i y)(x_n, y_n) \frac{h_n^i}{i!} \frac{\binom{p}{i}}{\binom{p+q}{i}} - \sum_{j=1}^{q} \, (-1)^j \, (D^j y)(x_{n+1}, y_{n+1}) \frac{h_n^j}{j!} \frac{\binom{q}{j}}{\binom{p+q}{j}},$$

and may be written with

$$F(z) = \sum_{i=0}^{p} \, (D^i y)(x_n, y_n) \frac{h_n^i}{i!} \frac{\binom{p}{i}}{\binom{p+q}{i}} - \sum_{j=1}^{q} \, (-1)^j \, (D^j y)(x_{n+1}, z) \frac{h_n^j}{j!} \frac{\binom{q}{j}}{\binom{p+q}{j}}$$

as

$$y_{n+1} = F(y_{n+1}).$$

As a result of the Lipschitz conditions

$$\| \, F(u) - F(v) \, \| \leq \sum_{j=1}^{q} \, L_j \frac{|h_n|^j}{j!} \frac{\binom{q}{j}}{\binom{p+q}{j}} \, \| \, u - v \, \| \, .$$

It is therefore that

Theorem 2.1. *If f is sufficiently often differentiable and $(D^j y)(x,z)$ satisfy the Lipschitz conditions* (2.3)*, in a neighbourhood of (x_n, y_n), then in the case of $q > 0$ there exists for all stepsizes h_n with*

$$\sum_{j=1}^{q} \, L_j \frac{|h_n|^j}{j!} \frac{\binom{q}{j}}{\binom{p+q}{j}} < 1$$

a unique solution y_{n+1} of (2.2)*, which can be obtained by iteration.* ■

In [7] and [5], (2.2) is derived in different ways as collocation method. This formula is also arrived at if for example in [4] the generalised Taylor formula (traced back to Hermite) is used in $y(x)$ and the remainder is left out. Method (2.2) has the order of consistency $p + q$. In [5] a continuous approximation of the solution trajectory, which is uniformly of the same order, is given.

Special cases of this method are:

$p = 1$,	$q = 0$:	explicit Euler
$p = 0$,	$q = 1$:	implicit Euler
$p = 1$,	$q = 1$:	trapezoidal rule

3. Non-autonomous equations

The method is A-stable for $q-2 \le p \le q$ and L-stable for $q-2 \le p \le q-1$. The implicit Euler method is BN-stable and hence also AN-stable, but

Theorem 3.1. *Except for the implicit Euler method, the implicit Taylor series methods* (2.2) *are not AN-stable.*

The testequation for AN-stability is

$$y'(x) = g(x)\,y, \quad \text{with} \quad \operatorname{Re} g(x) \le 0. \tag{3.1}$$

For $i > 0$, it follows from (3.1) that

$$(D^i y)(x,z) = \sum_{j=0}^{i-1} \binom{i-1}{j} g^{(j)}(x)\,(D^{i-1-j}y)(x,z) = g_i(x)\,z. \tag{3.2}$$

If we also put

$$g_0(x) \equiv 1,$$

the Taylor series method (2.2) for solving an IVP for the equation (3.1) can be written as

$$\sum_{j=0}^{q} (-1)^j g_j(x_{n+1}) \frac{h_n^j}{j!} \frac{\binom{q}{j}}{\binom{p+q}{j}}\, y_{n+1} = \sum_{i=0}^{p} g_i(x_n) \frac{h_n^i}{i!} \frac{\binom{p}{i}}{\binom{p+q}{i}}\, y_n. \tag{3.3}$$

For a real function $g(x)$ for which in the interval of integration

$$g^{(i)}(x) \begin{cases} \le 0 & \text{if} \quad i = \text{even} \\ \ge 0 & \text{if} \quad i = \text{odd} \end{cases} \tag{3.4}$$

is true, it follows from (3.2) that

$$g_i(x) \begin{cases} \ge 0 & \text{if} \quad i = \text{even} \\ \le 0 & \text{if} \quad i = \text{odd} \end{cases}$$

and

$$\sum_{j=0}^{q} (-1)^j g_j(x_{n+1}) \frac{h_n^j}{j!} \frac{\binom{q}{j}}{\binom{p+q}{j}} \ge 1$$

for all stepsizes $h_n > 0$ for which x_{n+1} lies within the interval of integration. Therefore (3.3) has an unique solution y_{n+1}. Of course, (3.4) is a highly demanding condition, and theorem 2.1 only guarantees the existence of a solution if the products $|h_n|^j L_j$ are small. In the stiff case this is not a reasonable assumption.

Let $\langle \cdot, \cdot \rangle$ be an inner product. The system (1.1a) is said to be dissipative, if

$$\langle f(x,y) - f(x,z), y - z \rangle \le 0$$

holds. As already mentioned, there are implicit integration methods for which the algebraic equations for dissipative systems are solvable for all stepsizes. The implicit Euler method and the trapezoidal rule are such methods. But

Theorem 3.2. *For $q \ge 2$ the algebraic equations of A-stable implicit Taylor series methods* (2.2) *for dissipative systems are not always solvable for all stepsizes.*

In the following section we should first prove theorem 3.2. The proof for theorem 3.1 follows then from these findings.

4. Proof of Theorems 3.1 and 3.2

To prove theorems 3.1 and 3.2, we construct and analyse scalar, real IVPs (1.1) with right-hand sides

$$f(x,y) = \lambda\, g(x)\, y, \quad \text{with} \quad \lambda < 0 \quad \text{and} \quad g(x) \geq 0. \tag{4.1}$$

Such differential equations are dissipative.

For the examination of method (2.2), we should here distinguish between cases

$$\begin{aligned} &q = 2, \quad 0 \leq p \leq 2,\\ &q = 3, \quad 1 \leq p \leq 3,\\ &q = \text{even} \geq 4, \quad q-2 \leq p \leq q,\\ &q = \text{odd} \geq 5, \quad q-2 \leq p \leq q, \end{aligned}$$

and for simplicity's sake we always choose

$$x_n = 0, \qquad \text{and} \qquad h_n = x_{n+1} = 1. \tag{4.2}$$

a) For $q = 2,\ 0 \leq p \leq 2$ we deduce from (2.2)

$$\begin{aligned} & y_{n+1} - \big(D^1 y\big)(x_{n+1}, y_{n+1}) h_n \frac{2}{p+2} \\ &\qquad + \big(D^2 y\big)(x_{n+1}, y_{n+1}) h_n^2 \frac{1}{(p+1)(p+2)} = \sum_{i=0}^{p} \big(D^i y\big)(x_n, y_n) \frac{h_n^i}{i!} \frac{\binom{p}{i}}{\binom{p+q}{i}}. \end{aligned} \tag{4.3}$$

If we choose

$$g(x) = x^{6\,(p+1)} \tag{4.4}$$

for the right hand side (4.1), (4.4) it follows

$$\big(D^i y\big)(0, y_n) \equiv 0 \qquad \text{for} \qquad 0 \leq i \leq p$$

and

$$\begin{aligned} \big(D^1 y\big)(1, y_{n+1}) &= \lambda\, y_{n+1}, \\ \big(D^2 y\big)(1, y_{n+1}) &= \lambda\,(\,6\,(p+1) + \lambda)\, y_{n+1}. \end{aligned}$$

Hence for parameters (4.2) it follows from (4.3)

$$\left[1 - \lambda \frac{2}{p+2} + \lambda\,(\,6\,(p+1) + \lambda)\, \frac{1}{(p+1)(p+2)}\right] y_{n+1} = y_n.$$

With

$$z_1(\lambda) = 1 + \frac{4}{p+2}\lambda + \frac{1}{(p+1)(p+2)}\lambda^2$$

this can be written as

$$z_1(\lambda)\cdot y_{n+1} = y_n. \tag{4.5}$$

Since

$$\lambda_1 = -2\,(p+1) - \sqrt{(p+1)(3p+2)}$$

is a solution of $z_1(\lambda)=0$, (4.5) doesn't normally have a solution for y_{n+1}.

b) In the case $q=3,\ 1\le p\le 3$ we choose

$$g(x) = x^{6p}. \tag{4.6}$$

It follows from (4.1) and (4.6) that

$$\bigl(D^i y\bigr)(0,y_n)\equiv 0 \qquad \text{for} \qquad 0\le i\le p$$

and

$$\begin{aligned}
\bigl(D^1 y\bigr)(1,y_{n+1}) &= \lambda\, y_{n+1},\\
\bigl(D^2 y\bigr)(1,y_{n+1}) &= \lambda(6p+\lambda) y_{n+1},\\
\bigl(D^3 y\bigr)(1,y_{n+1}) &= \lambda\,\bigl(\,6p(6p-1)+18p\lambda+\lambda^2\bigr)\, y_{n+1}.
\end{aligned}$$

For the parameters (4.2) we therefore derive

$$\begin{aligned}
y_{n+1} - \lambda y_{n+1}\frac{1}{1!}\frac{\binom{3}{1}}{\binom{p+3}{1}} + \lambda(6p+\lambda)y_{n+1}\frac{1}{2!}\frac{\binom{3}{2}}{\binom{p+3}{2}}& \\
-\,\lambda\,\bigl(\,6p(6p-1)+18p\lambda+\lambda^2\bigr)\, y_{n+1}\frac{1}{3!}\frac{\binom{3}{3}}{\binom{p+3}{3}} &= y_n
\end{aligned} \tag{4.7}$$

from (2.2). After some simple transformations, (4.7) can with the aid of

$$z_2(\lambda) = 1 - \frac{21p^2-15p}{(p+1)(p+2)(p+3)}\lambda - \frac{15p-3}{(p+1)(p+2)(p+3)}\lambda^2 - \frac{1}{(p+1)(p+2)(p+3)}\lambda^3$$

be written as

$$z_2(\lambda)\cdot y_{n+1} = y_n.$$

With

$$\begin{aligned}
\alpha_1 &= \frac{\pi}{3} - \frac{1}{3}\ \arctan\left(\frac{1}{7}\sqrt{59}\right),\\
\alpha_2 &= \frac{\pi}{3} - \frac{1}{3}\ \arctan\left(\frac{2}{429}\sqrt{10735}\right),\\
\alpha_3 &= \frac{\pi}{3} - \frac{1}{3}\ \arctan\left(\frac{1}{817}\sqrt{110545}\right),
\end{aligned}$$

$$\lambda_2 = \begin{cases} -4 + 6\,sin(\alpha_1) - 2\sqrt{3}\,cos(\alpha_1) \approx -2.310 & \text{if} \quad p = 1 \\ -9 - \sqrt{3}\sqrt{61}\,sin(\alpha_2) - \sqrt{61}\,cos(\alpha_2) \approx -24.445 & \text{if} \quad p = 2 \\ -14 - \sqrt{3}\sqrt{146}\,sin(\alpha_3) - \sqrt{146}\,cos(\alpha_2) \approx -37.966 & \text{if} \quad p = 3 \end{cases}$$

is a solution of $z_2(\lambda) = 0$.

c) For $q = \text{even} \geq 4,\ q - 2 \leq p \leq q$ we choose

$$g(x) = x^{2(p+1)}(x-1)^{q-2}. \tag{4.8}$$

From (4.1) and (4.8) it follows that

$$\big(D^i y\big)(x,y) = \lambda \sum_{k=0}^{i-1} \binom{i-1}{k} \big(g^{(k)}\,\big(D^{i-1-k}y\big)\big)(x,y), \qquad i = 1, 2, \ldots \tag{4.9}$$

and

$$g^{(k)}(x) = \begin{cases} \sum_{l=max(0,k+2-q)}^{min(2p+2,k)} \binom{k}{l} \dfrac{(2p+2)!}{(2p+2-l)!} x^{2p+2-l} \dfrac{(q-2)!}{(q-2-k+l)!}(x-1)^{q-2-k+l}, \\ \qquad\qquad\qquad\qquad \text{for} \quad max(0,k+2-q) \leq min(2p+2,k) \\ 0, \qquad \text{otherwise} \end{cases}$$

with

$$g^{(k)}(0) = 0 \qquad \text{for} \qquad k = 0, 1, \ldots, q-1$$

and

$$g^{(k)}(1) = \begin{cases} 0 & \text{for} \quad 0 \leq k \leq q-3, \\ (q-2)! & \text{for} \quad k = q-2, \\ 2(p+1)(q-1)! & \text{for} \quad k = q-1. \end{cases}$$

With this we further derive

$$\big(D^i y\big)(0, y_n) \equiv 0 \qquad \text{if} \qquad i = 1, \ldots, p$$

and

$$\big(D^j y\big)(1, y_{n+1}) \equiv 0 \qquad \text{if} \qquad j = 1, \ldots, q-2,$$

$$\big(D^{q-1} y\big)(1, y_{n+1}) = \lambda \binom{q-2}{q-2} \big(g^{(q-2)}\,\big(D^0 y\big)\big)(1, y_{n+1}),$$

$$= \lambda (q-2)!\, y_{n+1},$$

$$(D^q y)(1, y_{n+1}) = \lambda \left[\binom{q-1}{q-2} \left(g^{(q-2)}\left(D^1 y\right)\right)(1, y_{n+1}) + \binom{q-1}{q-1} \left(g^{(q-1)}\left(D^0 y\right)\right)(1, y_{n+1}) \right],$$

$$= \lambda \binom{q-1}{q-1} \left(g^{(q-1)}\left(D^0 y\right)\right)(1, y_{n+1}),$$

$$= 2\lambda(p+1)(q-1)!\, y_{n+1}$$

from (4.9). For the parameters (4.2), (2.2) results in

$$y_{n+1} - \lambda(q-2)!\, y_{n+1} \frac{1}{(q-1)!} \frac{\binom{q}{q-1}}{\binom{p+q}{q-1}} + 2\lambda(p+1)(q-1)!\, y_{n+1} \frac{1}{q!} \frac{\binom{q}{q}}{\binom{p+q}{q}} = y_n. \tag{4.10}$$

With

$$z_3(\lambda) = 1 + \frac{(p+1)!(q-2)(q-2)!}{(p+q)!}\, \lambda,$$

(4.10) can be written as

$$z_3(\lambda) \cdot y_{n+1} = y_n.$$

But

$$\lambda_3 = -\frac{(p+q)!}{(p+1)!(q-2)(q-2)!}.$$

is a solution of $z_3(\lambda) = 0$.

d) $q = \text{odd} \geq 5,\ q-2 \leq p \leq q:$ Let α be an even number with

$$\alpha > 1 + \frac{2(p+1)q}{q-1}, \tag{4.11}$$

$$\beta = -\frac{2(q-2)\left[\alpha(q-1) - (p+1)q\right]}{(p+1)(p+2)(q-1)q + \alpha(q-2)\left[(\alpha-1)(q-1) - 2(p+1)q\right]}, \tag{4.12}$$

and we choose

$$g(x) = x^\alpha (x-1+\beta)^2 (x-1)^{q-3}. \tag{4.13}$$

It follows from (4.11) that

$$(\alpha-1)(q-1) - 2(p+1)\, q > 0, \tag{4.14}$$

so that for the here considered p and q, β is well defined. If we let

$$u(x) = x^\alpha (x-1+\beta)^2,$$
$$v(x) = (x-1)^{q-3},$$

then we derive

$$g^{(k)}(x) = \begin{cases} \sum_{l=max(0,k+3-q)}^{min(\alpha+2,k)} \binom{k}{l} u^{(l)}(x)\, v^{(k-l)}(x), \\ \qquad\qquad \text{for} \quad max(0,k+3-q) \le min(\alpha+2,k) \\ 0, \qquad \text{otherwise} \end{cases} \tag{4.15}$$

from (4.13), with

$$\begin{aligned} u^{(l)}(0) &= 0 \qquad \text{if} \qquad l = 0, 1, \ldots, \alpha-1, \\ u^{(0)}(1) &= \beta^2, \\ u^{(1)}(1) &= \alpha\beta^2 + 2\beta, \\ u^{(2)}(1) &= \alpha(\alpha-1)\beta^2 + 4\alpha\beta + 2, \end{aligned}$$

and

$$\begin{aligned} v^{(l)}(1) &= 0 \qquad \text{for} \qquad l = 0, \ldots, q-4, \\ v^{(q-3)}(1) &= (q-3)!\ . \end{aligned}$$

Because of (4.11), it is that $\alpha > 2p$, so that it follows from (4.15) that

$$g^{(k)}(0) = 0 \qquad \text{for} \qquad k = 0, 1, \ldots, p-1$$

and

$$\begin{aligned} g^{(k)}(1) &= 0 \qquad \text{for} \qquad k = 0, 1, \ldots, q-4, \\ g^{(q-3)}(1) &= \binom{q-3}{0} u^{(0)}(1)\, v^{(q-3)}(1) = \beta^2 (q-3)!\ , \\ g^{(q-2)}(1) &= \binom{q-2}{1} u^{(1)}(1)\, v^{(q-3)}(1) = \beta(\alpha\beta+2)(q-2)!\ , \\ g^{(q-1)}(1) &= \binom{q-1}{2} u^{(2)}(1)\, v^{(q-3)}(1) = \frac{1}{2}\left[\alpha(\alpha-1)\beta^2 + 4\alpha\beta + 2\right](q-1)!\ . \end{aligned}$$

Therefore from (4.9) we derive

$$\big(D^i y\big)(0, y_n) \equiv 0 \qquad \text{for} \qquad i = 1, 2, \ldots, p$$

and

$$\begin{aligned} \big(D^j y\big)(1, y_{n+1}) &\equiv 0 \qquad \text{for} \qquad j = 1, \ldots, q-3, \\ \big(D^{q-2} y\big)(1, y_{n+1}) &= \lambda \binom{q-3}{q-3} \big(g^{(q-3)}(D^0 y)\big)(1, y_{n+1}), \\ &= \lambda\beta^2 (q-3)!\, y_{n+1}, \end{aligned}$$

$$\begin{aligned}
\big(D^{q-1}y\big)(1,y_{n+1}) &= \lambda\Bigg[\binom{q-2}{q-3}\,\big(g^{(q-3)}(D^1y)\big)(1,y_{n+1})\\
&\qquad\qquad + \binom{q-2}{q-2}\,\big(g^{(q-2)}(D^0y)\big)(1,y_{n+1})\Bigg],\\
&= \lambda\binom{q-2}{q-2}\,\big(g^{(q-2)}(D^0y)\big)(1,y_{n+1}),\\
&= \lambda\beta(\alpha\beta+2)\,(q-2)!\,y_{n+1},\\
(D^qy)(1,y_{n+1}) &= \lambda\Bigg[\binom{q-1}{q-3}\,\big(g^{(q-3)}(D^2y)\big)(1,y_{n+1})\\
&\qquad + \binom{q-1}{q-2}\,\big(g^{(q-2)}(D^1y)\big)(1,y_{n+1})\\
&\qquad\qquad + \binom{q-1}{q-1}\,\big(g^{(q-1)}(D^0y)\big)(1,y_{n+1})\Bigg],\\
&= \lambda\binom{q-1}{q-1}\,\big(g^{(q-1)}(D^0y)\big)(1,y_{n+1}),\\
&= \frac{\lambda}{2}\,\big[\alpha(\alpha-1)\beta^2+4\alpha\beta+2\big]\,(q-1)!\,y_{n+1}.
\end{aligned}$$

Hence method (2.2) is

$$\begin{aligned}
(4.16)\quad y_{n+1} - \lambda\beta^2(q-3)!\,y_{n+1}\frac{1}{(q-2)!}\frac{\binom{q}{q-2}}{\binom{p+q}{q-2}}&\\
+\lambda\beta(\alpha\beta+2)\,(q-2)!\,y_{n+1}\frac{1}{(q-1)!}\frac{\binom{q}{q-1}}{\binom{p+q}{q-1}}&\\
-\frac{\lambda}{2}\,\big[\alpha(\alpha-1)\beta^2+4\alpha\beta+2\big]\,(q-1)!\,y_{n+1}\frac{1}{q!}\frac{\binom{q}{q}}{\binom{p+q}{q}} &= y_n
\end{aligned}$$

for parameters (4.2), and after a simple change

$$\begin{aligned}
\Bigg\{1-\frac{p!\,(q-3)!}{2\,(p+q)!}\Big\{\Big[(p+1)(p+2)(q-1)\,q+\alpha\,(q-2)\big((\alpha-1)(q-1)-2\,(p+1)\,q\big)\Big]\beta^2&\\
+4\,(q-2)\big[\alpha(q-1)-(p+1)\,q\big]\beta+2\,(q-2)(q-1)\Big\}\,\lambda\Bigg\}\,y_{n+1} &= y_n.
\end{aligned}$$

We substitute β using (4.12), and after further changes, with

$$a_4(p,q)=\frac{p!\,(q-2)!}{(p+q)!},$$

$$b_4(\alpha,p,q) = (p+1)\,q\,[p\,(q\,(q-2)-1)-2] + \alpha(q-2)(q-1)\,[(\alpha+1)(q-1)-2\,(p+1)\,q]\,,$$

$$c_4(\alpha,p,q) = (p+1)(p+2)(q-1)\,q + \alpha\,(q-2)\,[(\alpha-1)(q-1)-2\,(p+1)\,q],$$

and

$$z_4(\lambda) = 1 + a_4(p,q)\cdot\frac{b_4(\alpha,p,q)}{c_4(\alpha,p,q)}\cdot\lambda,$$

(4.16) can be expressed as

$$z_4(\lambda)\cdot y_{n+1} = y_n.$$

But $z_4(\lambda)=0$ has the solution

$$\lambda_4 = -\frac{c_4(\alpha,p,q)}{a_4(p,q)\cdot b_4(\alpha,p,q)}.$$

By reason of (4.14), for the here looked at p, q and α, it is that

$$\lambda_4 < 0.$$

This proves theorem 3.2 .

e) Out of the methods (2.2), only the implicit Euler method and the trapezoidal rule are A-stable for $q=1$. The implicit Euler method is BN- and therefore also AN-stable. It is known of the trapezoidal rule that it is not AN-stable. Hence, to prove theorem 3.1 we only need to look at the methods (2.2) with $q\geq 2$, $q-2\leq p\leq q$.

If only one numerical method is AN-stable, when solving an initial value problem (1.1) with right-hand side (4.1), $|y_{n+1}|\leq|y_n|$ must be true for all x_n, y_n and all stepsizes $h_n>0$. Again we look at the examples constructed above. For those examples, method (2.2) may be written in the form of

$$z_i(\lambda)\cdot y_{n+1} = y_n, \qquad i=1,\dots,4.$$

$z_i(\lambda)$ are polynomials in λ with $z_i(\lambda_i)=0$. For all $i=1,\dots,4$ $\quad\lambda_i<0\quad$ is true. Therefore, for each sufficiently small $0<\varepsilon<1$, there exists a

$$\hat\lambda_i = \hat\lambda_i(\varepsilon) < 0,$$

so that

$$|z_i(\hat\lambda_i)|\leq\varepsilon, \quad \text{for} \quad i=1,\dots,4,$$

and

$$|y_{n+1}| \geq \frac{1}{\varepsilon}|y_n| > |y_n|$$

holds. This proves theorem 3.1 . ■

5. Autonomous equations

It is well known (cf. e.g.[8]) that the concepts of BN-stability (based on non-autonomous systems) and B-stability (based on autonomous systems) are equivalent for irreducible Runge-Kutta methods. This equivalency is proved by showing that for these methods B-stability is equivalent to algebraic stability. 'It is a deep and difficult mathematical result' (cited from [8]). Up to now for the Taylor series methods such an equivalency is not known. Here we want to show, by constructing the corresponding examples, that B-stable Taylor series methods of higher orders do not exist.

By the usual trick of appending the time, or the indepedent variable, as an extra component to the vector $y(x)$, every non-autonomous system of differential equations can be transformed into an autonomous system. The Taylor series method (2.2) is invariant with respect to this transformation in the sense that the resulting approximations are the same, provided the algebraic equations are solved exactly at each step. If one transforms the differential equations, which we constructed in order to prove the theorems 3.1 and 3.2, into autonomous systems, those will, however, not be dissipative automomous systems. But it is not difficult to find corresponding dissipative autonomous equations.

Theorem 5.1. *Except for the implicit Euler method, the implicit Taylor series methods* (2.2) *are not B-stable.*

Proof. To prove this theoren we want to distinguish between cases

$$p = 0, \quad q = 2,$$
$$p = 1, \quad q = 2,$$
$$p = 1, \quad q = 3,$$

and

$$p, q \geq 2.$$

a) We look at the differential equation

$$y' = -(y^3 + 1)$$

with

$$(D^2 y)(z) = 3\,z^2\,(z^3 + 1),$$
$$(D^3 y)(z) = -3\,z\,(5z^3 + 2)(z^3 + 1).$$

We choose two different initial values y_n and $\hat{y}_n$ and calculate the corresponding new approximations y_{n+1} and $\hat{y}_{n+1}$. For simplicity's sake we always use $h_n = 1$ as stepsize.

For $p = 0$, $q = 2$ and the initial values

$$y_n = 1 \quad \text{and} \quad \hat{y}_n = \frac{45}{64},$$

the Taylor series method (2.2) gives the new approximations of

$$y_{n+1} = 0 \quad \text{and} \quad \hat{y}_{n+1} = -\frac{1}{2}.$$

For $p = 1$, $q = 2$ and the initial values

$$y_n = -\left(\frac{3+\sqrt{5}}{2}\right)^{\frac{1}{3}} - \left(\frac{3+\sqrt{5}}{2}\right)^{-\frac{1}{3}} \approx -2.103803403$$

and

$$\hat{y}_n = -\left(\frac{331+3\sqrt{10353}}{128}\right)^{\frac{1}{3}} - \left(\frac{331+3\sqrt{10353}}{128}\right)^{-\frac{1}{3}} \approx -2.292579289$$

we arrive, thaks to the computer algebra program Maple, at

$$y_{n+1} = 0 \quad \text{and} \quad \hat{y}_{n+1} = \frac{1}{2}.$$

For $p = 1$, $q = 3$ and the initial values

$$y_n = -\left(\frac{18+2\sqrt{33}}{9}\right)^{\frac{1}{3}} - \frac{4}{3}\left(\frac{18+2\sqrt{33}}{9}\right)^{-\frac{1}{3}} \approx -2.382975767$$

and

$$\hat{y}_n = -\left(\frac{2037}{512} + \frac{\sqrt{285767241}}{4608}\right)^{\frac{1}{3}} - \frac{4}{3}\left(\frac{2037}{512} + \frac{\sqrt{285767241}}{4608}\right)^{-\frac{1}{3}} \approx -2.646914051$$

we again arrive at

$$y_{n+1} = 0 \quad \text{and} \quad \hat{y}_{n+1} = \frac{1}{2}.$$

For all the above cases, $|y_{n+1} - \hat{y}_{n+1}| > |y_n - \hat{y}_n|$ is true, so that the methods are not B-stable.

b) In the case of $p, q \geq 2$ we look at the differential equation $y' = f(y)$ with

$$f(y) = \begin{cases} -y & \text{if} \quad |y-1| \geq 2\varepsilon \\ -1 & \text{if} \quad |y-1| \leq \varepsilon \\ f_1(y) & \text{if} \quad 1-2\varepsilon \leq y \leq 1-\varepsilon \\ f_2(y) & \text{if} \quad 1+\varepsilon \leq y \leq 1+2\varepsilon\ , \end{cases}$$

$0 < \varepsilon \ll 1$ and the functions f_1 and f_2 are chosen so that f is non-increasing and $p+q$ times differentiable. After the lemma 5.1 below, such f_1, f_2 can always be found. This equation was used earlier to prove that Rosenbrock methods are not B-stable (cf. e.g. [12], [13], or [10]).

If $|z-1| \leq \varepsilon$ then it is true that

$$(D^1 y)(z) = -1 \quad \text{and} \quad (D^i y)(z) = 0 \quad \text{if} \quad i = 2, 3, \ldots \tag{5.1}$$

and if $|z-1| \geq 2\varepsilon$ it is true that

$$(D^i y)(z) = (-1)^i z \quad \text{if} \quad i = 1, 2, \ldots \tag{5.2}$$

For $y_n = 0$ the Taylor series method gives for all stepsizes $h_n > 0$ the approximation $y_{n+1} = 0$. If we take

$$\hat{y}_n = \frac{1 + \frac{q}{p+q} h_n}{\sum_{i=0}^{p} (-1)^i \frac{h_n^i}{i!} \frac{\binom{p}{i}}{\binom{p+q}{i}}}$$

and h_n is chosen so that $|\hat{y}_n| < 1$ and $|\hat{y}_n - 1| \geq 2\varepsilon$, then the Taylor series method results in the approximation $\hat{y}_{n+1} = 1$ because of (5.1) and (5.2). So $|y_{n+1} - \hat{y}_{n+1}| > |y_n - \hat{y}_n|$ is once again true. ∎

Theorem 5.2. *For $q \geq 2$ the algebraic equations of A-stable implicit Taylor series methods* (2.2) *for dissipative systems do not always have a unique solution for all stepsizes.*

Proof. If we put

$$F(p,q,h,z) = \sum_{j=0}^{q} (-1)^j \left(D^j y\right)(z) \frac{h^j}{j!} \frac{\binom{q}{j}}{\binom{p+q}{j}}$$

and

$$G(p,q,h,z) = \sum_{i=0}^{p} \left(D^i y\right)(z) \frac{h^i}{i!} \frac{\binom{p}{i}}{\binom{p+q}{i}}$$

then the method (2.2) for autonomous systems can be expressed as

$$F(p,q,h_n,y_{n+1}) = G(p,q,h_n,y_n). \tag{5.3}$$

For the dissipative equation

$$y' = -y + \frac{1}{2}\sin(2\,y),$$

$$F(p,q,h_n,0) = G(p,q,h_n,0)$$

is true for all $p, q \geq 0$ and all $h_n > 0$, and we have

$$F(0,2,5,2) \approx 63.054857, \quad F(0,2,5,3) \approx 20.261709, \quad G(0,2,5,30) = 30,$$

$$F(1,2,5,2) \approx 26.315621, \quad F(1,2,5,3) \approx 13.986749, \quad G(1,2,5,-30) \approx 20.254009,$$

$$F(2,2,5,2) \approx 16.139812, \quad F(2,2,5,3) \approx 11.109798, \quad G(2,2,5,5) \approx 12.019150.$$

Therefore (5.3) does not always have a unique solution for $q = 2$ and $h_n = 5$.

For $q \geq 3$ it is true that, according to lemma 5.2, at $|y_{n+1}| \to \infty$

$$F(p,q,h_n,y_{n+1}) = \mathcal{O}\left(\cos\left(2\,y_{n+1} + \operatorname{sign}\left(\left[\frac{q}{2}\right] - \frac{q}{2}\right)\frac{\pi}{2}\right) y_{n+1}^{q-1}\right),$$

so that (5.3) does not have a unique solution for $y_n = 0$ and a fixed h_n. ∎

Lemma 5.1. *The Hermite interpolation polynomials f_1 and f_2, defined by*

$$f_1(1-2\varepsilon) = -(1-2\varepsilon), \quad f_1'(1-2\varepsilon) = -1, \quad f_1''(1-2\varepsilon) = \cdots = f_1^{(p+q)}(1-2\varepsilon) = 0,$$

$$f_1(1-\varepsilon) = -1, \quad f_1'(1-\varepsilon) = \cdots = f_1^{(p+q)}(1-\varepsilon) = 0$$

and

$$f_2(1+\varepsilon) = -1, \quad f_2'(1+\varepsilon) = \cdots = f_2^{(p+q)}(1+\varepsilon) = 0,$$

$$f_2(1+2\varepsilon) = -(1+2\varepsilon), \quad f_2'(1+2\varepsilon) = -1, \quad f_2''(1+2\varepsilon) = \cdots = f_2^{(p+q)}(1+2\varepsilon) = 0,$$

with $\varepsilon > 0$*, are non-increasing functions in* $[1-2\varepsilon, 1-\varepsilon]$ *respectively* $[1+\varepsilon, 1+2\varepsilon]$.

Proof. If we use $r = p+q$ as an abreviation, then it is true (cf. e.g. [1]) that

$$f_1(y) = \sum_{j=0}^{1} \sum_{k=0}^{r-j} y_{1j} \frac{1}{j!} \frac{1}{k!} \left[\frac{(y-1+2\,\varepsilon)^{r+1}}{\Omega_1(y)} \right]^{(k)}_{y=1-2\varepsilon} \frac{\Omega_1(y)}{(y-1+2\,\varepsilon)^{r+1-j-k}} - 1,$$

with

$$y_{10} = 2\,\varepsilon, \quad y_{11} = -1$$

and

$$\Omega_1(y) = (y-1+2\,\varepsilon)^{r+1}(y-1+\varepsilon)^{r+1}.$$

Because of

$$\left[\frac{(y-1+2\,\varepsilon)^{r+1}}{\Omega_1(y)} \right]^{(k)}_{y=1-2\varepsilon} = \left[\frac{1}{(y-1+\varepsilon)^{r+1}} \right]^{(k)}_{y=1-2\varepsilon}$$
$$= (-1)^k \frac{(r+k)!}{r!} \frac{1}{(-\varepsilon)^{r+1+k}} = (-1)^{r+1} \frac{(r+k)!}{r!} \frac{1}{\varepsilon^{r+1+k}}$$

it is then true that

$$\begin{aligned} f_1(y) &= 2\,\varepsilon \sum_{k=0}^{r} \frac{1}{k!} (-1)^{r+1} \frac{(r+k)!}{r!} \frac{1}{\varepsilon^{r+1+k}} (y-1+2\,\varepsilon)^k (y-1+\varepsilon)^{r+1} \\ &\quad - \sum_{k=0}^{r-1} \frac{1}{k!} (-1)^{r+1} \frac{(r+k)!}{r!} \frac{1}{\varepsilon^{r+1+k}} (y-1+2\,\varepsilon)^{1+k} (y-1+\varepsilon)^{r+1} - 1, \\ &= \left[2 \sum_{k=0}^{r} \frac{(r+k)!}{k!} \frac{(y-1+2\,\varepsilon)^k}{\varepsilon^k} - \sum_{k=0}^{r-1} \frac{(r+k)!}{k!} \frac{(y-1+2\,\varepsilon)^{1+k}}{\varepsilon^{1+k}} \right] \\ &\quad \times \frac{(-1)^{r+1}}{r!} \frac{(y-1+\varepsilon)^{r+1}}{\varepsilon^r} - 1, \\ &= \left\{ 2 \cdot r! + \sum_{k=1}^{r} \frac{(r+k-1)!}{(k-1)!} \left[\frac{2(r+k)}{k} - 1 \right] \frac{(y-1+2\,\varepsilon)^k}{\varepsilon^k} \right\} \\ &\quad \times \frac{(-1)^{r+1}}{r!} \frac{(y-1+\varepsilon)^{r+1}}{\varepsilon^r} - 1, \\ f_1(y) &= \left\{ 2 \cdot r! + \sum_{k=1}^{r} \frac{(r+k-1)!}{(k-1)!} \frac{2\,r+k}{k} \frac{(y-1+2\,\varepsilon)^k}{\varepsilon^k} \right\} \frac{(-1)^{r+1}}{r!} \frac{(y-1+\varepsilon)^{r+1}}{\varepsilon^r} - 1. \end{aligned}$$

From this it follows that

$$f_1'(y) = \left\{ \sum_{k=1}^{r} \frac{(r+k-1)!}{(k-1)!} (2\,r+k) \frac{(y-1+2\,\varepsilon)^{k-1}}{\varepsilon^k} \right\} \frac{(-1)^{r+1}}{r!} \frac{(y-1+\varepsilon)^{r+1}}{\varepsilon^r}$$
$$+ \left\{ 2\cdot r! + \sum_{k=1}^{r} \frac{(r+k-1)!}{(k-1)!} \frac{2\,r+k}{k} \frac{(y-1+2\,\varepsilon)^k}{\varepsilon^k} \right\} \frac{(-1)^{r+1}}{r!} (r+1) \frac{(y-1+\varepsilon)^r}{\varepsilon^r},$$

and because

$$(y-1+\varepsilon)^{r+1} = (y-1+2\varepsilon-\varepsilon)(y-1+\varepsilon)^r = (y-1+2\,\varepsilon)(y-1+\varepsilon)^r - \varepsilon\,(y-1+\varepsilon)^r,$$

$$f_1'(y) = \left\{ \sum_{k=1}^{r} \frac{(r+k-1)!}{(k-1)!} (2\,r+k) \frac{(y-1+2\,\varepsilon)^k}{\varepsilon^k} \right.$$
$$\left. - \sum_{k=1}^{r} \frac{(r+k-1)!}{(k-1)!} (2\,r+k) \frac{(y-1+2\,\varepsilon)^{k-1}}{\varepsilon^{k-1}} \right\} \frac{(-1)^{r+1}}{r!} \frac{(y-1+\varepsilon)^r}{\varepsilon^r}$$
$$+ \left\{ 2\cdot r! + \sum_{k=1}^{r} \frac{(r+k-1)!}{(k-1)!} \frac{2\,r+k}{k} \frac{(y-1+2\,\varepsilon)^k}{\varepsilon^k} \right\} \frac{(-1)^{r+1}}{r!} (r+1) \frac{(y-1+\varepsilon)^r}{\varepsilon^r},$$
$$= \left\{ r! + \sum_{k=1}^{r-1} \left[\frac{(r+k-1)!}{(k-1)!} (2\,r+k) + \frac{(r+k)!}{k!} (2\,r+k+1) \right.\right.$$
$$\left. + \frac{(r+k-1)!}{(k-1)!} \frac{2\,r+k}{k} (r+1) \right] \frac{(y-1+2\,\varepsilon)^k}{\varepsilon^k}$$
$$\left. + \left[\frac{(2\,r-1)!}{(r-1)!} 3\,r + \frac{(2\,r-1)!}{(r-1)!} \frac{3\,r}{r} (r+1) \right] \frac{(y-1+2\,\varepsilon)^r}{\varepsilon^r} \right\} \frac{(-1)^{r+1}}{r!} \frac{(y-1+\varepsilon)^r}{\varepsilon^r},$$
$$f_1'(y) = \left\{ r! + \sum_{k=1}^{r-1} \frac{(r+k-1)!}{k!} \frac{(y-1+2\,\varepsilon)^k}{\varepsilon^k} \right.$$
$$\left. + 3\,\frac{(2\,r-1)!}{(r-1)!} (2\,r+1) \frac{(y-1+2\,\varepsilon)^r}{\varepsilon^r} \right\} \frac{(-1)^{r+1}}{r!} \frac{(y-1+\varepsilon)^r}{\varepsilon^r}.$$

For $y \in [1-2\,\varepsilon, 1-\varepsilon]$ $f_1'(y) \le 0$ is true. The polynomial $f_2(y)$ can be arrived at after mapping $f_1(y)$ twice: $f_2(y) = -[2 + f_1(2-y)]$. ■

Lemma 5.2. *For the function*

$$f(y) = -y + \frac{1}{2}\sin(2\,y), \tag{5.4}$$

for the differential operator D which is defined by (2.1), it is true for $r \ge 3$ that

$$(D^r y)(z) = a_r\, cos\left(2\,z + \mathrm{sign}\left(\left[\frac{r}{2}\right] - \frac{r}{2}\right)\frac{\pi}{2}\right) z^{r-1} + \sum_{i=0}^{r-2} f_{ri}(\sin(2\,z), \cos(2\,z))\, z^i,$$

where $a_r \neq 0$ is an integer, $[a]$ denotes the integer part of $a \in \mathbb{R}$ and $f_{ri}(x,y)$ are polynomials in x and y.

Proof. From (2.1) and (5.4) it follows that

$$\left(D^2 y\right)(z) = (-1+\cos(2\,z))(-z+\frac{1}{2}\sin(2\,z))$$

and

$$\left(D^3 y\right)(z) = -2\sin(2\,z)\,z^2 + [1+2\cos(2\,z)-3\cos^2(2\,z)]\,z + \sin(2\,z)\cos(2\,z)\,[\cos(2\,z)-1],$$

so that the lemma for $r=3$ is correct. If we assume its correctness for a $r>3$, than it follows from

$$\begin{aligned}
\left[\frac{d}{dy} cos\left(2\,y + \operatorname{sign}\left(\left[\frac{r}{2}\right]-\frac{r}{2}\right)\frac{\pi}{2}\right) y^{r-1}\right] f(y) = &\left[-2\sin\left(2\,y + \operatorname{sign}\left(\left[\frac{r}{2}\right]-\frac{r}{2}\right)\frac{\pi}{2}\right) y^{r-1}\right.\\
&\left. + (r-1)\cos\left(2\,y + \operatorname{sign}\left(\left[\frac{r}{2}\right]-\frac{r}{2}\right)\frac{\pi}{2}\right) y^{r-2}\right]\left[-y+\frac{1}{2}\sin(2\,y)\right],\\
= 2\sin\left(2\,y + \operatorname{sign}\left(\left[\frac{r}{2}\right]-\frac{r}{2}\right)\frac{\pi}{2}\right) y^{r} &- (r-1)\cos\left(2\,y + \operatorname{sign}\left(\left[\frac{r}{2}\right]-\frac{r}{2}\right)\frac{\pi}{2}\right) y^{r-1}\\
&- \sin\left(2\,y + \operatorname{sign}\left(\left[\frac{r}{2}\right]-\frac{r}{2}\right)\frac{\pi}{2}\right) y^{r-1}\\
&+ \frac{r-1}{2}\sin(2\,y)\cos\left(2\,y + \operatorname{sign}\left(\left[\frac{r}{2}\right]-\frac{r}{2}\right)\frac{\pi}{2}\right) y^{r-2},\\
= 2\,(-1)^r\cos\left(2\,y + \operatorname{sign}\left(\left[\frac{r+1}{2}\right]-\frac{r+1}{2}\right)\frac{\pi}{2}\right) y^{r}&\\
- \left[(r-1)\cos\left(2\,y + \operatorname{sign}\left(\left[\frac{r}{2}\right]-\frac{r}{2}\right)\frac{\pi}{2}\right) \right.&\left.+ \sin\left(2\,y + \operatorname{sign}\left(\left[\frac{r}{2}\right]-\frac{r}{2}\right)\frac{\pi}{2}\right)\right] y^{r-1}\\
&+ \frac{r-1}{2}\sin(2\,y)\cos\left(2\,y + \operatorname{sign}\left(\left[\frac{r}{2}\right]-\frac{r}{2}\right)\frac{\pi}{2}\right) y^{r-2}
\end{aligned}$$

and

$$\begin{aligned}
\left[\frac{d}{dy}\sin^i(2\,y)\cos^j(2\,y)\,y^k\right] f(y) = &\left[2i\sin^{i-1}(2\,y)\cos^{j+1}(2\,y)\,y^k\right.\\
&\left.-2j\sin^{i+1}(2\,y)\cos^{j-1}(2\,y)\,y^k + k\sin^i(2\,y)\cos^j(2\,y)\,y^{k-1}\right]\left[-y+\frac{1}{2}\sin(2\,y)\right],\\
= 2\Big[j\sin^{i+1}(2\,y)\cos^{j-1}(2\,y) - i\sin^{i-1}(2\,y)\cos^{j+1}(2\,y)\Big]\,y^{k+1}&\\
+ \Big[i\sin^{i}(2\,y)\cos^{j+1}(2\,y) - j\sin^{i+2}(2\,y)\cos^{j-1}(2\,y) &- k\sin^i(2\,y)\cos^j(2\,y)\Big]\,y^k\\
&+ \frac{k}{2}\sin^{i+1}(2\,y)\cos^j(2\,y)\,y^{k-1}
\end{aligned}$$

that it is also correct for $r+1$. ■

6. Summary, open problems and future work

In this paper we have considered a family of implicit one-step integrators for the numerical solution of initial value problems of ordinary differential equations. It was shown that these methods, exept for the implicit Euler method, are not AN-stable nor B-stable, and that the algebraic system whose root is the next approximate solution point of the ODE are not always or at least not always uniquely solvable for large stepsizes.

There are of course a number of numerical methods in use which are, at most, A-stable. The here given examples are, without any practical background, constructed artificially only to prove the theorems. It could be, however, that the here introduced results can explain certain difficulties with the so-called HOP-code (cf. e.g. [6]). I think that for a well founded assessment of these metods, further numerical tests are needed.

If a method has good linear, but no non-linear stability properties, then we can investigate the solution of problems with only small nonlinear perturbations, so-called semi-linear problems. This is currently under investigation.

The here investigated methods can be regarded as a special case of the method

$$\sum_{j=0}^{q} \beta_j h_n^j \left(D^j y\right)(x_{n+1}, y_{n+1}) = \sum_{i=0}^{p} \alpha_i h_n^i \left(D^i y\right)(x_n, y_n), \tag{6.1}$$

where the α_i and β_j are free parameters (cf. e.g. [11], [9] or [7]). One may therefore ask wether the α_i and β_j can be chosen in such a way that better stability- or solvability properties can be reached. First investigations make it look as if this is not possible. For example the following theorem holds:

Theorem 6.1. *For $q = 2$ and $\beta_2 \neq 0$, there exists no AN-stable method* (6.1) *and there is no such method whose algebraic equations for dissipative systems are always solvable for all stepsizes $h_n > 0$.*

Proof. Let r be an integer with

$$r \geq \frac{1}{\sqrt{|\beta_2|}} + \left|\frac{\beta_1}{\beta_2}\right|. \tag{6.2}$$

For $q = 2$, the right-hand side (4.1) with $g(x) = x^{2r}$ and parameters (4.2) we deduce

$$[1 + \beta_1\lambda + \beta_2(2r\lambda + \lambda^2)]y_{n+1} = y_n,$$

from (2.2). With

$$z_5(\lambda) = \beta_2\,\lambda^2 + (\beta_1 + 2r\beta_2)\lambda + 1,$$

this formula can be expressed as

$$z_5(\lambda)\,y_{n+1} = y_n.$$

$$\lambda_5 = \frac{-(\beta_1 + 2r\beta_2) - \sqrt{(\beta_1 + 2r\beta_2)^2 - 4\beta_2}}{2\beta_2}$$

is a root of $z_5(\lambda)$. Because of (6.2) we have $\lambda_5 < 0$. ■

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Hans-Eberhard Scholz
FB Mathematik und Informatik
Martin-Luther Universität
Halle-Wittenberg
Postfach
D-06099 Halle/Saale
Germany
e-mail: eberhard@mail.mathematik.uni-halle.de