Martin–Luther–Universität
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LINEAR PARTIAL DIFFERENTIAL ALGEBRAIC EQUATIONS
PART I: INDEXES, CONSISTENT BOUNDARY/INITIAL CONDITIONS

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Abstract. Linear partial differential algebraic equations (PDAEs) of the form $Au_t(t,x) + Bu_{xx}(t,x) + Cu(t,x) = f(t,x)$ are studied where at least one of the matrices $A,\ B \in \mathbb{R}^{n\times n}$ is singular. For these systems we introduce a differential time index and a differential space index. These indexes are characterized by means of a Fourier and a Laplace transform, respectively. We show that in contrast to problems with regular matrices A and B the initial conditions and/or boundary conditions for problems with singular matrices A and B have to fulfill certain consistency conditions. The problem of an index jump is also discussed. Furthermore, we introduce a perturbation index pair and show the relation between the uniform differential time index and the differential space index. Finally, we give a consistent representation of the solution of an initial boundary value problem for a PDAE.

AMS subject classification: 35E99, 35M10

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1. Introduction

There are numerous mathematical models in science and engineering which lead to systems of equations of different type, e.g. the systems may consist of

- parabolic and elliptic differential equations or of
- parabolic, elliptic and ordinary differential equations or of
- elliptic and ordinary differential equations and algebraic equations.

There are many possibilities for the combination of equations of different character. In applications these systems must, in general, be supplemented by suitable initial conditions (ICs) and boundary conditions (BCs). Under some assumptions such systems can be reduced by a Laplace or by a Fourier transformation to a sequence of differential algebraic systems (DAEs). Hence the original system is also called a partial differential algebraic equation (PDAE).

We mention two examples for such systems.

Example 1.1. The *modeling of a population dynamic* of n species in dependence of m ideally distributed food sources ([11],[16])

$$\frac{\partial u_j}{\partial t} = D\,\Delta\, u_j \;+\; f_j(u,v), \qquad j = 1,..,n,$$
$$\frac{\partial v_i}{\partial t} = g_i(u,v), \qquad i = 1,..,m\ ,$$

where $u = (u_1, \ldots, u_n)^\top$ is the species density and $v = (v_1, \ldots, v_m)^\top$ the food source density vector. The quantities $D > 0$, f_j, g_i and suitable initial and boundary conditions are assumed to be given.

This system can also be interpreted as a model of a reaction-diffusion system. Here u_j may be seen as a concentration of a diffusing substance while the concentration v_i may be considered as a substance whose particles can not diffuse.

Example 1.2. *Smelting of solid materials in an electric furnace.* The modeling of the temperature distribution by the heat conduction equation and the electrical charge conservation leads to the PDAE ([8],[12])

$$\rho\, c\,\frac{\partial T}{\partial t} = \operatorname{div}(K\,\nabla T) + \sigma(T)(\nabla\Phi)^\top\nabla\Phi \qquad \text{in } \Omega \subset \mathbb{R}^k,\ k = 1, 2 \text{ or } 3,\ t > 0,$$
$$0 = \operatorname{div}(\sigma(T)\,\nabla\Phi) \qquad \text{in } \Omega$$

for the temperature T and the electrical potential Φ. Suitable initial and boundary conditions have to be prescribed. The density ρ, the specific heat c, the heat conductivity K and the electrical conductivity $\sigma = \sigma(T)$ are assumed to be known.

In the first example parabolic differential equations are coupled with ordinary differential equations. In the second example a parabolic and an elliptic equation must be solved simultaneously.

In this paper we investigate linear PDAEs with constant coefficients of the form

$$A\, u_t(t,x) + B\, u_{xx}(t,x) + C\, u(t,x) = f(t,x) \tag{1.1}$$

where $t \in (0, t_e)$ and $x \in (-l, l)$, $A, B, C \in \mathbb{R}^{n\times n}$, $u, f : [0, t_e] \times [-l, l] \to \mathbb{R}^n$. We are interested in cases where at least one of the matrices A and B is singular. The two special cases $A = 0$ or $B = 0$ lead to ordinary differential equations or DAEs which are not considered here. Therefore in this paper we assume that none of the matrices A or B is the zero matrix.

Furthermore, for all $t \in [0, t_e]$ we require BCs for the components u_j of u for all $j \in \mathfrak{M}_{BC} \subseteq \{1, 2, .., n\}$ ($\mathfrak{M}_{BC}$ is specified in subsection 2.1, see also the following example 1.3) of the form

$$\mathrm{R_B}\ u_j(t,x) := \frac{\partial^\kappa u_j}{\partial x^\kappa}(t, \pm l) \;=\; 0, \quad \kappa \;=\; 0 \quad \text{or} \quad 1. \tag{1.2}$$

Beside these, ICs

$$u(0,x) = g(x) \quad \text{for} \quad x \in [-l, l] \tag{1.3}$$

are assumed to be given. The components g_j of g for all $j \in \mathfrak{M}_{IC} \subseteq \{1, 2, .., n\}$ can be prescribed ($\mathfrak{M}_{IC}$ is defined in subsection 2.2). For the components u_k, $k \notin \mathfrak{M}_{BC}$, BCs of only Dirichlet type are in order. In general, the BCs for u_k, $k \notin \mathfrak{M}_{BC}$, and the ICs for u_i, $i \notin \mathfrak{M}_{IC}$,

must be determined with the help of the PDAE (1.1), the BCs for u_j, $j \in \mathfrak{M}_{BC}$, and the ICs for u_m, $m \in \mathfrak{M}_{IC}$ (of course, there may be cases which need not all these informations). For the calculation of these consistent ICs and BCs a coupled system of equations can be generated (see subsections 2.1 and 2.2, examples are also given in sections 3 and 4).
Furthermore, we require the compatibility conditions (component wise)

$$\mathrm{R_B}\ g(x) = \mathrm{R_B}\, u(0,x) \tag{1.4}$$

between the ICs and the BCs.

In contrast to problems with regular matrices A and B (e.g. parabolic problems) the IC (1.3) and/or the BC (1.2) for problems with singular matrices A and/or B have to fulfill certain supplementary conditions (so-called consistency conditions). The following example illustrates this.

Example 1.3. Let $u = (u_1, u_2, u_3)^\top$ and the PDAE

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=A} \begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{pmatrix} + \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=B} \begin{pmatrix} u_{1xx} \\ u_{2xx} \\ u_{3xx} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & c_1 & 0 \\ 0 & 0 & c_2 \\ 0 & c_3 & 0 \end{pmatrix}}_{=C} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$u_j(t,-1) = u_j(t,1) \;=\; 0,\; j \in \mathfrak{M}_{BC}$$

$$u_i(0,x) = g_i(x)\, \sin(\pi x),\; i \in \mathfrak{M}_{IC}$$

be given for $t \in (0, t_e)$, $x \in (-1, 1)$ with $a, b > 0$, $c_i \neq 0$ and sufficient smooth functions $f_i(t,x)$, $i = 1,2,3$. The term $\sin(\pi x)$ in the IC for u_i guarantees the compatibility (1.4) with the BCs for $i \in \mathfrak{M}_{BC}$. In the index set $\mathfrak{M}_{BC}$ we collect those indexes of the components of u for which an arbitrary BC can be prescribed ($\mathfrak{M}_{IC}$ is characterized in an analogous manner). In contrast to problems with regular matrices A, B (e.g. parabolic systems where, in general, $\mathfrak{M}_{BC} = \mathfrak{M}_{IC} = \{1, ..., n\}$), here these sets are proper subsets of $\{1,2,3\}$ because we get immediately for the solution components

$$u_2(t,x) = \frac{1}{c_3} f_3(t,x)$$

$$u_3(t,x) = \frac{1}{c_2}\left(f_2(t,x) - \frac{a}{c_3} f_{3t}(t,x) + \frac{b}{c_3} f_{3xx}(t,x) \right) .$$

There is no free parameter left so that the initial and boundary values for the components $u_2(t,x), u_3(t,x)$ are directly given by the initial and boundary values of the right hand functions and their derivatives. Therefore we can not prescribe ICs and BCs for the components $u_2(t,x), u_3(t,x)$, and hence we define $\mathfrak{M}_{BC} = \mathfrak{M}_{IC} = \{1\}$. The solution component $u_1(t,x)$ is the solution of the parabolic initial boundary value problem

$$u_{1t} \;= u_{1xx} + f_1 - \frac{c_1}{c_3} f_3\,, \quad u_1(t,-1) \;=\; u_1(t,1) \;=\; 0\;,\; u_1(0,x) \;=\; g_1(x) \sin(\pi x)\;,$$

where we can assign $g_1(x)$ (the choice of the Dirichlet BC is also a free one).

This example clarifies that problems of the form (1.1) with singular matrices essentially differ from such with regular matrices A, B. It is analogous to the statement that ODEs are not DAEs [14].

This paper is organized as follows. In the next section we introduce the basic notions of the differential spatial index by means of a Laplace transform (subsection 2.1) and the uniform differential time index by a Fourier transform (subsection 2.2), respectively. These indexes are used to characterize the PDAE in several contexts. If B is a zero matrix, then the uniform differential time index is identical with the well known index of a DAE. Furthermore, in subsection 2.3 a perturbation index pair is defined. Especially, with this index pair one can characterize the sensitivity of a numerical solution against small perturbations with respect to space and time on the right hand side of the PDAE. It is a result of this paper that the perturbation index pair can be expressed in terms of the differential spatial index and the uniform differential time index.

Of importance for the treatment of PDAEs is the fact that, as illustrated in the foregoing example, in general, any ICs and BCs cannot be prescribed for all components of the solution vector u. To overcome this difficulty we define in section 2 the fundamental index sets $\mathfrak{M}_{IC}$ and $\mathfrak{M}_{BC}$. We show that these can be determined from the matrices A, B and C. If we know these two sets, then we know for which components of the solution vector u of the PDAE ICs and BCs can be assigned.

In section 3 we apply the tools of section 2 to construct (under the assumption that an unique solution exists) a general consistent (Fourier) representation of the solution of the IBVP (1.1) – (1.4). This is done as well for homogeneous Dirichlet BCs as for homogeneous Neumann BCs. The special case that the pencil (A, B) is regular (which is often realized in applications) is considered in section 4. In this case it may be easier to transform the PDAE (1.1) in a suitable manner discussed there.

2. Indexes of the PDAE

As in the case of DAEs which can be characterized conveniently by the differential and the perturbation index (cf. [6], [9]) it is also useful to introduce indexes for PDAEs. These describe special properties of the systems (with respect to both the analytical solution and the numerical treatment [13]). In contrast to (ordinary) DAEs one must distinguish between a time index and a spatial index for PDAEs. The definition of these indexes will be done by means of the Fourier and the Laplace transform, respectively. On this basis we define in this section a space index and an uniform time index. Beside these a perturbation index for the PDAE is introduced. It characterizes the influence of small perturbations on a solution of the PDAE (if it exists).

In the sequel the following assumptions are used:

(I) : The initial boundary value problem (IBVP) (1.1)-(1.4) has one and only one solution.

(II) : Each component of the solution vector u satisfies a growth condition of the form

$$|u_i(t, x)| \leq M\, e^{\alpha t}\ , \quad i = 1(1)n,$$

(M and α are independent of x).

(III) : The matrix pencil $(B, \xi A + C)$, $\mathrm{Re}(\xi) > \alpha$, is regular.

(IV) : The matrix pencil $(A, \mu_k B + C)$ is regular for all k. μ_k is an eigenvalue of the operator $\frac{\partial^2}{\partial x^2}$ with prescribed BCs (1.2).

(V) : The right hand vector $f(t, x)$ and the initial vector $g(x)$ are sufficiently smooth.

2.1. Laplace transform and the spatial index. Let now $t_e = \infty$ and $y : [0, \infty) \to \mathbb{R}$ be continuous with (growth condition)

$$|y(t)| \leq M\, e^{\alpha t} \ , \quad t \in [0, \infty), \ 0 < M < \infty, \ \alpha > 0.$$

Define (as usual) the Laplace transform of $y(t)$ by

$$y_\xi := \int_0^\infty e^{-t\,\xi}\, y(t)\, dt \ , \quad \mathrm{Re}(\xi) > \alpha.$$

Assumption (II) implies that equation (1.1) can be transformed into

$$B\, u_\xi''(x) + (\xi A + C)\ u_\xi(x) = f_\xi(x) + A\, g(x) \ , \quad \mathrm{Re}(\xi) > \alpha \ , \tag{2.1}$$

where $g(x)$ is the initial vector (1.3). If B is a singular matrix, then eq. (2.1) is a DAE depending on the parameter ξ. Under assumptions (I) – (V) and with properly chosen BCs, eq. (2.1) has an unique solution.

In order to define a spatial index we need the Kronecker normal form of the DAE (2.1) [5]. Assumption (III) guarantees that there are nonsingular matrices $P_{L,\xi}, Q_{L,\xi} \in \mathbb{R}^{n\times n}$ such that

$$P_{L,\xi}\, B\, Q_{L,\xi} \ = \ \begin{pmatrix} I_{m_1} & 0 \\ 0 & N_{L,\xi} \end{pmatrix} \quad , \quad P_{L,\xi}\ (\xi A + C)\ Q_{L,\xi} \ = \ \begin{pmatrix} R_{L,\xi} & 0 \\ 0 & I_{m_2} \end{pmatrix} . \tag{2.2}$$

$R_{L,\xi} \in \mathbb{R}^{m_1\times m_1}$ is a square matrix and $N_{L,\xi} \in \mathbb{R}^{m_2\times m_2}$ is a nilpotent Jordan chain matrix where $m_1 + m_2 = n$. $I_k \in \mathbb{R}^{k\times k}$ with $k \in \{m_1, m_2\}$ is an unit matrix. The Riesz index (or nilpotency) of $N_{L,\xi}$ is denoted by $\nu_{L,\xi}$ (i.e. $N_{L,\xi}^{\nu_{L,\xi}} = 0$, $N_{L,\xi}^{\nu_{L,\xi}-1} \neq 0$). With these relations eq. (2.1) can be transformed into the decoupled system of equations

$$v_\xi''(x) + R_{L,\xi}\, v_\xi(x) = r_{\xi,1}(x) \tag{2.3}$$

$$N_{L,\xi}\, w_\xi''(x) + w_\xi(x) = r_{\xi,2}(x) \ . \tag{2.4}$$

Here we introduced the definitions $(v_\xi^\top(x), w_\xi^\top(x))^\top := Q_{L,\xi}^{-1} u_\xi(x)$ and $(r_{\xi,1}^\top(x), r_{\xi,2}^\top(x))^\top := P_{L,\xi}(f_\xi(x) + A g(x))$. The differential equation (2.3) shows that we have to solve a system of differential equations of second order having the dimension m_1. In general, we have to require for each component $v_{\xi,j}$ of v_ξ a BC, i.e. $\mathfrak{M}_{BC}$ has m_1 elements. Eq. (2.4) is equivalent to an algebraic equation. To show this we derive the following equations.

$$\begin{aligned} w_\xi(x) &= r_{\xi,2}(x) - N_{L,\xi} w_\xi''(x) \\ &= r_{\xi,2}(x) - N_{L,\xi} r_{\xi,2}''(x) + N_{L,\xi}^2 w_\xi^{(4)}(x) \\ &\ \vdots \\ w_\xi(x) &= r_{\xi,2}(x) - N_{L,\xi} r_{\xi,2}''(x) + \ldots (-1)^{\nu_{L,\xi}-1} N_{L,\xi}^{\nu_{L,\xi}-1} r_{\xi,2}^{(2\nu_{L,\xi}-2)}(x) + (-1)^{\nu_{L,\xi}} \underbrace{N_{L,\xi}^{\nu_{L,\xi}}}_{=\,0} w_\xi^{(2\nu_{L,\xi})}(x) \end{aligned} \tag{2.5}$$

From eq.(2.5) we obtain formally for each component of the vector $w_\xi \in \mathbb{R}^{m_2}$ an expression in terms of the given vector $r_{\xi,2}$ and its derivatives with respect to x up to the order $2\nu_{L,\xi} - 2$. This means that any BCs cannot be prescribed for the components of w_ξ.

The matrices $P_{L,\xi}, Q_{L,\xi}, R_{L,\xi}, N_{L,\xi}$ (and $m_1, m_2, \nu_{L,\xi}$) depend on the parameter ξ so that they may change, in general, their structure depending on ξ. For which components of $u_\xi(x)$ (and finally of $u(t,x)$) we can prescribe a BC depends on the structure of the matrix $Q_{L,\xi}$. In this paper we will assume that $\mathrm{Re}(\xi)$ can be chosen such that the index set $\mathfrak{M}_{BC}$ is independent of the Laplace parameter ξ.

From eq. (2.5) we get a hint for a definition of a space index analogous to the definition of the differential index for DAEs. One differentiation of this equation with respect to x yields

$$w'_\xi(x) = r'_{\xi,2}(x) - N_{L,\xi} r'''_{\xi,2}(x) + \ldots (-1)^{\nu_{L,\xi}-1} N_{L,\xi}^{\nu_{L,\xi}-1} r_{\xi,2}^{(2\nu_{L,\xi}-1)}(x)\ . \tag{2.6}$$

I. e. in order to get an explicit differential equation for $w_\xi(x)$ one needs $2\nu_{L,\xi}-1$ differentiations with respect to the variable x. The equivalence of the transformed PDAE (2.1) and the decoupled system (2.3), (2.4) with the differential equation (2.6) for $w_\xi(x)$ suggests to define the differential spatial index $\nu_{d,x}$ under assumptions (II), (III) as follows

Definition 2.1. Let $\alpha^* \in \mathbb{R}^+$ be a number such that

1. $\alpha^* > \alpha$ and
2. the matrix $N_{L,\xi}$ with nilpotency $\nu_{L,\xi} \geq 1$ does not change the structure for all $\xi \in \mathcal{C}$ with $\mathrm{Re}(\xi) > \alpha^*$.

Then

$$\nu_{d,x} := 2\nu_L - 1 \qquad \text{with } \nu_L := \nu_{L,\xi}\ , \mathrm{Re}(\xi) > \alpha^*,$$

is called the *differential spatial index* of the PDAE (1.1). If $\nu_L = 0$ then the differential spatial index of PDAE is defined to be zero.

Remark 2.1. Under the assumptions (I)-(V) there exists $\alpha^* \in \mathbb{R}^+$. □

The differential spatial index $\nu_{d,x}$ shows directly which differentiation properties with respect to the variable x the function $r_{\xi,2}(x)$ (and hence $f(t,x)$, $g(x)$) must have at least in order to convert the Laplace transformed PDAE (2.1) into an explicit system of differential equations. The index of the DAE (2.1) is determined by the matrix pencil $(B, \xi A + C)$. Under the assumptions (I) – (V) the boundary value problem (BVP) (2.1) with the corresponding boundary conditions (1.2) is solvable uniquely if the matrix $R_{L,\xi}$ in (2.2) is negative definite[1].

Remark 2.2. Since n is small in many applications (such as $n = 2, 3$ or 4) the nilpotency $\nu_{L,\xi}$ of $N_{L,\xi}$ can be determined easily. In the case that $N_{L,\xi}$ consists of only one block, it is $\nu_{L,\xi} = m_2$. □

2.2. Fourier transform and the time index. Beside the space index defined in the previous section a time index for PDAEs can be introduced too.

In the following the PDAE (1.1) is multiplied by a scalar function $\phi_k(x)$ and integrated with respect to x on the interval $[-l, l]$. The functions $\phi_k(x)$, $k = 1, 2, \ldots$, are orthonormal eigenfunctions of the operator $\frac{\partial^2}{\partial x^2}$ with eigenvalue μ_k. $\phi_k(x)$ fulfills the same BC as $u_j(t,x)$,

[1]This assumption can be reduced, but it is chosen for simplification.

$j \in \mathfrak{M}_{BC}$, i.e. the homogeneous condition (1.2). Defining the finite Fourier transform of a vector valued function $\chi(t,x)$ by

$$\hat{\chi}_k(t) = \frac{1}{l}\int_{-l}^{l} \chi(t,x)\phi_k(x)\,dx$$

we get

$$A\hat{u}_k'(t) + (\mu_k\, B + C)\,\hat{u}_k(t) = \hat{f}_k(t) + B\rho_k(t) \;=:\; \bar{f}_k(t) \tag{2.7}$$

with $\rho_k(t) = (\rho_{k1}(t), .., \rho_{kn}(t))^\top$ and

$$\begin{aligned} \rho_{kj}(t) &= 0 && \text{for } j \in \mathfrak{M}_{BC}, \\ \rho_{kj}(t) &= \frac{1}{l}\Big[\phi_k'(x)u_j(t,x) - \phi_k(x)u_{x,j}(t,x)\Big]_{x=-l}^{x=l} && \text{for } j \notin \mathfrak{M}_{BC}, \end{aligned}$$

which results from partial integration of the term $\int_{-l}^{l} u_{xx}(t,x)\phi_k(x)\,dx$.

If the matrix A is singular (as it may be here), eq. (2.7) is a DAE depending on the parameter μ_k which can be uniquely solved with suitable ICs under the assumptions (IV) and (V). Analogous to the case of the Laplace transform in section 2.1 the assumption (IV) implies that there exist regular matrices $P_{F,k}, Q_{F,k}$ such that [5]

$$P_{F,k}\;A\;Q_{F,k} \;=\; \begin{pmatrix} I_{n_1} & 0 \\ 0 & N_{F,k} \end{pmatrix} \quad , \quad P_{F,k}\;(\mu_k B + C)\;Q_{F,k} \;=\; \begin{pmatrix} R_{F,k} & 0 \\ 0 & I_{n_2} \end{pmatrix} \tag{2.8}$$

with $R_{F,k} \in \mathbb{R}^{n_1\times n_1}$. $N_{F,k} \in \mathbb{R}^{n_2\times n_2}$ is again a nilpotent Jordan chain matrix with Riesz index $\nu_{F,k}$. Of course $n_1+n_2 = n$. These relations imply that eq. (2.1) is equivalent to the decoupled system of equations

$$y_k'(t) + R_{F,k}\;y_k(t) = s_{k,1}(t) \tag{2.9}$$

$$N_{F,k}\;z_k'(t) + z_k(t) = s_{k,2}(t) \tag{2.10}$$

where $(y_k^\top(t), z_k^\top(t))^\top := Q_{F,k}^{-1}\hat{u}_k(t)$, $(s_{k,1}^\top(t), s_{k,2}^\top(t))^\top := P_{F,k}\bar{f}_k(t)$. The solution of eq. (2.9) is given by

$$y_k(t) = e^{-R_{F,k}t}y_k(0) + \int_0^t e^{-R_{F,k}(t-\tau)}\;s_{k,1}(\tau)\,d\tau \tag{2.11}$$

while the solution of eq. (2.10) can be written immediately (cf. eq. (2.5)) as

$$z_k(t) = \sum_{i=0}^{\nu_{F,k}-1} (-N_{F,k})^i\; s_{k,2}^{(i)}(t)\;. \tag{2.12}$$

This representation shows that, in general, we can not prescribe any IC $z_k(0)$ because this equation implies that the initial values of $z_k(t)$ are given by the function on the right. Furthermore, if the solution component $z_k(t)$ must have a certain smoothness, then the right hand side function f has for indexes $\nu_{F,k} \geq 2$ to fulfill strong requirements concerning its differentiation properties.

After these preliminaries we can define the following index.

Definition 2.2. The PDAE (1.1) is said to have the *uniform differential time index* $\nu_{d,t} = \nu_F$ if for all k the pencil $(A, \mu_k B + C)$ is regular and has the Riesz index $\nu_{F,k} = \nu_F$ (independent of k) and if the Jordan chain matrix $N_{F,k}$ has the same structure for all k.

From eq. (2.12), it can be concluded that the time index $\nu_{d,t}$ gives the information which differentiation properties with respect to the variable t the function $s_{k,2}(t)$ (i.e. also $f(t,x)$) must have at least. Furthermore, eq. (2.9) shows that n_1 is equal to the number of ICs which can be prescribed for a solution of the PDAE (1.1).

Remark 2.3. We note that in a second paper [13] the two differential indexes are defined from an other point of view. The definition given there uses only algebraic properties (the proper quality) of certain matrices being constructed with the matrices A, B, and C. $\square$

Remark 2.4. If the matrices A, B, C have a special structure it can happen that the pencil $(A, \mu_k B + C)$ does not have the same index for all k. This can only occur for a finite number of different k. We call that an *index jump* (see example 2.3). Analogous it can also happen that the structure of $N_{F,k}$ changes for some k (see example 2.4). Problems like these differ qualitatively from those with an uniform time index such that it is advantageous to assign them to an other class of PDAEs. We note that the index definitions in [3], [4] do not distinguish between problems with and without index jump and changing structure of $N_{F,k}$. If we transform the PDAE (1.1) by the Laplace transform, we can avoid an index jump and a change of the structure of $N_{L,\xi}$ by choosing the Laplace parameter ξ as described in definition 2.1. An analogous behavior can not appear in the case of the Fourier transformation of the PDAE (1.1) (since the parameters μ_k, $k = 1, 2, ...$, are fixed discrete quantities). $\square$

Remark 2.5. For $\nu_{d,t} = 0$ and $\nu_{d,x} = 0$ the system (1.1) is a PDE. $\square$

Example 2.1. For the PDAE of example 1.3 it is easily shown that $\nu_L = 2$, i.e. $\nu_{d,x} = 3$, $\nu_{d,t} = 2$.

Example 2.2. Let in the PDAE (1.1) $n = 2$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -b_1 & -b_2 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & 0 \end{pmatrix} .$$

For $b_2 = 0$ we get the indexes $\nu_{L,\xi} = 2$ and $\nu_{d,x} = 3$. On the other hand for $b_2 \neq 0$ it follows $\nu_L = 1$ and $\nu_{d,x} = 1$. For all b_2 it is $\nu_{d,t} = 2$.

The following examples illustrate the case of the index jump and a change of the structure of $N_{F,k}$, respectively.

Example 2.3. Let be $n = 2$, A singular, B regular and homogeneous Dirichlet BCs $u(t, -l) = u(t, l) = 0$ for $t \in [0, t_e]$ be given. In the special case that B is regular we get in (2.7) that $\rho_k(t) = 0$. Let the matrices $A, B, C \in \mathbb{R}^{2\times 2}$ be

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; B \;=\; \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \; C \;=\; \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} .$$

The matrix pencil $(A, \mu_k B + C)$ is regular if either $\{b_4\, \mu_k = -c_4$ and $(b_2\, \mu_k + c_2)(b_3\, \mu_k + c_3) \neq 0\}$ or $\{b_4\, \mu_k \neq -c_4\}$. Let ν be such that the matrix $\nu A + (\mu_k B + C)$ is regular. Then the Riesz index of the matrix pencil is (cf.[2], [7])

$$\operatorname{ind}(A, \mu_k B + C) = \operatorname{ind}\left([\nu A + (\mu_k B + C)]^{-1} A\right) = \begin{cases} 1 & \text{for } b_4\,\mu_k \neq -c_4 \\ 2 & \text{for } b_4\,\mu_k = -c_4\ . \end{cases}$$

We get

- uniform time index 1 if $b_4\,\mu_k \neq -c_4$ for all $k = 1, 2, ..,$
- uniform time index 2 if $b_4 = c_4 = 0$,
- index jump if $\mu_k = -\frac{c_4}{b_4}$ for one k.

Let the matrix $\mu_k B + C$ be regular for all $k > 0$, i.e. $D_k := \det(\mu_k B + C) \neq 0$. Using the Jordan normal form of $(\mu_k B + C)^{-1} A$ we can give the following representation of the solution of the DAE (2.7) (cf. [15]). For the case that $b_4\,\mu_k \neq -c_4$, i.e. if the pencil $(A, \mu_k B + C)$ has the Riesz index 1, the solution $\hat{u}_k = (\hat{u}_{k1}, \hat{u}_{k2})^\top$ is

$$\hat{u}_k(t) = \hat{u}_{k1}(0) \begin{pmatrix} 1 \\ -\frac{b_3\,\mu_k + c_3}{b_4\,\mu_k + c_4} \end{pmatrix} e^{-\frac{D_k}{b_4\,\mu_k + c_4} t} + \begin{pmatrix} 0 \\ \frac{1}{b_4\,\mu_k + c_4} \end{pmatrix} \hat{f}_{k2}(t)$$
$$+ \begin{pmatrix} 1 \\ -\frac{b_3\,\mu_k + c_3}{b_4\,\mu_k + c_4} \end{pmatrix} \int_0^t \left(\hat{f}_{k1}(\tau) - \frac{b_2\,\mu_k + c_2}{b_4\,\mu_k + c_4} \hat{f}_{k2}(\tau)\right) e^{\frac{D_k}{b_4\,\mu_k + c_4}(\tau - t)}\, d\tau\ .$$

with $\hat{f}_k = (\hat{f}_{k1}, \hat{f}_{k2})^\top$. For $t = 0$ must be

$$\hat{u}_k(0) = \begin{pmatrix} \hat{u}_{k1}(0) \\ \hat{u}_{k2}(0) \end{pmatrix} = \hat{u}_{k1}(0) \begin{pmatrix} 1 \\ -\frac{b_3\,\mu_k + c_3}{b_4\,\mu_k + c_4} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{b_4\,\mu_k + c_4} \end{pmatrix} \hat{f}_{k2}(0).$$

From this we get in this case a consistency condition between the initial values and the right hand side function

$$\hat{f}_{k2}(0) = (b_3\,\mu_k + c_3)\hat{u}_{k1}(0) + (b_4\,\mu_k + c_4)\hat{u}_{k2}(0)\ .$$

If we prescribe $\hat{u}_{k1}(0)$, then $\hat{u}_{k2}(0)$ is uniquely determined by $\hat{u}_{k1}(0)$ and $\hat{f}_{k2}(0)$ and can not be prescribed arbitrarily.

The solution of (2.7) for $b_4\,\mu_k = -c_4$ (for Riesz index 2) is given by

$$\hat{u}_k(t) = \begin{pmatrix} \dfrac{\hat{f}_{k2}(t)}{b_3\,\mu_k + c_3} \\ \dfrac{(b_3\,\mu_k + c_3)\hat{f}_{k1}(t) - (b_1\,\mu_k + c_1)\hat{f}_{k2}(t) - \hat{f}'_{k2}(t)}{(b_2\,\mu_k + c_2)(b_3\,\mu_k + c_3)} \end{pmatrix},$$

and the consistency conditions are

$$\hat{u}_{k1}(0) = \frac{\hat{f}_{k2}(0)}{b_3\,\mu_k + c_3}$$
$$\hat{u}_{k2}(0) = \frac{(b_3\,\mu_k + c_3)\hat{f}_{k1}(0) - (b_1\,\mu_k + c_1)\hat{f}_{k2}(0) - \hat{f}'_{k2}(0)}{(b_2\,\mu_k + c_2)(b_3\,\mu_k + c_3)}$$

what means that, in general, we can prescribe neither $\hat{u}_{k1}(0)$ nor $\hat{u}_{k2}(0)$.

Interpreting the solution $\hat{u}_k(t)$ of (2.7) as the kth Fourier coefficient of $u(t, x)$, i.e. we assume that $u(t,x)$ and the right hand side function $f(t,x)$ have a Fourier series expansion with respect to the $\phi_k(x)$, it is

$$u(t,x) = \sum_{k=1}^{\infty} \hat{u}_k(t)\phi_k(x).$$

Then for the uniform differential time index 1 of the PDAE (1.1) all DAEs (2.7) have differential index 1. That means that for all these initial value problems we can prescribe $\hat{u}_{k1}(0)$, and we can calculate a consistent $\hat{u}_{k2}(0)$ uniquely from $\hat{u}_{k1}(0)$ and the given right hand functions. Prescribing initial values (1.3) for the first component of $g(x)$ in this example (i.e. $\mathfrak{M}_{IC} = \{1\}$), we get consistency between the initial values and the right hand side functions. Analogously we can not prescribe any initial condition in the case of an uniform time index 2 (i.e. $\mathfrak{M}_{IC} = \emptyset$) because it is completely determined by the right hand side.

It is difficult to prescribe for $\hat{u}_k(t)$ consistent ICs (1.3) in the case of an index jump. In the example it is $\mu_j = -\frac{c_4}{b_4}$ for one $j > 0$, and the jth Fourier coefficient of $u_1(t,0)$ is given by the right hand side function. All other Fourier coefficients $\hat{u}_{k1}(0)$, $k \neq j$, are free parameters. So, without a Fourier analysis, we can not determine which IC for the PDAE (1.1) can be prescribed.

Example 2.4. Let $n = 4$ and the matrices A, B, C be given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & c \end{pmatrix}.$$

The matrix pencil $(A, \mu_k B + C)$ has Riesz index 2 for all $k \in \mathbb{N}^+$. Let $c = \mu_{\bar{k}}$ for one $\bar{k} \in \mathbb{N}^+$. The Kronecker transformations (2.8) of the pencil $(A, \mu_{\bar{k}} B + C)$ and the pencils $(A, \mu_k B + C)$ lead to $(a = 1 - \mu_k, b = \mu_k - c)$

$$P_{F,\bar{k}}\, A \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-c & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{Q_{F,\bar{k}}} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{N_{F,\bar{k}}}, \qquad P_{F,k}\, A \underbrace{\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & a & 0 \\ b & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}}_{Q_{F,k}} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

with the lower right 3×3 block being $N_{F,k}$,

i.e. the matrix $N_{F,\bar{k}} \in \mathbb{R}^{4\times4}$ has an other structure as the matrix $N_{F,k} \in \mathbb{R}^{3\times3}$, $k \neq \bar{k}$. It follows from (2.12) that $u_{\bar{k}}(0)$ is determined by the right hand side function but we get from $u_k(t) = Q_{F,k}(y_k^\top, z_k^\top)^\top$ that for all $k \neq \bar{k}$ any value of the third component of $u_k(0)$ can be chosen. Again it is not clear in which way we can assign ICs for the PDAE (1.1).

2.3. Perturbation index of the PDAE. In addition to the differential indexes $\nu_{d,x}$ and $\nu_{d,t}$ it is useful to have also a perturbation index of the PDAE (1.1). For these equations there are several variants of a perturbation index which are defined by means of Fourier transforms [1],[4]. Among others these indexes can be used to characterize the PDAE with respect to methods for its numerical solution (e.g. the method of lines, MOL). In this paper a variant of a perturbation index is chosen which is a generalization of the well known perturbation index for DAEs [9]. We emphasize that our perturbation index is really a perturbation index pair defined in the $(t,x)-$space. This is different from the differential spatial index $\nu_{d,x}$ and the uniform differential time index $\nu_{d,t}$ which are defined after having the PDAE Laplace and Fourier transformed, respectively.

Let for short $I := [0,t]$, $0 < t \le t_e$. Then we give the

Definition 2.3. The PDAE (1.1) has under the assumption (I) the *perturbation index pair* $[m,k]_p$ along a solution $u(t,x)$ on $[0,t_e] \times [-l,l]$, if m and k are both the smallest positive integers such that, for all $\hat{u}(t,x)$ having a defect

$$A\hat{u}_t(t,x) + B\hat{u}_{xx}(t,x) + C\hat{u}(t,x) \;=\; f(t,x) + \delta(t,x)$$

there is for Dirichlet BCs (prescribed for u_j, $j \in \mathfrak{M}_{BC}$) on $[0,t_e] \times [-l,l]$ an estimate

$$\begin{aligned}\|\hat{u}(t,x) - u(t,x)\| \;\le C_0 &\left(\|\hat{u}(0,x) - u(0,x)\| + \sum_{i=0}^{m-1}\sum_{j=0}^{k-1} \max_{\tau \in I} \left\| \frac{\partial^{i+j}\delta(\tau,x)}{\partial\tau^i \partial x^j} \right\| \right) \\ &+ C_1 \max_{\tau\in I} \left(\|\hat{u}(\tau,-l) - u(\tau,-l)\| + \|\hat{u}(\tau,l) - u(\tau,l)\| \right)\end{aligned}$$

whenever the expression on the right hand side is sufficiently small. As usual δ is interpreted as a perturbation in the PDAE (1.1). C_0 and C_1 are assumed to be finite non negative constants independent of t, x and δ.

For Neumann BCs ($\kappa = 1$ in eq.(1.2)) the second row on the right hand side is replaced by

$$\begin{aligned} &C_1 \sum_{j\in\mathfrak{M}_{BC}} \max_{\tau\in I} \left(\left\| \frac{\partial(\hat{u}_j(\tau,-l) - u_j(\tau,-l))}{\partial x} \right\|_{sc} + \left\| \frac{\partial(\hat{u}_j(\tau,l) - u_j(\tau,l))}{\partial x} \right\|_{sc} \right) \\ &+C_2 \sum_{j\notin\mathfrak{M}_{BC}} \max_{\tau\in I} \left(\|\hat{u}_j(\tau,-l) - u_j(\tau,-l)\|_{sc} + \|\hat{u}_j(\tau,l) - u_j(\tau,l)\|_{sc} \right)\end{aligned}$$

where $\|\,.\,\|_{sc}$ is a suitable norm of a scalar and again C_2 a finite non negative constant. If the $x-$dependence is not present, then all $x-$derivatives vanish and C_1, C_2 may be chosen as zero. Then the definition is identical with the definition of the perturbation index for DAEs, cf. e.g. [9].

Under some assumptions, there is a relation between the perturbation index pair $[m,k]_p$ and the two differential indexes defined in this paper. To show this we suppose a perturbation is given, and we determine $\hat{u}$ by means of an initial boundary value problem with the perturbed PDAE given in the definition.

Lemma 2.1. Assume $t_e = \infty$. Let $\hat{u}$ be sufficiently smooth and suppose it satisfies a growth condition (cf. assumption (II)). Furthermore, let $\nu_{d,x}$ be the differential spatial index of the PDAE and suppose that there is also an uniform differential time index $\nu_{d,t}$. Then the perturbation index pair of the PDAE is given by

$$[m,k]_p \;=\; [\nu_{d,t}\,,\,\nu_{d,x}].$$

Proof. It is enough to show that the highest derivatives of the perturbation δ with respect to t and x occuring in $\hat{u}(t,x)$ are of order $\nu_{d,t} - 1$ and $\nu_{d,x} - 1$, respectively. For this aim we make use of two different representations of the function $\hat{u}$ which, by assumption, exists and is unique (one representation is given by a Fourier series, the other by a Laplace transform technic).

First we consider the derivatives of δ with respect to t by replacing f by $f+\delta$ in the PDAE (1.1).

According to section 2.2 the highest time derivative of the Fourier transform $\bar{f}_k(t)+\delta_k(t)$ is of order $\nu_{F,k}-1$ (see eq. (2.12)). Since the inverse Fourier transform does not change the order of the time derivative and because the time index is assumed to be uniform (i.e. $\nu_{F,k}=\nu_F$) we get the result that the highest time derivative of δ in $(t,x)-$space is of order ν_F-1. But this means $m=\nu_F=\nu_{d,t}$ (cf. Definition 2.2). In the second part we consider in $\hat{u}(t,x)$ the highest derivative of $\delta(t,x)$ with respect to x. To do this we use now the Laplace transform $\hat{u}_\xi(x)=Q_{L,\xi}\left(\hat{v}_\xi^\top(x),\ \hat{w}_\xi^\top(x)\right)^\top$ of $\hat{u}(t,x)$ ($\hat{u}_\xi$ exists by the assumptions of the Lemma). From assumptions (II), (III) and eq. (2.5) we see that the highest $x-$derivative of the Laplace transform δ_ξ is of order $2\nu_L-2$. Since the inverse Laplace transform does not change the order of the $x-$derivatives and from Definition 2.3 we conclude immediately that $k=2\nu_L-1$ must be chosen. According to Definition 2.1 this is identical with $\nu_{d,x}$, i.e. $k=\nu_{d,x}$. Taking further into account that the highest time derivative of δ occuring in $\hat{u}$ is of order ν_F-1 we get the Lemma. ⊠

Example 2.5. For the PDAE of example 1.3 we choose $t_e=\infty$ and suppose that u and $\hat{u}$ both satisfy (component wise) a growth condition (cf. assumption (II)). Then the assumptions of Lemma 2.1 are valid. From example 2.1 we get

$$[m,k]_p \;=\; [2,3].$$

We also see from the solution of the PDAE that there are no mixed derivatives (with respect to time and space) of $\delta(t,x)$ appearing in $\hat{u}(t,x)$. In general, the question whether there are mixed derivatives of δ appearing in the estimate of $\|\hat{u}(t,x)-u(t,x)\|$ or not can be decided easily by considering the Laplace transform $u_\xi(x)$. This follows from the representation of the solution investigated in section 3.

3. Consistent representation of the solution

In this section we discuss the question how the solution of the IBVP (1.1)-(1.4) can be represented under the assumption that it has an uniform time index $\nu_{d,t}$ (cf. definition 2.2). In order to get a suitable representation we must take into account the fact that, in general, any ICs and BCs can not be prescribed for all components $u_i(t,x)$, $i=1(1)n$. The term "consistent representation of the solution" means we are interested in a representation of the solution of the PDAE (1.1) which uses ICs and BCs only for components u_i, $i\in\mathfrak{M}_{IC}$, and u_j, $j\in\mathfrak{M}_{BC}$, respectively.

In subsection 2.1 we already pointed out some results on proper BCs. We got the result that we can prescribe BCs for $v_\xi(x)$, but not for $w_\xi(x)$. We have to be careful in determining the set $\mathfrak{M}_{BC}$. We easily see that this set has m_1 elements, and for m_2 components of $u_\xi(x)$ we can not prescribe any BC. We mention that it is not always unique for which components of u BCs can be assigned. Analogously, we saw that $\mathfrak{M}_{IC}$ has n_1 elements which are not always determined uniquely. The following example illustrates the determination of the sets $\mathfrak{M}_{IC}$ and $\mathfrak{M}_{BC}$.

Example 3.1. Let

$$A=\begin{pmatrix}1&1\\0&0\end{pmatrix},\quad B=\begin{pmatrix}-1&0\\-1&0\end{pmatrix},\quad C=\begin{pmatrix}c_1&c_2\\-1&0\end{pmatrix},\quad c_1,c_2>0$$

then $\nu_{d,t} = 1$ ($n_1 = n_2 = 1$), $\nu_{d,x} = 1$ ($m_1 = m_2 = 1$) and

$$P_{F,k} = \begin{pmatrix} \frac{1}{c_2} & \frac{c_1 - c_2 - \mu_k}{c_2(\mu_k - 1)} \\ 0 & \frac{1}{\mu_k - 1} \end{pmatrix}, \quad Q_{F,k} = \begin{pmatrix} 0 & -1 \\ c_2 & 1 \end{pmatrix}$$

$$P_{L,\xi} = \begin{pmatrix} 0 & \frac{1}{\xi + c_2} \\ \frac{1}{\xi + c_2} & -\frac{1}{\xi + c_2} \end{pmatrix}, \quad Q_{L,\xi} = \begin{pmatrix} -(\xi + c_2) & 0 \\ \xi + c_1 - 1 & 1 \end{pmatrix} .$$

This implies the transformations

$$\begin{pmatrix} \hat{u}_{k1} \\ \hat{u}_{k2} \end{pmatrix} = Q_{F,k} \begin{pmatrix} y_k \\ z_k \end{pmatrix} = \begin{pmatrix} -z_k \\ c_2 y_k + z_k \end{pmatrix}, \tag{3.1}$$

$$\begin{pmatrix} u_{\xi 1} \\ u_{\xi 2} \end{pmatrix} = Q_{L,\xi} \begin{pmatrix} v_\xi \\ w_\xi \end{pmatrix} = \begin{pmatrix} -(\xi + c_2) v_\xi \\ (\xi + c_1 - 1) v_\xi + w_\xi \end{pmatrix}. \tag{3.2}$$

We know from section 2 that, in general, the initial value for z_k and the boundary values for w_ξ can not be prescribed. Eq. (3.1) fixes the component $u_1(t,x)$ to be that one for which we can not prescribe an IC, i.e. $\mathfrak{M}_{IC} = \{2\}$. On the other hand since a BC for v_ξ can be assigned, we conclude from (3.2) that $\mathfrak{M}_{BC} = \{1\}$. Thus a BC for $u_{\xi 2}$ (or $u_2(t,x)$) can not be prescribed because the BC for $u_{\xi 2}$ must be computed in the manner shown for the second component in eq. (3.2).

In the following we consider for $t \in [0,\infty)$, $x \in [-l,l]$, a formal representation of the solution $u(t,x)$ of the IBVP (1.1)-(1.3). First homogeneous Dirichlet BCs (1.2), i.e. $u_j(t,\pm l) = 0$, $j \in \mathfrak{M}_{BC}$, are assumed. The solution is supposed to be representable in series form

$$u(t,x) = \sum_{k=0}^{\infty} \tilde{u}_k(t) \phi_k(x) + d(t,x) \tag{3.3}$$

with $\tilde{u}_k : [0,\infty) \to \mathbb{R}^n$, $\phi_k : [-l,l] \to \mathbb{R}$, $d : [0,\infty) \times [-l,l] \to \mathbb{R}^n$ (cf. 2.2). Especially the components d_j of the vector valued function d are such that

$$d_j = 0 \text{ for } j \in \mathfrak{M}_{BC} \ .$$

The remaining components $u_j(t,\pm l)$, $j \notin \mathfrak{M}_{BC}$, may be computed by means of the PDAE (for illustration see example 3.2). Let $d(t,x)$ be a smooth vector function with

$$d(t,\pm l) = u(t,\pm l)$$

and $U(t,x)$ be a new vector function defined by

$$U(t,x) := u(t,x) - d(t,x) \ . \tag{3.4}$$

Obviously, U satisfies

$$U(t,-l) = U(t,l) = 0. \tag{3.5}$$

Inserting $u = U + d$ into eq. (1.1) we get the new PDAE

$$A\, U_t(t,x) + B\, U_{xx}(t,x) + C\, U(t,x) = F(t,x) \tag{3.6}$$

where $F := f - A\,d_t - B\,d_{xx} - C\,d$. The IC is

$$U_j(0,x) = g_j(x) - d_j(0,x) \quad j \in \mathfrak{M}_{IC}\ . \tag{3.7}$$

The BC (3.5) is satisfied by the expansion

$$U(t,x) = \sum_{k=1}^{\infty} U_k(t)\phi_k(x) \ = \ \sum_{k=1}^{\infty} U_k(t)\cos(\tfrac{k\pi}{2l}x). \tag{3.8}$$

Supposing $F(t,x)$ and $U(0,x)$ can be represented analogously as Fourier cosine series it is easy to see that $U_k(t)$ must be a solution of the DAE

$$A\,U_k'(t) + (\mu_k\,B + C)\,U_k(t) = F_k(t), \quad k = 1,2,..., \tag{3.9}$$

$$\mu_k = -\left(\tfrac{k\pi}{2l}\right)^2, \qquad F_k(t) = \frac{1}{l}\int_{-l}^{l} F(t,x)\cos(\tfrac{k\pi}{2l}x)\,dx\ , \tag{3.10}$$

with consistent ICs. According to section 2, under conditions (IV) and (V) (where f must be replaced by F) there is an unique solution of this DAE for $k = 1,2,\ldots$ Therefore, u can formally be represented in the form (3.3) for Dirichlet BCs $u_j(t,\pm l) = 0$, $j \in \mathfrak{M}_{BC}$, and given (consistent) ICs. In this sense, the representation of the solution in the form (3.3) is a consistent one.

Example 3.2. Let $n = 2$, $u = (u_1, u_2)^\top$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u_t + \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} u_{xx} + \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} u \ = \ f. \tag{3.11}$$

In order to construct the correct BCs necessary for the transformation (3.4) we first Laplace transform the corresponding PDAE. Then it can be written

$$-M_\xi^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_{1\xi}'' \\ u_{2\xi}'' \end{pmatrix} + \begin{pmatrix} u_{1\xi} \\ u_{2\xi} \end{pmatrix} \ = \ M_\xi^{-1}\tilde{f}_\xi, \qquad \xi \neq 0.$$

M_ξ is the regular matrix $M_\xi = \xi A + C$ and $\tilde{f}_\xi = f_\xi + Ag(x)$ (cf. subsection 2.1). Suppose $\beta \neq \xi$. Then in the next step we perform a Kronecker transformation of the last PDAE with the matrices $P_\xi = \left(\begin{smallmatrix} 1 & -1 \\ -\frac{\beta}{\xi} & 1 \end{smallmatrix}\right)$ and $Q_\xi = P_\xi^{-1}$. The new equation is

$$\begin{pmatrix} R_\xi & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_\xi'' \\ w_\xi'' \end{pmatrix} + \begin{pmatrix} v_\xi \\ w_\xi \end{pmatrix} \ = \ P_\xi^{-1}M_\xi^{-1}\tilde{f}_\xi, \quad R_\xi = \tfrac{\beta-\xi}{\xi^2}, \quad \begin{pmatrix} v_\xi \\ w_\xi \end{pmatrix} = P_\xi^{-1}\begin{pmatrix} u_{1\xi} \\ u_{2\xi} \end{pmatrix}$$

or explicitly $v_\xi = \frac{\xi}{\xi-\beta}\,(u_1+u_2)$, $w_\xi = \frac{\xi}{\xi-\beta}\left(\frac{\beta}{\xi}u_1 + u_2\right)$. From these formulas it is seen immediately that $\mathfrak{M}_{BC}$ is defined not uniquely. We can choose either $\mathfrak{M}_{BC} = \{1\}$ or $\mathfrak{M}_{BC} = \{2\}$. Taking the first case we infer from the original equation (3.11) that the BC for u_2 can be calculated by solving the ODE

$$\frac{d}{dt}u_2(t,\pm l) = f_2(t,\pm l) - \beta u_1(t,\pm l) \tag{3.12}$$

with a given IC $u_2(0, \pm l)$. After solving this simple ordinary initial value problem we have consistent BCs for both components $u_1(t, \pm l)$ and $u_2(t, \pm l)$. Thus the transformation (3.4) may be done and U can be represented in the form (3.8).

Secondly, the case of homogeneous Neumann BCs (1.2), i.e.

$$\frac{\partial u_j}{\partial x}(t, -l) \;=\; \frac{\partial u_j}{\partial x}(t, l) \;=\; 0\,, \quad j \in \mathfrak{M}_{BC}, \tag{3.13}$$

is considered. Let $d(t, x)$ again be a smooth vector function but now defined such that

$$\begin{aligned} \frac{\partial d_j(t, \pm l)}{\partial x} \;&=\; 0\,, \quad j \in \mathfrak{M}_{BC}, \\ d_j(t, \pm l) \;&=\; 0\,, \quad j \notin \mathfrak{M}_{BC}. \end{aligned}$$

Defining with this $d(t, x)$ again $U(t, x)$ by eq. (3.4) we see that U satisfies the PDAE (3.6) with the same right hand side F. However, the BCs are now

$$U_j(t, \pm l) \;=\; 0\,, \quad j \notin \mathfrak{M}_{BC}, \tag{3.14}$$

$$\frac{\partial U_j(t, \pm l)}{\partial x} \;=\; 0\,, \quad j \in \mathfrak{M}_{BC}. \tag{3.15}$$

The IC is easily derived from eq. (3.7). To construct a series expansion for $U(t, x)$ we assume that $F(t, x)$ and $U(0, x)$ can be represented as general Fourier series

$$F(t, x) \;=\; \sum_{k=0}^{\infty} \Big(F_k(t)\phi_k(x) + \tilde{F}_k(t)\tilde{\phi}_k(x)\Big),$$

$$U(0, x) \;=\; \sum_{k=0}^{\infty} \Big(U_k(0)\phi_k(x) + \tilde{U}_k(0)\tilde{\phi}_k(0)\Big)$$

where $\phi_k(x) = \cos(\frac{k\pi}{2l}x)$ and $\tilde{\phi}_k(x) = \sin(\frac{k\pi}{2l}x)$. F_k, U_k and $\tilde{F}_k$, $\tilde{U}_k$ are the corresponding Fourier coefficients of $F(t, x)$, $U(0, x)$ (compare eq. (3.10)). The solution $U(t, x)$ is written formally

$$U(t, x) \;=\; \sum_{k=0}^{\infty} \Big(U_k(t)\phi_k(x) + \tilde{U}_k(t)\tilde{\phi}_k(x)\Big). \tag{3.16}$$

Since the functions ϕ_k and $\tilde{\phi}_k$ are members of an orthonormal set we get the following DAE for $V_k(t) := U_k(t) + \tilde{U}_k(t)$:

$$A\frac{d}{dt}V_k(t) + (\mu_k B + C)V_k(t) \;=\; F_k(t) + \tilde{F}_k(t), \quad k = 0, 1, ... \tag{3.17}$$

with IC as defined above (as before $\mu_k = -\left(\frac{k\pi}{2l}\right)^2$). Suppose now the assumptions (IV) and (V) (where f and g must be replaced by F and $g - d$, respectively) are fulfilled. Then there is an unique solution of this DAE and U_k, $\tilde{U}_k$, $k = 0, 1, ...$ must satisfy

$$U_{k,j}(t) \;=\; \begin{cases} 0 & \text{for} \quad j \in \mathfrak{M}_{BC}, \\ V_{k,j} & \text{for} \quad j \notin \mathfrak{M}_{BC}, \end{cases}$$

and

$$\tilde{U}_{k,j}(t) \;=\; \begin{cases} V_{k,j} & \text{for} \quad j \in \mathfrak{M}_{BC}, \\ 0 & \text{for} \quad j \notin \mathfrak{M}_{BC}. \end{cases}$$

Inserting these expressions into the representation (3.16) we see that the BCs (3.14) and (3.15) are satisfied. Hence eq. (3.16) with eq. (3.4) is a formal consistent representation of the solution of the PDAE (1.1) under the BCs (3.13).

4. Boundary and initial conditions for regular pencils (A,B)

As shown in subsections 2.1 and 2.2 the Laplace and the Fourier transforms of the solution u of the PDAE (1.1) yield the index sets $\mathfrak{M}_{BC}$ and $\mathfrak{M}_{IC}$, respectively. Hence we know for which components of u a BC and an IC can be prescribed. However, in general it is practically not easy to solve the consistence equations (2.5) and (2.12) to obtain the BC for u_i, $i \notin \mathfrak{M}_{BC}$, and the IC for u_j, $j \notin \mathfrak{M}_{IC}$. In this subsection we develop a method which (under some assumptions) is easier in practice since it is not really necessary to solve the consistence equations. In fact the BCs and ICs which can not be prescribed must be calculated, e.g. in a numerical procedure [13].

To construct the method it is supposed that the matrix pencil (A, B) is regular. For instance, this assumption is fulfilled in example 2.3 if the matrix B has a proper form (e.g. $b_4 \neq 0$). On the other hand, the pencil (A, B) is not regular in example 1.3.

The regularity of the pencil (A, B) implies that the matrices A and B can be transformed with regular matrices P, $Q \in \mathbb{R}^{n\times n}$, such that the PDAE (1.1) takes the form

$$\underbrace{\begin{pmatrix} I_1^{(1)} & 0 \\ 0 & N^{(1)} \end{pmatrix}}_{PAQ} \begin{pmatrix} v_t \\ w_t \end{pmatrix} + \underbrace{\begin{pmatrix} R^{(1)} & 0 \\ 0 & I_2^{(1)} \end{pmatrix}}_{PBQ} \begin{pmatrix} v_{xx} \\ w_{xx} \end{pmatrix} + \tilde{C} \begin{pmatrix} v \\ w \end{pmatrix} \;=\; \tilde{f}. \tag{4.1}$$

where $(v(t,x)^\top, w(t,x)^\top)^\top := Q^{-1}u(t,x)$, $\tilde{f}(t,x) := Pf(t,x)$, $\tilde{C} := PCQ$. $I_j^{(1)}$ is a unit matrix of order $n_j^{(1)}, j = 1, 2$. We first consider a form of decoupling. Assuming $t_e = \infty$ we can write down the Laplace transform of this equation (cf. also subsection 2.1)

$$\begin{pmatrix} R^{(1)} & 0 \\ 0 & I_2^{(1)} \end{pmatrix} \begin{pmatrix} v_\xi'' \\ w_\xi'' \end{pmatrix} + \xi \begin{pmatrix} I_1^{(1)} & 0 \\ 0 & N^{(1)} \end{pmatrix} \begin{pmatrix} v_\xi \\ w_\xi \end{pmatrix} + \tilde{C} \begin{pmatrix} v_\xi \\ w_\xi \end{pmatrix} \;=\; \tilde{f}_\xi^{(1)} \;:=\; \tilde{f} + \begin{pmatrix} I_1^{(1)} & 0 \\ 0 & N^{(1)} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_{|t=0}$$

Under the assumption that the pencil $(PBQ, P(\xi A + C)Q)$ is regular for $\mathrm{Re}(\xi) > \alpha$ (cf. subsection 2.1) this equation can again be Kronecker transformed with regular square $(n, n)-$matrices P_ξ, Q_ξ into the decoupled system

$$\begin{pmatrix} I_{L1}^{(2)} & 0 \\ 0 & N_L^{(2)} \end{pmatrix} \begin{pmatrix} \overline{v}_\xi'' \\ \overline{w}_\xi'' \end{pmatrix} + \begin{pmatrix} R_L^{(2)} & 0 \\ 0 & I_{L2}^{(2)} \end{pmatrix} \begin{pmatrix} \overline{v}_\xi \\ \overline{w}_\xi \end{pmatrix} \;=\; \tilde{f}_\xi^{(2)} \tag{4.2}$$

where

$$\begin{pmatrix} I_{L1}^{(2)} & 0 \\ 0 & N_L^{(2)} \end{pmatrix} = P_\xi \begin{pmatrix} R^{(1)} & 0 \\ 0 & I_2^{(1)} \end{pmatrix} Q_\xi, \quad \begin{pmatrix} R_L^{(2)} & 0 \\ 0 & I_{L2}^{(2)} \end{pmatrix} = P_\xi \left(\xi \begin{pmatrix} I_1^{(1)} & 0 \\ 0 & N^{(1)} \end{pmatrix} + \tilde{C} \right) Q_\xi \tag{4.3}$$

and $(\overline{v}_\xi^\top, \overline{w}_\xi^\top)^\top := Q_\xi^{-1}(v_\xi^\top, w_\xi^\top)^\top$, $\tilde{f}_\xi^{(2)} := P_\xi \tilde{f}_\xi^{(1)}$. As before $I_{Lj}^{(2)}$, $j = 1, 2$, is a unit matrix of order $n_{Lj}^{(2)}$. If $R^{(1)}$ is singular, then $N_L^{(2)}$ is a nilpotent matrix of order $n_{L2}^{(2)}$. The decoupled system (4.2) is similar to the system in eqs. (2.3), (2.4). It is seen from eq. (4.2) that $n_{L1}^{(2)}$ BCs for the components of $\overline{v}_\xi$ are needed. Since the rank of a $(n, n)-$matrix is not changed by multiplication with a regular $(n, n)-$matrix, we infer from the definition of $R^{(1)}$ and the first of eqs. (4.3)

$$rank(B) = rank \begin{pmatrix} R^{(1)} & 0 \\ 0 & I_2^{(1)} \end{pmatrix} = r^{(1)} + n_2^{(1)} = n_{L1}^{(2)} + rank(N_L^{(2)}) \tag{4.4}$$

where $r^{(1)}$ is defined by $r^{(1)} := rank(R^{(1)})$.

In order to get an analogous result for the number of ICs necessary for solving the PDAE (1.1) we first perform a Fourier transform (assumed to exist) of eq. (4.1), cf. subsection 2.2. Supposing that the pencil

$$\left(\begin{pmatrix} I_1^{(1)} & 0 \\ 0 & N^{(1)} \end{pmatrix}, \mu_k \begin{pmatrix} R^{(1)} & 0 \\ 0 & I_2^{(1)} \end{pmatrix} + \tilde{C} \right) \tag{4.5}$$

is regular we perform a second (Kronecker) transform with regular $(n, n)-$matrices P_{Fk}, Q_{Fk}. The result is an equation of the form given in eqs. (2.9), (2.10). It is rewritten here

$$\begin{pmatrix} I_{Fk1}^{(2)} & 0 \\ 0 & N_{Fk}^{(2)} \end{pmatrix} \begin{pmatrix} y_k'(t) \\ z_k'(t) \end{pmatrix} + \begin{pmatrix} R_{Fk}^{(2)} & 0 \\ 0 & I_{Fk2}^{(2)} \end{pmatrix} \begin{pmatrix} y_k(t) \\ z_k(t) \end{pmatrix} = s_k \tag{4.6}$$

where analogous to eq. (4.3)

$$\begin{pmatrix} I_{Fk1}^{(2)} & 0 \\ 0 & N_{Fk}^{(2)} \end{pmatrix} = P_{Fk} \begin{pmatrix} I_1^{(1)} & 0 \\ 0 & N^{(1)} \end{pmatrix} Q_{Fk}, \quad \begin{pmatrix} R_{Fk}^{(2)} & 0 \\ 0 & I_{Fk2}^{(2)} \end{pmatrix} = P_{Fk} \left(\mu_k \begin{pmatrix} R^{(1)} & 0 \\ 0 & I^{(1)} \end{pmatrix} + \tilde{C} \right) Q_{Fk} \tag{4.7}$$

and $(y_k^\top, z_k^\top)^\top = Q_{Fk}^{-1} \hat{u}_k$. s_k and μ_k are defined as in subsection 2.2. From the set of equations (4.7) we conclude that (analogous to eqs. (4.4) for BCs) the relations

$$rank(A) = rank \begin{pmatrix} I_1^{(1)} & 0 \\ 0 & N^{(1)} \end{pmatrix} = n_1^{(1)} + rank(N^{(1)}) = n_{Fk1}^{(2)} + rank(N_{Fk}^{(2)}) \tag{4.8}$$

are valid for the ICs. Thus we have proved the following

Lemma 4.1. Suppose the matrix pencil (A, B) is regular. Assume further that

(1) the pencil $(PBQ, P(\xi A + C)Q)$ is regular for $\mathrm{Re}(\xi) > \alpha$,
(2) the pencil (4.5) is regular for $k = 1, 2, ...,$

(3) the matrices $N_{Fk}^{(2)}$ and $R_{Fk}^{(2)}$ are independent of k (i.e. the index k can be omitted in these matrices and we write $I_{Fk1}^{(2)} = I_{F1}^{(2)}$, $N_{Fk}^{(2)} = N_F^{(2)}$ etc.).

Then the total number of BCs (for the Laplace transformed variables) and of ICs (for the Fourier transformed variables) necessary for solving the PDAE (1.1) is

$$n_{L1}^{(2)} = r^{(1)} + n_2^{(1)} - rank(N_L^{(2)}) \tag{4.9}$$

$$n_{F1}^{(2)} = n_1^{(1)} + rank(N^{(1)}) - rank(N_F^{(2)}), \tag{4.10}$$

respectively.

In practice it is convenient to solve the PDAE (4.1) instead of eq. (1.1). To do this one must know the components of v and w for which BCs and ICs can be prescribed. In the notation of the foregoing subsections we need $\mathfrak{M}_{BC}$ and $\mathfrak{M}_{IC}$. But this can, in general, be decided by looking at the transformation formulas of the components of the vectors $(v_\xi^\top, w_\xi^\top)^\top = Q_\xi(\overline{v}_\xi^\top, \overline{w}_\xi^\top)^\top$ and $\hat{u}_k = Q_{Fk}(y_k^\top, z_k^\top)^\top$. With these relations and with Lemma 4.1 one can determine (possibly not in an unique manner) $\mathfrak{M}_{BC}$ and $\mathfrak{M}_{IC}$.

We emphasize that it is not necessary in a numerical procedure to solve eqs.(4.2) and (4.6) for $(\overline{v}_\xi^\top, \overline{w}_\xi^\top)^\top$ and $(y_k^\top, z_k^\top)^\top$, respectively. Hence one must not at all transform back any of the quantities $(\overline{v}_\xi^\top, \overline{w}_\xi^\top)^\top$ or $(y_k^\top, z_k^\top)^\top$ into the original $(t, x)-$space. All what is needed are the two index sets $\mathfrak{M}_{BC}$ and $\mathfrak{M}_{IC}$ for eq. (4.1).

The foregoing results can be used effectively for the numerical solution of the PDAE (1.1), e.g. by means of the method of lines (MOL) [13]. The reason is the semi explicit structure (with respect to $\frac{\partial}{\partial t}$) of eq. (4.1). The BCs and ICs for the components of v and w whose indices are not contained in the two index sets can be computed straightforwardly in that method.

We note that a semi explicit PDAE of the form (4.1) can be obtained also on other ways. For example, transform the matrix A (assumed to be singular) into the Jordan normal form $\left(\begin{smallmatrix} R & 0 \\ 0 & N \end{smallmatrix}\right)$. Again N means a nilpotent matrix. Suppose now R to be in diagonal form. Then we get a PDAE of the form of eq. (4.1) which has a semi explicit structure. However, the coefficient matrix of the second partial derivative with respect to x is now much more complicated. However, the investigation of this case can be performed in the same manner as done above.

As already mentioned, in general it is not clear without a detailed analysis for which components of a PDAE ICs and BCs must be prescribed. The following examples show how this information can be obtained.

Example 4.1. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -b \\ -1 & -b \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

with $b \in \mathbb{R}$, $b \neq 1$. Obviously the pencil (A, B) is regular. First eq. (4.1) is generated. It is easy to find regular matrices P and Q,

$$P = \frac{1}{1-b} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} -b & -1 \\ 1 & 1 \end{pmatrix},$$

giving $n_1^{(1)} = 1$, $n_2^{(1)} = 1$, $r^{(1)} = 0$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_t \\ w_t \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_{xx} \\ w_{xx} \end{pmatrix} + \tilde{C} \begin{pmatrix} v \\ w \end{pmatrix} = \tilde{f}.$$

The second step is the Laplace transformation of this PDAE and then its transformation to the Kronecker normal form (4.2). Denoting the elements of $\tilde{C}$ again with $\tilde{c}_i$, $i = 1, 2, 3, 4$, matrices P_ξ, Q_ξ are

$$P_\xi = \begin{pmatrix} \frac{\tilde{c}_3}{\xi + \tilde{c}_1} & -1 \\ \frac{1}{\tilde{c}_2} & 0 \end{pmatrix}, \quad Q_\xi = \frac{1}{\xi + \tilde{c}_1} \begin{pmatrix} \tilde{c}_2 & \tilde{c}_2 \\ -(\xi + \tilde{c}_1) & 0 \end{pmatrix}$$

One finds $I_{L1}^{(2)} = 1$, $R_L^{(2)} = \frac{(\xi + \tilde{c}_1)\tilde{c}_4 - \tilde{c}_2\tilde{c}_3}{\xi + \tilde{c}_1}$, $N_L^{(2)} = 0$, $I_{L2}^{(2)} = 1$. The relation $(v_\xi^\top, w_\xi^\top)^\top = Q_\xi(\overline{v}_\xi^\top, \overline{w}_\xi^\top)^\top$ yields

$$v_\xi = \frac{\tilde{c}_2}{\xi + \tilde{c}_1}\left(\overline{v}_\xi + \overline{w}_\xi\right), \qquad w_\xi = -\overline{v}_\xi.$$

Since we can pose one BC for $\overline{v}_\xi$ we see that any BC may be assigned for w_ξ (and of course w). The BC for v_ξ (and v) can then be calculated.

A completely analogous consideration can be done to get the proper ICs. The result is that any IC can be dictated for the variable v.

Example 4.2. Let A and C be defined as in the foregoing example and $B = \begin{pmatrix} -1 & -b \\ -1 & 0 \end{pmatrix}$.

Again the pencil (A, B) is regular. A pair of matrices P, Q is

$$P = \begin{pmatrix} 1 & -(1-b) \\ 0 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Here $n_1^{(1)} = 1$, $n_2^{(1)} = 1$ and $R^{(1)} = b$. For $b \neq 0$ we must prescribe an IC for v and a BC for v and for w. For $b = 0$ a BC can be posed only for w.

Concluding remark: The investigations in this paper are also the background for a numerical solution of the IBVP (1.1) – (1.4) by discretization methods. This is considered in a second paper [13]. Especially the results concerning the indexes, the index sets $\mathfrak{M}_{BC}$ and $\mathfrak{M}_{IC}$ and the construction of consistent BCs and ICs are needed for a numerical solution of problems considered in this paper.

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