STRUCTURAL ASPECTS OF ORDERED POLYMATROIDS

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ABSTRACT. This paper generalizes some aspects of polymatroid theory to partially ordered sets. The investigations are mainly based on Faigle and Kern, Submodular Linear Programs on Forests, Mathematical Programming **72** (1996).

A slightly modified concept of submodularity is introduced. As a consequence the main results do not require any assumptions concerning the underlying partially ordered groundset of the polymatroid. The partial orders are not required to be rooted forests.

We consider two different basis concepts for ordered polymatroids. These are Core(f), the set of all elements with maximal cardinality and Max(f), the set of all maximal feasible elements. Both concepts are equivalent for unordered polymatroids. The sets Core(f) and Max(f) are completely described by facet-inducing inequalities. Furthermore it is shown by an example that Max(f) is in general not a polyhedral set.

1. INTRODUCTION AND NOTATIONS

This paper studies some structural aspects of a class of polyhedra called ordered polymatroids. Ordered polymatroids are defined by submodular rank functions with respect to the set of all antichains of a finite partially odered set (poset).

Linear functions defined in the set of all feasible points of a polymatroid can be efficiently optimized by the greedy algorithm. Recently, this aspect of the theory of polymatroids and submodular polyhedra could be extended to the more general framework of partially ordered sets (see Faigle, Kern [3]).

Polymatroids provide a polyhedral abstraction of certain properties of matroids. All elements of the base polytope of a polymatroid are equicardinal. This is a polyhedral abstraction of the well-known fact that all bases of a matroid have the same number of elements. Polymatroids were introduced by Edmonds in 1970 (see [1]). The polymatroids form a subclass of the class of ordered polymatroids and are called unordered polymatroids for our purposes.

In this paper the question of feasibility of Greedy vectors in ordered polymatroids and the facial structure of ordered polymatroids are discussed.

Throughout this paper we work with the vector space $\mathbb{R}^E$ where E is a finite set. The set E is the **groundset** of the polymatroid. The vectors of $\mathbb{R}^E$ are indexed by the elements of E. In our model the set E is endowed with a partial order $P = (E, \preceq)$, i.e. a reflexive, transitive and antisymmetric binary relation. A partial order P is **trivial** if there exist no order-relations between pairs of elements of E.

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A subset $I \subseteq E$ with property

$$a \in I \text{ and } b \prec a \implies b \in I$$

is called **order-ideal**. A subset I which consists of pairwise incomparable elements w.r.t. P is called **antichain**. An antichain is **maximal** if it cannot be augmented by an element $e \in E$ such that $A \cup \{e\}$ is again an antichain. An antichain is called **minimal** if it contains only minimal elements of the groundset.

We denote by $2^{(E,\preceq)}$ the set of all antichains of the poset P and by

$$\mathrm{id}(A) = \{e \in E : \ e \preceq a \text{ for some } a \in A\}$$

the order-ideal generated by the antichain A. The symbol I^+ denotes the antichain of maximal elements of a subset $I \subseteq E$ and

$$x^+(I) := x(I^+) = \sum_{e \in I^+} x_e$$

is used for vectors $x \in \mathbb{R}^E$ and subsets $I \subseteq E$. Similarly, I^- denotes the set of all minimal elements of a subset $I \subseteq E$. Especially, E^- describes the set of all minimal elements of the groundset E. By

$$\begin{aligned} A \vee B &:= (\mathrm{id}(A) \cup \mathrm{id}(B))^+ \\ A \wedge B &:= (\mathrm{id}(A) \cap \mathrm{id}(B))^+ \end{aligned}$$

the two operations **maximum** and **minimum** of the antichains A and B are defined. Our definition of submodularity of a function uses the **reduced minimum** of the antichains A and B which is defined by

$$A \sqcap B := (A \wedge B) \cap (A \cup B).$$

A function $f : \ 2^{(E,\preceq)} \longrightarrow \mathbb{R}$ is **submodular** if

$$f(A) + f(B) \geq f(A \vee B) + f(A \sqcap B) \tag{1}$$

holds for all $A, B \in 2^{(E,\preceq)}$. The polyhedron

$$\mathbb{P}(f) = \{x \in \mathbb{R}^E : \ x(A) \leq f(A) \text{ for all } A \in 2^{(E,\preceq)}\}$$

is called **ordered polymatroid**. Unordered polymatroids are obtained from this definition if the underlying poset $P = (E, \preceq)$ is chosen to be the trivial order on the groundset E.

2. Linear optimization in ordered polymatroids

In order to prepare the greedy algorithm for solving linear optimization problems over ordered polymatroids primal Greedy vectors are introduced.

Definition 2.1 (primal Greedy vectors). *Let $\pi = (e_1, e_2, \ldots, e_n)$ be a linear extension of the poset $P = (E, \preceq)$. Then the result of the procedure*

PRIMAL GREEDY

$$\begin{aligned} x_{e_1} &:= f(e_1) \\ x_{e_2} &:= f(\{e_1, e_2\}^+) - x^+(\{e_1\} \setminus \mathrm{id}(e_2)) \\ &\ \ \vdots \\ x_{e_n} &:= f(\{e_1, e_2, \ldots, e_n\}^+) - x^+(\{e_1, e_2, \ldots, e_{n-1}\} \setminus \mathrm{id}(e_n)) \end{aligned}$$

is called **Greedy vector** *with respect to f and π.*

Remark 2.1. Let us assume that the order P is trivial. Then we obtain

$$\begin{aligned}\{e_1, e_2, \dots, e_{i-1}\}^+ \setminus \mathrm{id}(e_i) &= \{e_1, e_2, \dots, e_{i-1}\} \setminus \{e_i\} \\ &= \{e_1, e_2, \dots, e_{i-1}\}\end{aligned}$$

for all $i \in \{1, \dots, n\}$ in Definition 2.1 and get back the well-known formula

$$x_{e_i} = f(\{e_1, e_2, \dots, e_i\}) - f(\{e_1, e_2, \dots, e_{i-1}\}) \quad (i \in \{1, \dots, n\})$$

for the greedy algorithm applied to an unordered polymatroid $\mathbb{P}(f)$.

Recall that $x = (x_{e_1}, x_{e_2}, \dots, x_{e_n})$ maximizes the linear function $c^T x$ with $c \in \mathbb{R}^E_+$ over the unordered polymatroid $\mathbb{P}(f)$ provided that the weight-coefficients c_{e_i} are ordered s.t.

$$c_{e_1} \geq c_{e_2} \geq \dots \geq c_{e_n}.$$

For this fact we refer to Chapter 10 of Grötschel, Lovász and Schrijver [5, p. 307]. $\diamondsuit$

The following theorem is a consequence of contraction properties of ordered polymatroids (see Faigle, Kern [2, p. 9] or Krüger [7, chapter 4]).

Theorem 2.1 (feasibility of Greedy vectors). *Primal Greedy vectors are feasible elements of ordered polymatroids.*

Proof. The proof is done by induction on the number of elements of the groundset E. Let $x = (x_1, x_2, \dots, x_n)$ be the primal Greedy vector w.r.t. the linear extension $\pi = (e_1, e_2, \dots, e_n)$ of $P = (E, \preceq)$ and the ordered polymatroid $\mathbb{P}(f)$. We assume that the vector $x' = (x_2, \dots, x_n)$ is a feasible element of the polyhedron $\mathbb{P}(f')$ which is defined by the function

$$\begin{aligned} f' : \quad 2^{(E \setminus \{e_1\}, \preceq)} &\longrightarrow \mathbb{R} \\ f'(A) &= \begin{cases} f(A \cup e_1) - f(e_1) & \text{if } A \cup e_1 \text{ is an antichain,} \\ f(A) & \text{otherwise} \end{cases}\end{aligned}$$

on the groundset $E \setminus \{e_1\}$. The polyhedron $\mathbb{P}(f')$ is an ordered polymatroid because f' is a submodular function w.r.t. the antichains of the reduced groundset $E \setminus \{e_1\}$. It remains to prove

$$x(A) \leq f(A) \text{ for all } A \in 2^{(E, \preceq)}.$$

Therefore the following 3 cases are distinguished.

(1) $e_1 \in A$. By the induction assumption we have

$$x'(A \setminus e_1) \leq f'(A \setminus e_1) \leq f(A) - f(e_1) = f(A) - x_1$$

which is equivalent to $x(A) \leq f(A)$.

(2) $e_1 \notin A$ and $A \cup e_1$ is an antichain. Since f is submodular, we have

$$x(A) = x'(A) \leq f(A \cup e_1) - f(e_1) \leq f(A).$$

(3) $e_1 \notin A$ and $A \cup e_1$ is not an antichain. The inequality $x(A) \leq f(A)$ follows from $x'(A) = x(A)$ and $f'(A) = f(A)$. $\diamondsuit$

Remark 2.2. The proof of Theorem 2.1 is due to Faigle, Kern (see [2, p. 9]). They proved feasibility of Greedy vectors w.r.t. polymatroids defined on rooted forests (see also [3, p. 202, Theorem 4.1]). A poset $P = (E, \preceq)$ is a **rooted forest** if each element of the groundset E has at most one upper neighbour. In [2] and [3] functions $f: \quad 2^{(E,\preceq)} \longrightarrow \mathbb{R}$ with the property

$$f(A)+f(B) \geq f(A \vee B)+f(A \wedge B) \tag{2}$$

for all $A, B \in 2^{(E,\preceq)}$ are considered.

In contrast to (1) the minimum $A \wedge B$ of two antichains A and B is used in the right side of (2) instead of the reduced minimum $A \sqcap B$. In [3, p. 204, Example 4.1] an example is given that a primal Greedy vector is not feasible if $\mathbb{P}(f)$ is defined w.r.t. a function with property (2).

The properties (1) and (2) are equivalent if the poset $P=(E,\preceq)$ is a rooted forest. For rooted forests we have

$$A \sqcap B = A \wedge B \text{ for } A, B \in 2^{(E,\preceq)}.$$

The proof of Theorem 2.1 in general does not work for partial orders P which are not rooted forests. The function f' in the proof of Theorem 2.1 does not necessarily satisfy (2) if f satisfies (2). $\diamondsuit$

Krüger showed that primal Greedy vectors are exactly the vertices of the polyhedron $\mathbb{P}(f)$ (see [6, p. 4]).

Faigle and Kern developed a dual-primal greedy algorithm (see [3, p. 200]) for solving the linear optimization problem

$$\begin{array}{rcl} \sum_{e\in E} c_e x_e & \longrightarrow & \max \\ x(A) & \leq & f(A) \text{ for all } A \in 2^{(E,\preceq)} \end{array} \tag{3}$$

with an abitrary vector $c \in \mathbb{R}_+^E$ of weight coefficients and a submodular function f. The set of all vectors of the $\mathbb{R}^E$ which satisfy the restrictions of the problem (3) is an ordered polymatroid $\mathbb{P}(f)$ defined on the poset P. The greedy algorithm of Faigle and Kern first solves the dual program of (3) which is given by

$$\begin{array}{rcl} \sum_{A\in 2^{(E,\leq)}} f(A) y_A & \longrightarrow & \min \\ \sum_{A^+\ni p_j} y_A & \geq & c_j \\ y_A & \geq & 0. \end{array} \tag{4}$$

Notice that there is a variable y_A in (4) for each antichain A of the poset P. As a preparation for the dual part of the greedy algorithm we define

$$c_{\min} = \min\{c(e) : \; e \in E^+\}.$$

DUAL GREEDY

Initialize:	$y_A \leftarrow 0$ for all $A \in 2^{(E,\preceq)}$
Iterate:	WHILE $E \neq \emptyset$
	determine some $e_{\min}$ s.t. $c_{e_{\min}} = c_{\min}$
	$\bar{y}_E \leftarrow c_{\min}$
	$c(e) \leftarrow (c_e - c_{\min})$ for all $e \in E^+$
	$E \leftarrow E \setminus \{e_{\min}\}$.

The result of **DUAL GREEDY** is a feasible vector $\bar{y}$ w.r.t. the constraints of (4) and a linear extension $\pi = (e_1, e_2, \ldots, e_n)$ of P. Each element e_i with

$i \in \{1, \ldots, n\}$ occurs as a $e_{\min}$ in one iteration step of **DUAL GREEDY**. The order of the indizes in the permutation π is reversed, i.e. $e_n = e_{\min}$ where $e_{\min}$ is taken from the first iteration step.

The primal part of the greedy algorithm computes the primal Greedy vector $\bar{x}$ w.r.t. f and π according to Definition 2.1 above. The optimality of the primal Greedy vector $\bar{x}$ and of the vector $\bar{y}$ determined in **DUAL GREEDY** w.r.t. the optimization problems (3) and (4), respectively, is proved by using Linear Programming Duality. For the vectors x and $\bar{y}$ the following holds

$$\sum_{e \in E} c_e \bar{x}_e = \sum_{A \in 2^{(E,\leq)}} f(A) \bar{y}_A$$

(see Faigle, Kern [3, p. 198, Lemma 2.1]).

3. A contraction theorem and some related lemmas

In following sections we make use of the dominance relation between pairs of vectors $x, y \in \mathbb{R}^E$. A vector $x \in \mathbb{R}^E$ is **dominated** by another vector $y \in \mathbb{R}^E$ if the relations

$$\begin{aligned} x_e &\leq y_e \quad && \text{for all } \ e \in E \\ x_e &< y_e \quad && \text{for at least one } \ e \in E \end{aligned}$$

hold. Then we write $x \lneqq y$.

An element x of an ordered polymatroid $\mathbb{P}(f)$ is **maximal** if it is not dominated by another $y \in \mathbb{P}(f)$. The set of all maximal elements of $\mathbb{P}(f)$ is denoted by $\mathrm{Max}(f)$ and is included in $\mathrm{Core}(f)$ as a subset. The **Core polytope** $\mathrm{Core}(f)$ consists of all elements which have maximal cardinality in the polyhedron $\mathbb{P}(f)$. The **cardinality** of a vector $x \in \mathbb{R}^E$ is the sum

$$x(E) = \sum_{e \in E} x_e$$

of all components of x. For unordered polymatroids both concepts are equivalent, i.e. $\mathrm{Max}(f) = \mathrm{Core}(f)$ holds. For a proof of this fact see Fujishige [4, p. 28, Theorem 2.3].

The contraction operation reduces the groundset of an ordered polymatroid and is useful for induction proofs. Let $\mathbb{P}(f)$ be an ordered polymatroid and let $M \subseteq E^-$ be an antichain which only contains minimal elements of the groundset. Now, we choose a vector $\bar{x} \in \mathbb{R}^M$ such that

$$\bar{x}(A) \leq f(A) \text{ for all } A \subseteq M \tag{5}$$

holds. Since M is an antichain, the inequalities in formula (5) define an unordered polymatroid. The unordered polymatroid defined by (5) is called **reduction** of $\mathbb{P}(f)$ by the antichain M and is denoted by $\mathbb{P}(f)|^M$.

Next, we define the **contraction polyhedron** w.r.t. $\bar{x}$ by

$$\mathbb{P}_{(\bar{x},\cdot)}(f) := \{x \in \mathbb{R}^{E \setminus M} : (\bar{x}, x) \in \mathbb{P}(f)\}$$

and the set

$$\mathrm{Max}_{(\bar{x},\cdot)}(f) := \{x \in \mathbb{R}^{E \setminus M} : (\bar{x}, x) \in \mathrm{Max}(f)\}.$$

Obviously

$$\mathrm{Max}_{(\bar{x},\cdot)}(f) \subseteq \mathbb{P}_{(\bar{x},\cdot)}(f).$$

We want to show that contraction polyhedra again are ordered polymatroids. Therefore, the **contraction function**

$$
\begin{array}{rrcl}
 & f_{(\bar{x},\cdot)}: & 2^{(E\setminus M,\preceq)} & \longrightarrow \mathbb{R} \\
(6) & f_{(\bar{x},\cdot)}(A) & := & min\,\{f(A\cup T)-\bar{x}(T) : T\subseteq M \text{ und} \\
 & & & A\cup T \text{ is an antichain}\}
\end{array}
$$

is introduced. This function assigns a real value to each antichain M of the reduced groundset $E\setminus M$.

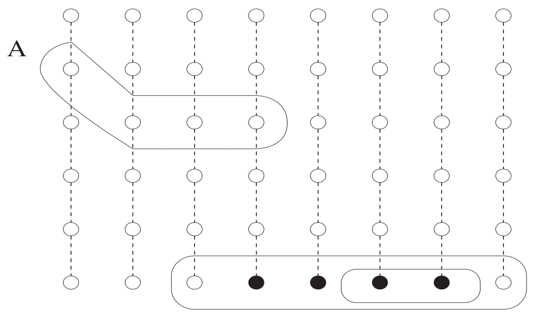

FIGURE 1. The definition of the contraction function. The black points illustrate the vector $\bar{x}$.

Lemma 3.1. *The contraction function $f_{(\bar{x},\cdot)}$ is a submodular function w. r. t. the lattice $2^{(E\setminus M,\preceq)}$ of antichains.*

Proof. In order to show submodularity of the contraction function let us consider two antichains $A,B\in 2^{(E,\preceq)}$ and let us assume that the minima in (6) applied to $f_{(\bar{x},\cdot)}(A)$ and $f_{(\bar{x},\cdot)}(B)$ are achieved by the antichains T_A and T_B, respectively. Now let two antichains $P,Q\subseteq M$ be defined by

$$
\begin{aligned}
P &:= \{e\in T_A\cup T_B : \nexists \bar{e}\in A\cup B \text{ mit } e\prec\bar{e}\}\\
Q &:= \{e\in T_A\cup T_B : \exists \bar{e}\in A\cup B \text{ mit } e\prec\bar{e}\}\cup(T_A\cap T_B).
\end{aligned}
$$

This choice of P and Q yields

$$
\begin{aligned}
(A\cup T_A)\vee(B\cup T_B) &= (A\vee B)\cup P\\
(A\cup T_A)\sqcap(B\cup T_B) &= (A\sqcap B)\cup Q
\end{aligned}
\tag{7}
$$

and

$$
x(T_A)+x(T_B)=x(P)+x(Q). \tag{8}
$$

Using (7) and (8) we obtain

$$
\begin{aligned}
&f_{(\bar{x},\cdot)}(A)+f_{(\bar{x},\cdot)}(B)\\
&\quad= f(A\cup T_A)-\bar{x}(T_A)+f(B\cup T_B)-\bar{x}(T_B)\\
&\quad\geq f((A\cup T_A)\vee(B\cup T_B))-\bar{x}(P)+f((A\cup T_A)\sqcap(B\cup T_B))-\bar{x}(Q)\\
&\quad= f((A\vee B)\cup P)-\bar{x}(P)+f((A\sqcap B)\cup Q)-\bar{x}(Q)\\
&\quad\geq f_{(\bar{x},\cdot)}(A\vee B)+f_{(\bar{x},\cdot)}(A\sqcap B).
\end{aligned}
$$

$\Diamond$

Theorem 3.1 (Contraction). *The contraction polyhedron $\mathbb{P}_{(\bar{x},\cdot)}(f)$ is an ordered polymatroid with groundset $E \setminus M$ and rank function $f_{(\bar{x},\cdot)}$, i.e.*

$$\mathbb{P}_{(\bar{x},\cdot)}(f) = \mathbb{P}(f_{(\bar{x},\cdot)}). \tag{9}$$

The equality

$$\mathrm{Max}_{(\bar{x},\cdot)}(f) = \mathrm{Max}(f_{(\bar{x},\cdot)}) \tag{10}$$

holds for the maximal elements of the contraction polyhedron.

Proof. Since $f_{(\bar{x},\cdot)}$ is submodular it remains to prove (9) and (10). Equality (9) holds because the conditions

$$x(A) + \bar{x}(T) \leq f(A \cup T) \text{ for all } T \subseteq M \text{ such that } A \cup M \text{ is an antichain}$$

and

$$x(A) \leq f_{(\bar{x},\cdot)}(A)$$

are equivalent for any fixed $x \in \mathbb{R}^{E\setminus M}$ and arbitrary antichains $A \subseteq E \setminus M$.

For the proof of (10) consider an element $x \in \mathrm{Max}_{(\bar{x},\cdot)}(f)$, i.e. $(\bar{x}, x) \in \mathrm{Max}(f)$. Assume that there is a $y \in \mathbb{P}_{(\bar{x},\cdot)}(f)$ s.t. $y \gneqq x$. Then we have $(\bar{x}, y) \in \mathbb{P}(f)$ and $(\bar{x}, y) \gneqq (\bar{x}, x)$. This contradicts $(\bar{x}, x) \in \mathrm{Max}(f)$.

Conversely, let $x \in \mathrm{Max}(f_{(\bar{x},\cdot)})$. Especially we have $(\bar{x}, x) \in \mathbb{P}(f)$. Assume that there is a $(\bar{y}, y) \in \mathbb{P}(f)$ s.t. $(\bar{x}, x) \lneqq (\bar{y}, y)$. Since each basis of the unordered polymatroid $\mathbb{P}(f)|^M$ is a maximal element of $\mathbb{P}(f)|^M$, the dominance relation $\bar{x} \lneqq \bar{y}$ between $\bar{x}$ and $\bar{y}$ is impossible. Therefore we have $\bar{x} = \bar{y}$ and $x \lneqq y$. This is a contradiction to the maximality of x in the contraction polyhedron $\mathbb{P}_{(\bar{x},\cdot)}(f) = \mathbb{P}(f_{(\bar{x},\cdot)})$. ♢

In the sequel we will frequently use the following lemma, which simplifies the computation of values of the contraction function.

Lemma 3.2. *Let $\bar{x}$ in (6) be a basis of the reduction $\mathbb{P}(f|^M)$, i.e. $\bar{x}(M) = f(M)$. Then*

$$f_{(\bar{x},\cdot)}(A) = f(A \vee M) - \bar{x}(M \setminus (A \sqcap M))$$

holds for arbitrary antichains $A \subseteq E \setminus M$. Especially, one obtains

$$f_{(\bar{x},\cdot)}(A) = f(A)$$

for maximal antichains A with respect to $P = (E, \preceq)$.

Proof. Let us assume that the minimum in (6) is obtained for $T \subseteq M$. Therefore we have

$$f_{(\bar{x},\cdot)}(A) = f(A \cup T) - \bar{x}(T).$$

By the submodularity of f we get

$$\begin{aligned} f_{(\bar{x},\cdot)}(A) &= f(A \cup T) - \bar{x}(T) - \bar{x}(M) + f(M) \\ &\geq f(A \vee M) + f((A \sqcap M) \cup T) - \bar{x}((A \sqcap M) \cup T) - \bar{x}(M \setminus (A \sqcap M)) \\ &\geq f(A \vee M) - \bar{x}(M \setminus (A \sqcap M)). \end{aligned}$$

On the other hand we have the converse inequality

$$\begin{aligned} f_{(\bar{x},\cdot)}(A) &\leq f(A \cup (M \setminus (A \sqcap M))) - \bar{x}(M \setminus (A \sqcap M)) \\ &= f(A \vee M) - \bar{x}(M \setminus (A \sqcap M)) \end{aligned}$$

by the definition of the contraction function. ♢

Theorem 3.2. *For a minimal antichain $M \subseteq E^-$ the statements*

(i) (u, v) *is a vertex of* $\mathbb{P}(f)$ *and* u *is a basis of* $\mathbb{P}(f)|^M$

(ii) u *is a vertex of* $\mathbb{P}(f)|^M$ *and* v *is a vertex of* $\mathbb{P}_{(u,\cdot)}(f)$

are equivalent. Furthermore, for a linear extension

$$\pi = (e_1, e_2, \dots, e_m, e_{m+1}, \dots, e_n)$$

of P *such that* $\pi|^M = (e_1, e_2, \dots, e_m)$ *is a permutation of* M *the statements*

(iii) $x = (u_1, u_2, \dots, u_m, v)$ *is a Greedy vector in* $\mathbb{P}(f)$ *w.r.t.* π

(iv) $u = (u_1, \dots, u_m)$ *is a Greedy vector in* $\mathbb{P}(f)|^M$ *w.r.t.* π^M *and* v *is a Greedy vector in* $\mathbb{P}_{(u,\cdot)}(f)$ *w.r.t.* $(e_{m+1}, \dots, e_n)$

are equivalent.

Proof. The proof relies on the definition of a primal Greedy vector (Definition 2.1), on the definition of a contraction polyhedron $\mathbb{P}_{(\bar{x},\cdot)}(f)$ of an ordered polymatroid $\mathbb{P}(f)$ and on Lemma 3.2. ♢

4. Faces of Ordered Polymatroids

The non-equivalence of the basis concepts $\mathrm{Max}(f)$ and $\mathrm{Core}(f)$ is an important difference between ordered and unordered polymatroids. The set $\mathrm{Max}(f)$ is not a polyhedral set in general. A first interesting fact concerning the structure of the set $\mathrm{Max}(f)$ is:

Theorem 4.1. *The set* $\mathrm{Max}(f)$ *is the union of all bounded faces of the polyhedron* $\mathbb{P}(f)$.

Proof. The recession cone of the polyhedron $\mathbb{P}(f)$ contains its directions to "infinity" and is equal to

$$\mathbb{R}^E_- = \{x \in \mathbb{R}^E : \; x \leq 0\}.$$

Using the recession cone of $\mathbb{P}(f)$ one easily shows that

- A maximal element cannot belong to the interior of an unbounded face of the polyhedron $\mathbb{P}(f)$.
- All elements of a bounded face of $\mathbb{P}(f)$ are maximal.

♢

In the following facet-inducing inequalities for all bounded faces of $\mathbb{P}(f)$ are derived. These results were obtained in Krüger [7, chapter 5].

First we need a definition.

Definition 4.1 (active antichains). *For a given* $x \in \mathbb{P}(f)$ *an antichain* A *is called active if* $x(A) = f(A)$ *holds. The set of all active antichains with respect to* x *is denoted by* $\mathcal{T}(x)$. *More formal this set is described by*

$$\mathcal{T}(x) = \{A \in 2^{(E,\preceq)} : \quad x(A) = f(A)\}.$$

Lemma 4.1. *The set* $\mathcal{T}(x)$ *with respect to an element* $x \in \mathbb{P}(f)$ *is closed with respect to the operations maximum and reduced minimum of antichains.*

Proof. Let $A, B \in \mathcal{T}(x)$. Because x is a feasible element of $\mathbb{P}(f)$, we have

$$\begin{aligned} f(A)+f(B) &\leq x(A)+x(B) \\ &= x(A \vee B)+x(A \sqcap B) \\ &\leq f(A \vee B)+f(A \sqcap B). \end{aligned}$$

On the other hand f is a submodular function and the converse directions of the inequalities also hold. This is only possible if $x(A \vee B) = f(A \vee B)$ and $x(A \sqcap B) = f(A \sqcap B)$. $\diamondsuit$

For the description of the set $\mathrm{Max}(f)$ a special kind of partitions of the poset P into antichains is considered.

Definition 4.2 (laminar partitions into antichains). *The set*

$$\mathcal{P} = \{P^1, P^2, \ldots, P^l\}$$

of antichains is called a laminar partition of the poset $P = (E, \preceq)$ if the following two conditions hold:

(i) $E = P^1 \dot\cup P^2 \dot\cup \ldots \dot\cup P^l$ *(the symbol $\dot\cup$ denotes the disjoint union)*

(ii) $e_i \in P^i, e_j \in P^j$ *und* $e_i \prec e_j \Longrightarrow i < j$.

A laminar partition $\mathcal{P}$ is called **irreducible** *if there exists no pair*

$$P^i, P^{i+1} \in \mathcal{P} \text{ such that } P^i \dot\cup P^{i+1} \in 2^{(E,\preceq)}.$$

Remark 4.1. Let

$$h = \max\{|C| : \ C \text{ is a chain of } P\}.$$

Then the recursive definition

$$\begin{aligned} S^h &:= E^+ \\ S^i &:= (E \setminus \{S^{i+1} \dot\cup \ldots \dot\cup S^h\})^+ \quad (i \in \{1, \ldots, h-1\}) \end{aligned}$$

yields the **standard partition** $\mathcal{S} = \{S^1, S^2, \ldots, S^h\}$ of the poset P.

The standard partition of a poset P is a special case of a laminar partition of P into antichains. $\diamondsuit$

We say that a primal Greedy vector x **follows** the laminar partition $\mathcal{P}$ of P if for $i < j$ each element p of the antichain P^i precedes each element q of the antichain P^j in the permutation π.

Using the following two lemmas we can construct a laminar partition from a given vector $x \in \mathrm{Max}(f)$.

Lemma 4.2. *For maximal elements of ordered polymatroids we have*

$$x(E^+) = f(E^+).$$

Proof. Due to the maximality of x there is an antichain X_e such that $e \in X_e$ and $x(X_e) = f(X_e)$. Since $\mathcal{T}(x)$ is closed with respect to "$\vee$" the antichain $X := \bigvee_{e \in E^+} X_e$ also satisfies $x(X) = f(X)$. As X contains each $e \in E^+$ the equality $X = E^+$ holds. $\diamondsuit$

Lemma 4.3. *If x is a maximal element of $\mathbb{P}(f)$ then there exists a nonempty antichain $M \subseteq E^-$ such that $x(M) = f(M)$.*

Proof. Because of Lemma 4.2 we know $x(E^+) = f(E^+)$. If E^+ only consists of minimal elements of the groundset the proof is complete.

Otherwise there is at least one element $e \in E$ such that $e \prec E^+$. As x is maximal there is an antichain X_e such that $e \in X_e$ and $x(X_e) = f(X_e)$. Since $\mathcal{T}(x)$ is closed with respect to "$\sqcap$", the antichain $X^1 := E^+ \sqcap X_e$ also satisfies $x(X^1) = f(X^1)$. Furthermore $X^1 \prec E^+$ holds. This reasoning can be repeated and results in a minimal antichain $M \subseteq E^-$ with $x(M) = f(M)$ because of the finiteness of the groundset E. $\diamondsuit$

Because the groundset E is the disjoint union of the antichains P^i each vector $x \in \mathbb{R}^E$ can be written as $x = (x^1, x^2, \ldots, x^l)$ such that $x^i \in \mathbb{R}^{P^i}$ for all $i \in \{1, \ldots, l\}$. The components of each x^i correspond to the elements of the antichain P^i in the partition $\mathcal{P}$. Now, by the recursion $f^1 := f, E^1 := E$

$$f^i := (f^{i-1})_{(x^{i-1},\cdot)},\ E^i := E^{i-1} \setminus P^{i-1} \quad (i = 2, \ldots, l)$$

l contraction functions are introduced. Each contraction function is defined with respect to the antichains of the reduced groundset E^i. Each of these E^i's can also be written as $E^i = E \setminus \{P^1 \cup P^2 \cup \ldots \cup P^{i-1}\}$. All functions f^i are submodular by Lemma 3.1. The corresponding ordered polymatroids $\mathbb{P}(f^i)$ on the groundsets E^i will be useful in the proofs of our main results.

Lemma 4.4. *Let $\mathcal{P} = \{P^1, P^2, \ldots, P^l\}$ be a laminar partition with respect to P and let $x = (x^1, x^2, \ldots, x^l) \in \mathbb{R}^E$ be a vector. Then the following statements are equivalent:*

(i) *Each x^i is a basis of $\mathbb{P}(f^i)|^{P^i}$, i.e. $x^i(P^i) = f^i(P^i)$ for all $i \in \{1, \ldots, l\}$.*
(ii) *The equalities $(x^1, x^2, \ldots, x^i)(P^1 \vee P^2 \vee \ldots \vee P^i) = f(P^1 \vee P^2 \vee \ldots \vee P^i)$ hold for all $i \in \{1, \ldots, l\}$.*

Proof. For a fixed $i \in \{1, \ldots, l\}$ the equivalence of the equalities

$$x^i(P^i) = f^i(P^i)$$

and

$$(x^j, \ldots, x^i)(P^j \vee \ldots \vee P^i) = f^j(P^j \vee \ldots \vee P^i)$$

for any $j \in \{1, \ldots, i-1\}$ is proved. Then, the equivalence of (i) and (ii) follows by choosing $j = 1$.

We apply Lemma 3.2 to $f^i(P^i)$ and obtain

$$f^i(P^i) = f^{i-1}(P^i \vee P^{i-1}) - x^{i-1}(P^{i-1} \setminus (P^i \sqcap P^{i-1})). \tag{11}$$

Additionally, we consider

$$(x^{i-1}, x^i)(P^{i-1} \vee P^i) = x^{i-1}(P^{i-1} \setminus (P^i \sqcap P^{i-1})) + x(P^i). \tag{12}$$

Formula 12 is a consequence of the definition of a laminar partition (Definition 4.2). Putting (11) and (12) together one obtains

$$(x^{i-1}, x^i)(P^{i-1}, P^i) = f^{i-1}(P^{i-1} \vee P^i).$$

This reasoning is repeated i-j-1 times. $\diamondsuit$

Theorem 4.2 (facet inducing inequalities). *For each laminar partition $\mathcal{P} = \{P^1, P^2, \ldots, P^l\}$ of the poset $P = (E, \preceq)$ the inequality*

$$x(P^1) + x(P^1 \vee P^2) + \ldots + x(P^1 \vee P^2 \vee \ldots \vee P^l) \leq \quad (13)$$
$$f(P^1) + f(P^1 \vee P^2) + \ldots + f(P^1 \vee P^2 \vee \ldots \vee P^l)$$

induces a face $\mathbb{B}(\mathcal{P})$ of the polyhedron $\mathbb{P}(f)$. The face $\mathbb{B}(\mathcal{P})$ consists of maximal elements of the polyhedron $\mathbb{P}(f)$.

Conversely, for each maximal $x \in \mathbb{P}(f)$ there exists a laminar partition $\mathcal{P}$ of the poset $P = (E, \preceq)$ such that $x \in \mathbb{B}(\mathcal{P})$.

Proof. First we show that (13) is a valid inequality for the polyhedron $\mathbb{P}(f)$ and for each laminar partition $\mathcal{P}$. For each $x \in \mathbb{P}(f)$ the inequalities

$$x(P^1 \vee P^2 \vee \ldots \vee P^i) \leq f(P^1 \vee P^2 \vee \ldots \vee P^i) \quad (14)$$

hold ($i \in \{1, \ldots, l\}$). If we sum up over all inequalities (14) we obtain (13).

Now we show that $\mathbb{B}(\mathcal{P})$ is non-empty. Therefore let $x = (x^1, x^2, \ldots, x^l)$ be a Greedy vector which follows the laminar partition $\mathcal{P}$. By the equivalence of (iii) and (iv) in Theorem 3.2 we conclude that $x(P^1) = f(P^1)$ and that $(x^2, \ldots, x^l)$ is a Greedy vector w. r. t. $\mathbb{P}(f^2)$ which follows the laminar partition $P^2 \dot\cup \ldots \dot\cup P^l$ of the reduced groundset $E \setminus P^1$. The same arguments yield $x^i(P^i) = f^i(P^i)$ for all $i \in \{1, \ldots, l\}$ which is equivalent to

$$x(P^1 \vee P^2 \vee \ldots \vee P^i) = f(P^1 \vee P^2 \vee \ldots \vee P^i)$$

for all $i \in \{1, \ldots, l\}$ by Lemma 4.4 and to

$$x(P^1) + x(P^1 \vee P^2) + \ldots + x(P^1 \vee P^2 \vee \ldots \vee P^l) =$$
$$f(P^1) + f(P^1 \vee P^2) + \ldots + f(P^1 \vee P^2 \vee \ldots \vee P^l).$$

Thus, we have shown that $x \in \mathbb{B}(\mathcal{P})$.

In order to prove the converse direction of the Theorem assume that a feasible $x = (x^1, x^2, \ldots, x^l)$ is maximal. By Lemma 4.3 there exists an antichain $P^1 \subseteq E^-$ s.t. $x(P^1) = f(P^1)$. Then, the vector $(x^2, \ldots, x^l)$ is a maximal element of the contraction polyhedron $\mathbb{P}_{(x^1, \cdot)}(f)$ by formula 10 in Theorem 3.1. Again, we can find an antichain $P^2 \subseteq (E^2)^-$ s.t. $x(P^2) = f^2(P^2)$. Repeatedly applying this argument we get $x(P^i) = f^i(P^i)$ for $i \in \{1, \ldots, l\}$ where the P^i's are the antichains of a laminar partition. We obtain

$$x(P^1 \vee P^2 \vee \ldots \vee P^i) = f(P^1 \vee P^2 \vee \ldots \vee P^i) \quad (15)$$

for $i \in \{1, \ldots, l\}$ by Lemma 4.4. Taking the sum over all equalities (15) we conclude that $x \in \mathbb{B}(\mathcal{P})$. ◊

Remark 4.2. Each $\mathbb{B}(\mathcal{P})$ is the intersection of $\mathbb{P}(f)$ with the hyperplane

$$\mathcal{H} = \{x \in \mathbb{P}(f) : \ x(P^1) + x(P^1 \vee P^2) + \ldots + x(P^1 \vee P^2 \vee \ldots \vee P^l) =$$
$$f(P^1) + f(P^1 \vee P^2) + \ldots + f(P^1 \vee P^2 \vee \ldots \vee P^l)\}.$$

Since feasible elements of the polyhedron $\mathbb{P}(f)$ satisfy

$$x(P^1 \vee \ldots \vee P^i) \leq f(P^1 \vee \ldots \vee P^i)$$

for all $i \in \{1, \ldots, l\}$ the elements of $\mathbb{B}(\mathcal{P})$ are exactly those elements of $\mathbb{P}(f)$ which satisfy

$$x(P^1 \vee \ldots \vee P^i) = f(P^1 \vee \ldots \vee P^i) \quad (16)$$

for all $i \in \{1, \dots, l\}$. $\diamondsuit$

Theorem 4.3. *The vertices of each face* $\mathbb{B}(\mathcal{P})$ *are exactly those Greedy vectors which follow the partition* $\mathcal{P}$ *of the poset* $P = (E, \preceq)$.

Proof. In the proof of Theorem 4.2 we have already shown that each Greedy vector $x = (x^1, x^2, \dots, x^l)$ which follows the partition $\mathcal{P}$ is an element of the face $\mathbb{B}(\mathcal{P})$. Therefore we only need to prove the converse direction of the Theorem.

As a consequence of Remark (4.2) each vertex of the face $\mathbb{B}(\mathcal{P})$ satisfies the conditions (ii) in Lemma 4.4 which are equivalent to the conditions (i) in Lemma 4.4, i.e.

$$(x^1, x^2, \dots, x^i)(P^1 \vee P^2 \vee \dots \vee P^i) = f(P^1 \vee P^2 \vee \dots \vee P^i)$$

for all $i \in \{1, \dots, l\}$. By the equivalence of (i) and (ii) in Lemma 3.2 we conclude that each x^i is a vertex of the unordered polymatroid $\mathbb{P}(f^i)|^{P^i}$. Thus each x^i is a Greedy vector in $\mathbb{P}(f^i)|^{P^i}$ w.r.t. a permutation π^i of the elements of the antichain $P^i \in \mathcal{P}$. Applying the equivalence of (iii) and (iv) in Lemma 3.2 we show that x is a Greedy vector in the ordered polymatroid $\mathbb{P}(f)$ w.r.t. the concatenation $\pi = (\pi^1, \pi^2, \dots, \pi^l)$.

If $(x^{i+1}, \dots, x^l)$ is a Greedy vector w.r.t the linear extension $(\pi^{i+1}, \dots, \pi^l)$ of $(E^{i+1}, \preceq)$ and x^i is a Greedy vector w.r.t. the permutation π^i of the antichain P^i then $(x^i, x^{i+1}, \dots, x^l)$ is a Greedy vector w.r.t. the linear extension $(\pi^i, \pi^{i+1}, \dots, \pi^l)$ of $(E^i, \preceq)$. The proof of the converse direction of the Theorem is obtained by choosing $i = 1$. $\diamondsuit$

At the end of the paper the question is answered whether laminar partitions completely describe the structure of the set $\mathrm{Max}(f)$. A crucial observation concerning this question is that each face $\mathbb{B}(\mathcal{P})$ w.r.t. a laminar partition $\mathcal{P}$ of P is a subset of another face $\mathbb{B}(\mathcal{P}')$ where $\mathcal{P}'$ is irreducible. This observation is now stated as a Lemma.

Lemma 4.5. *Let* $\mathcal{P} = \{P^1, \dots, P^i, P^{i+1}, P^l\}$ *be a laminar partition s.t.* $\mathbb{B}(\mathcal{P}') = \{P^1, \dots, P^i \vee P^{i+1}, \dots, P^l\}$ *again is a laminar partition then*

$$\mathbb{B}(\mathcal{P}) \subseteq \mathbb{B}(\mathcal{P}').$$

Proof. According to Remark 4.2 the face $\mathbb{B}(\mathcal{P})$ is defined by the equalities

$$x(P^1 \vee \dots \vee P^i) = x(P^1 \vee \dots \vee P^i) \tag{17}$$

for all $i \in \{1, \dots, n\}$. If the equality

$$x(P^1 \vee P^2 \vee \dots \vee P^i) = x(P^1 \vee P^2 \vee \dots \vee P^i)$$

is left out in the system (17) a system of $l - 1$ equalities is obtained which defines $\mathbb{B}(\mathcal{P}')$. $\diamondsuit$

We summarize our results concerning the facial structure of ordered polymatroids in the following theorem.

Theorem 4.4. *The set* $\mathrm{Max}(f)$ *of all maximal elements of an ordered polymatroid* $\mathbb{P}(f)$ *is described by*

$$\mathrm{Max}(f) = \bigcup \{\mathbb{B}(\mathcal{P}) : \mathcal{P} \textit{ is an irreducible laminar partition of } P = (E, \preceq)\},$$

i.e. $\mathrm{Max}(f)$ *is the union of all faces* $\mathbb{B}(\mathcal{P})$ *with respect to an irreducible laminar partition of the poset* $P = (E, \preceq)$ *into antichains.*

Remark 4.3.

(i) For the Core polytope of an ordered polymatroid $\mathbb{P}(f)$ we have

$$\mathrm{Core}(f) = \mathbb{B}(\mathcal{S}),$$

if $\mathcal{S}$ is the standard partition of the poset P.

(ii) From the Theorem 4.4 we can get back the equivalence of the two basis concepts Core(f) and Max(f) for an unordered polymatroid $\mathbb{P}(f)$ (see Section 1). The only laminar partition of a trivial poset P is

$$\mathcal{S} = \{E\}.$$

According to Theorem 4.4 we have

$$\mathrm{Max}(f) = \{x \in \mathbb{R}^E : \ x(E) = f(E)\}.$$

Thus, the sets Max(f) and Core(f) coincide with the base polytope of $\mathbb{P}(f)$. ♢

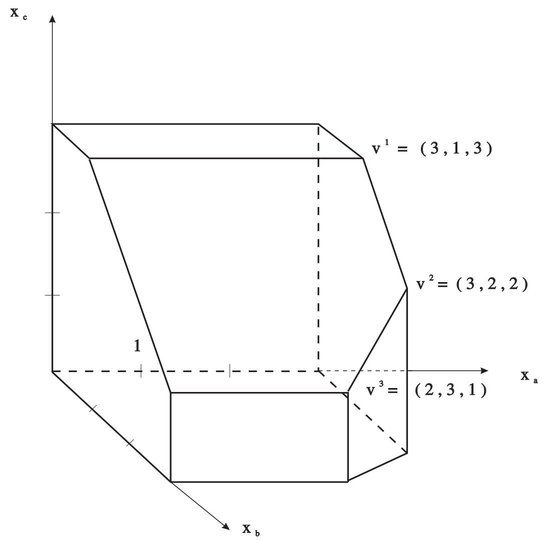

FIGURE 2. Example of an ordered polymatroid $\mathbb{P}(f)$. The polyhedron $\mathbb{P}(f)$ is restricted to the positive orthant $\mathbb{R}^E_+$.

Example 4.1. Let us consider the poset $P = (E, \preceq)$ on three elements $E = \{a, b, c\}$ with one relation $a \prec c$. Let the function f be defined by

$$\begin{array}{lcl} f(a) = f(b) = f(c) & = & 3 \\ f(\{a,b\}) & = & 5 \\ f(\{b,c\}) & = & 4. \end{array}$$

The corresponding polyhedron $\mathbb{P}(f)$ is shown in Figure 2. The set Core(f) is the connecting line of the vertices v^1 and v^2 and is a subset of

$$\mathrm{Max}(f) = conv(\{v^1, v^2\}) \cup conv(\{v^2, v^3\}). \tag{18}$$

The set Max(f) is not a polyhedral set as Figure 2 shows.

The poset P has two irreducible laminar partitions. These are the partitions

$$\begin{aligned}\mathcal{P} &= \{a\} \cup \{b, c\}\\ \mathcal{Q} &= \{a, b\} \cup \{c\}.\end{aligned}$$

Applying Theorem 4.2 to $\mathcal{P}$ and $\mathcal{Q}$ we get two inequalities

$$\begin{aligned}x_a + x_b + x_c &\leq 7\\ x_a + 2x_b + x_c &\leq 9.\end{aligned}$$

These inequalities induce two faces $\mathbb{B}(\mathcal{P}) = \mathrm{Core}(f) = conv(\{v^1, v^2\})$ and $\mathbb{B}(\mathcal{Q}) = conv(\{v^2, v^3\})$. The set $\mathrm{Max}(f)$ is completely described by (18) since Theorem 4.4 holds.

Now, Theorem 4.3 is verified for the polyhedron $\mathbb{P}(f)$. The linear extensions $\pi^1 = (a, c, b)$ and $\pi^2 = (a, b, c)$ of P follow the partition $\mathcal{P}$. The vertices v^1, v^2 of $\mathbb{P}(f)$ are Greedy vectors with respect to π^1, π^2. Indeed, v^1, v^2 are the vertices of $\mathbb{B}(\mathcal{P})$. Similarly, the vertices v^2, v^3 of $\mathbb{B}(\mathcal{Q})$ are exactly those Greedy vectors which follow the partition $\mathcal{Q}$.

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