Heat Kernels and maximal $\mathbf{L^p - L^q}$ Estimates for Parabolic Evolution Equations

Matthias Hieber
Universität Karlsruhe, Mathematisches Institut I, Englerstr. 2, D-76128 Karlsruhe, Germany

Jan Prüss
Universität Halle, Fachbereich Mathematik, Institut für Analysis, Theodor-Lieser Straße 5, D-06120 Halle, Germany

Abstract
Let A be the generator of an analytic semigroup T on $L^2(\Omega)$, where Ω is a homogeneous space with doubling property. We prove maximal $L^p - L^q$ a-priori estimates for the solution of the parabolic evolution equation $u'(t) = Au(t) + f(t), u(0) = 0$ provided T may be represented by a heat-kernel satisfying certain bounds (and in particular a Gaussian bound).

1991 Mathematics Subject Classification: 35K22, 58D25, 47D05

1 Introduction

Consider an inhomogeneous initial value problem of the form

$$(1.1)\qquad \begin{aligned} u'(t) &= Au(t) + f(t), \qquad t \in [0,\infty) \\ u(0) &= u_0, \end{aligned}$$

where A is the generator of an analytic semigroup on some Banach space X. It is a well known fact that, in general, the derivative u' of a solution u of the above Cauchy problem is less regular than the right hand side f, even though the problem is of parabolic type. This phenomena, sometimes called 'loss of regularity', causes problems in the treatment of quasilinear parabolic problems. It is therefore desirable to prove that this loss of regularity does not occur for a large class of elliptic operators A. In this case one speaks of maximal regularity. For several approaches to quasilinear parabolic problems based on the theory of maximal regularity we refer to the monographs [Am],[L-S-U],[Lu],[Pr] and the papers [Cl-Li],[vW].

In this paper we are concerned with maximal L^p-regularity. For detailed information on regularity properties of u within the framework of continuous functions we refer to the mongraph [Lu]. More precisely, given $p \in (1,\infty)$, we say that there is maximal L^p regularity provided for each $f \in L^p(I;X)$ there exists a unique $u \in W^{1,p}(I;X) \cap L^p(I;D(A))$ satisfying the above equation with $u_0 = 0$.

First results in this direction were obtained in the Hilbert space case. Indeed, de Simon [deS] proved in 1964 that there is maximal L^p regularity provided X is a Hilbert space. Shortly after this result it was Sobolevskii [Sob] who showed that the maximal regularity property is independent of p. In 1975, Da Prato and Grisvard [D-G] showed that there is maximal L^p regularity in real interpolation spaces.

An elegant approach to our problem is based on the theory of operator-valued Fourier multipliers. Indeed, the above cited results may be viewed as consequences of an $\mathcal{L}(H)$-valued version of Mihlin's theorem, a Banach space valued theorem on singular integrals due to Benedek, Calderón and Panzone [B-C-P] or a recent vector-valued multiplier result in Besov spaces due to Amann [Am]. Pisier [Pi] showed that an $\mathcal{L}(X)$-valued version of Mihlin's theorem holds true only if X is isomorphic to a Hilbert space. Hence, an approach to maximal L^p regularity based on Mihlin's theorem is necessarily restricted to the Hilbert space case.

In 1987 Lamberton [La] proved that there is maximal L^p regularity provided A generates a bounded analytic semigroup on $L^2(\Omega)$ which in addition acts as a contraction on $L^p(\Omega)$ for all $p \in [1,\infty]$. Observe that his approach is necessarily restricted to the case of second order differential operators and does in particular not allow to treat elliptic systems of operators.

Dore and Venni [D-V] proved that there is maximal L^p regularity for (1.1) provided A admits bounded imaginary powers of angle less than $\pi/2$. In practice it is not easy to verify that a given differential operator subject to certain boundary conditions has bounded imaginary powers on $L^p(\Omega)$. Recently Duong and Robinson [D-R] showed that A admits a bounded H^∞ calculus (and in particular that A has bounded imaginary powers) on $L^p(\Omega)$ provided this holds true on $L^2(\Omega)$ and the semigroup T may be represented by a kernel satisfying certain growth properties.

In this paper we show that, roughly speaking, we have maximal L^p regularity for (1.1) provided the kernel of semigroup T satisfies bounds similar to those introduced by Duong and Robinson. The proof of our result is based on the Calderón-Zygmund decomposition of an integrable function on the product space of a so-called homogeneous space with doubling property and a time interval. It was much inspired by the above cited result of Duong and Robinson and follows in essential parts the method employed there.

Our approach gives in particular an affirmative answer to the maximal L^p-regularity problem for large classes of

a) Elliptic differential operators of order m on $\mathbb{R}^n$ or on a bounded subset of $\mathbb{R}^n$ with smooth boundary subject to standard boundary conditions
b) Elliptic operators in divergence form subject to general boundary conditions
c) The Laplace-Beltrami operator on a complete Riemannian manifold with non-negative Ricci curvature
d) Schrödinger operators $\Delta + V$, where V belongs to the Kato class and
e) Fractional powers of elliptic operators.

Throughout this paper we denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X to Y whenever X and Y are Banach spaces. If A is a linear operator in X, we denote its domain by $D(A)$, its resolvent set by $\rho(A)$ and its spectrum by $\sigma(A)$. We frequently use the notation $[|argz| \leq \theta] := \{z \in \mathbb{C}; |argz| \leq \theta\}$. Finally, we denote by C various constants which may differ from occurence to occurence but are always independent of the free variables of a given formula.

2 Maximal regularity and operator-valued Fourier multipliers

Let X be a Banach space and let A be a linear, closed, densely defined operator in X. Let $I = [0,\infty)$ or $[0,T]$ for some $T > 0$ and let $f : I \to X$ be a function. We consider the inhomogeneous initial value problem

$$\begin{aligned} u'(t) &= Au(t) + f(t) \qquad (t \in I) \\ u(0) &= u_0. \end{aligned} \tag{2.1}$$

Let $p \in (1,\infty)$. Assuming that A generates an analytic semigroup on X, it is known that there exists a solution $u \in W^{1,p}(I;X) \cap L^p(I;D(A))$ satisfying (2.1) with $f = 0$ in the $L^p(I;X)$ sense if and only if

$$u_0 \in (D(A),X)_{\frac{1}{p},p}.$$

Starting from this situation, the inhomogeneous problem has been investigated by many authors. We refer to the monographs [Pr],[Lu] and [Am] for an overview of the results known. In this paper we consider only the case where $u_0 = 0$.

Given $p \in (1,\infty)$, A is said to belong to the *class* $MR(p,X)$ and we say that there is *maximal* L^p *regularity* for (2.1) if for each $f \in L^p(I;X)$ there exists a unique

$$u \in W^{1,p}(I;X) \cap L^p(I;D(A))$$

satisfying (2.1) with $u_0 = 0$ in the $L^p(I;X)$ sense. Of course, we consider $D(A)$ as a Banach space endowed with the graph norm. The closed graph theorem implies then that there exists a constant $C > 0$ such that

$$||u||_{L^p(I;X)} + ||u'||_{L^p(I;X)} + ||Au||_{L^p(I;X)} \leq C||f||_{L^p(I;X)}.$$

In this case we denote by R the operator on $L^p(I;X)$ defined by

$$Rf = u.$$

It is well known that there is maximal L^p regularity for (2.1) only if A generates an analytic semigroup (see e.g. [Co-La],[Pr]). If $I = [0,\infty)$, then the semigroup is even of negative exponential type. Indeed, choose $\mu \in [Rez > 0]$. Define $f_\mu \in L^p([0,\infty);\mathbb{C})$ by

$$f_\mu(t) = \begin{cases} e^{\mu t} & 0 \leq t \leq (Re\mu)^{-1} \\ 0 & t > (Re\mu)^{-1} \end{cases}$$

and let $x \in X$. Set

$$R_\mu x := Re\mu \int_0^\infty e^{-\mu t} R(f_\mu x)(t)dt = \frac{Re\mu}{\mu} \int_0^\infty e^{-\mu t} \frac{d}{dt} R(f_\mu x)(t)dt.$$

The maximal regularity property for (2.1) implies that there exists a constant $C > 0$ such that

$$||R_\mu||_{\mathcal{L}(L^p(I;X))} \leq \frac{C}{1+|\mu|}$$

for all $\mu \in [Rez > 0]$. Since R_μ is a left and right inverse of $(\mu - A)^{-1}$ it follows that $[Rez > 0] \subset \rho(A)$ and that there exists a constant $C > 0$ such that

$$||(\mu - A)^{-1}|| \leq \frac{C}{1+|\mu|} \qquad (\mu \in [Rez \geq 0]).$$

We note that for generators of analytic semigroups of negative exponential type the property of maximal L^p-regularity is independent of the interval on which we consider (2.1). Hence, without loss of generality, we assume that that $I = [0, \infty)$ and that A is the generator of an analytic semigroup of negative exponential type. Then there is maximal L^p regularity for (2.1) provided the operator $\mathcal{R}_{max}$ defined by

$$\mathcal{R}_{max} f \mapsto Au = A\int_0^t T(t-s)f(s)ds \tag{2.2}$$

acts as a bounded operator on $L^p(I;X)$. We now represent the operator $\mathcal{R}_{max}$ as convolution with the operator-valued kernel k given by

$$k(t) := \begin{cases} AT(t) & t > 0 \\ 0 & t \leq 0 \; . \end{cases}$$

Since $\sup_{t>0} ||tAT(t)|| \leq C$, we are dealing with the L^p boundedness of a singular integral operator whose kernel takes values in $\mathcal{L}(X)$. In the special case where $X = H$ is a Hilbert space the Fourier transform $\mathcal{F}$ acts as a bounded operator on $L^2(\mathbb{R}, H)$. Hence by taking Fourier transforms in (2.2) we obtain

$$\mathcal{F}(\mathcal{R}_{max} f)(s) = A(is - A)^{-1}(\mathcal{F}f)(s) \qquad (f \in L^2(\mathbb{R}, H)).$$

The analyticity of T implies that $\sup_{s\in\mathbb{R}} ||A(is - A)^{-1}|| < \infty$ and it follows that

$$\mathcal{R}_{max} \in \mathcal{L}(L^2(\mathbb{R}, H)). \tag{2.3}$$

In order to prove that $\mathcal{R}_{max}$ is a bounded operator on $L^p(\mathbb{R}, H)$, we use an $\mathcal{L}(H)$-valued version of the Mihlin multiplier theorem.

2.1 Proposition. *Let $1 < p < \infty$. Let $a \in C^1(\mathbb{R}, \mathcal{L}(H))$ such that*

$$||a||_{\mathcal{M}_p(\mathcal{L}(H))} := \sup_{\alpha\in\{0,1\}} \sup_{s\in\mathbb{R}} |s|^\alpha ||D^\alpha a(s)||_{\mathcal{L}(H)} < \infty.$$

Then there exists a constant $C > 0$ such that the operator T_a defined by $T_a f = \mathcal{F}^{-1}(a\mathcal{F}f)$ satisfies

$$||T_a||_{\mathcal{L}(L^p(\mathbb{R},H))} \leq C||a||_{\mathcal{M}_p(\mathcal{L}(H))}.$$

For a proof of this result we refer to [B-L;Thm.6.1.6]. Considering the function $a : \mathbb{R} \to \mathcal{L}(H)$ given by $a(s) = A(is - A)^{-1}$, we obtain in particular the following result which was proved first by de Simon [deS].

2.2. Corollary *[deS]. Let A be the generator of an analytic C_0-semigroup of negative exponential type on a Hilbert space H. Then there is maximal L^p regularity for (2.1) on $[0,\infty)$.*

In case that X is a Banach space, the following result on vector-valued Fourier multipliers is known.

2.3 Proposition *[B-C-P]. Let X be a Banach space and $p \in [1,\infty]$. Let $k : \mathbb{R}^n \to \mathcal{L}(X)$ be measurable and assume that $k \in L^1_{loc}(\mathbb{R}^n \backslash \{0\})$. Let $S \in \mathcal{L}(L^p(\mathbb{R}^n; X))$ such that*

$$(Sf)(x) = \int_{\mathbb{R}^n} k(x-y) f(y) dy \qquad (x \notin supp(f))$$

is well defined for $f \in L^\infty(\mathbb{R}^n; X)$ with compact support. Suppose there exists a constant $C > 0$ such that

$$\int_{|x|>2|y|} ||k(x-y) - k(x)|| dx \leq C$$

for all $y \in \mathbb{R}^n$. Then $S \in \mathcal{L}(L^q(\mathbb{R}^n; X))$ for all $q \in (1,\infty)$.

The above Proposition 2.3 implies in particular that the maximal L^p-regularity property is independent of $p \in (1,\infty)$. This fact was proved in [Sob],[Co-La] and [C-V].

2.4 Proposition *[Sob]. Let $1 < p < \infty$. Let A be the generator of an analytic C_0-semigroup of negative exponential type on a Banach space X. If there is maximal L^p regularity on $[0,\infty)$ for (2.1), then there is maximal L^q regularity on $[0,\infty)$ for $q \in (1,\infty)$.*

For the convience of the reader we give a proof of Proposition 2.4 based on Proposition 2.3. Let $I = [0,\infty)$. By assumption, $\mathcal{R}_{max}$ is a bounded operator on $L^p(I; X)$ and hence by standard arguments also on $L^p(\mathbb{R}; X)$. Note that $k \in L^1_{loc}(\mathbb{R} \backslash \{0\})$. Since $\sup_{t>0} ||t^2 A^2 T(t)|| \leq C < \infty$ we obtain

$$\begin{aligned}
\int_{t>2|s|} ||k(t-s) - k(t)|| dt &= \int_{t>2|s|} || \int_t^{t-s} A^2 T(\tau) d\tau || dt \\
&\leq C \int_{t>2|s|} | \int_t^{t-s} \tau^{-2} d\tau | dt \\
&= C \int_{t>2|s|} |\frac{1}{t} - \frac{1}{t-s}| dt \leq C'
\end{aligned}$$

for a suitable constant C' independent of s. Proposition 2.3 implies that $\mathcal{R}_{max} \in \mathcal{L}(L^q(\mathbb{R}; X))$ for $q \in (1,\infty)$. This implies that $\mathcal{R}_{max} \in \mathcal{L}(L^q(I; X))$. □

3 The main result and examples

It is the aim of this paper to prove that a large class of generators of analytic semigroups T on $L^q(\Omega)$ satisfy the maximal L^p regularity property provided Ω satisfies certain geometric and the kernel of T satisfies certain growth conditions. More precisely, denote by Ω a topological space and let d be a metric on $\Omega \times \Omega$. For $x \in \Omega$ and $\rho > 0$ set

$$B(x,\rho) := \{y \in \Omega; d(x,y) < \rho\}.$$

We assume that Ω has the *doubling property* , i.e. we assume the existence of a Borel mesure μ on Ω and a constant $c_2 > 0$ such that

$$|B(x, 2\rho)| \leq c_2 |B(x, \rho)| \tag{3.1}$$

for all $x \in \Omega$. Here we set $|A| := \mu(A)$ for a μ-measurable subset A of Ω. The doubling property (3.1) implies that there exist constants $C, n > 0$ such that

$$|B(x, \lambda\rho)| \leq C\lambda^n |B(x, \rho)| \tag{3.2}$$

for all $x \in \Omega$ and all $\lambda \geq 1$. In the following we restrict ourselves to spaces Ω which are also subject to the condition

$$ess \sup_{x\in\Omega} |B(x, \rho)| \leq C ess \inf_{x\in\Omega} |B(x, \rho)| \tag{3.3}$$

for some $C > 0$ and all $\rho > 0$. Examples of topological spaces satisfying (3.1) and (3.3) include $\mathbb{R}^n$, bounded subsets of $\mathbb{R}^n$ with Lipschitz boundary, nilpotent Lie groups or compact manifolds.

Assume that the strongly continuous semigroup T on $L^2(\Omega)$ may be represented as an integral operator, i.e. we have

$$(T(t)f)(x) := \int_\Omega K(t, x, y)f(y)dy \qquad (x - a.e., t > 0, f \in L^2(\Omega)), \tag{3.4}$$

where the kernel $K(\cdot, \cdot, \cdot) : (0, \infty) \times \Omega \times \Omega \to \mathbb{C}$ satisfies the assumption that the above integral exists for almost all $x \in \Omega$ and all $t > 0$. We say that K satisfies a *Poisson bound of order* $m > 0$ if

$$|K(t, x, y)| \leq P(t, x, y) \wedge P(t, y, x) \tag{3.5}$$

for all $x, y \in \Omega$ and all $t > 0$, where

$$P(t, x, y) = |B(x, t^{1/m})|^{-1} p(\frac{d(x, y)^m}{t}) \tag{3.6}$$

and p is a bounded, decreasing, continuous and strictly positive function satisfying

$$\lim_{r\to\infty} r^{n+\delta} p(r^m) = 0 \tag{3.7}$$

for some $\delta > 0$. Here n is defined as in (3.2). Observe that the above bounds on the kernel K imply the existence of C_0-semigroups T_p on $L^p(\Omega)$ $(1 \leq p < \infty)$ satisfying

$$T_p(t)f = T(t)f \qquad (f \in L^2 \cap L^p, t \geq 0).$$

Evidentially, we may represent T_p also by the kernel K. Denote by A_p the generator of T_p. We now state the main theorem of this paper.

3.1 Theorem. *Let $1 < p, q < \infty$ and $I = [0, \infty)$. Let (Ω, μ, d) be a topological space satisfying (3.1) and (3.3) and let A be the generator of an analytic C_0-semigroup T on $L^2(\Omega)$ of negative exponential type. Assume that T may be represented by a kernel satisfying a Poisson bound of order $m > 0$.*
Then $A \in MR(p; L^q(\Omega))$, i.e. for each $f \in L^p(I; L^q(\Omega))$ there exists a unique solution

$$u \in W^{1,p}(I; L^q(\Omega)) \cap L^p(I; D(A_p))$$

of (2.1) with $u_0 = 0$ in the $L^p(I; L^q(\Omega))$ sense. Moreover, there exists a constant $C > 0$ such that

$$\int_0^\infty ||u(t)||^q_{L^p} dt + \int_0^\infty ||u'(t)||^q_{L^p} dt + \int_0^\infty ||Au(t)||^q_{L^p} dt \leq C \int_0^\infty ||f(t)||^q_{L^p} dt \tag{3.8}$$

for all $f \in L^p(I; L^q(\Omega))$.

Following [Cl-Li], a-priori-estimates of type (3.8) may be used to give an elegant proof for the existence of a unique local solution of certain quasilinear parabolic problems. Indeed, let $X = L^p(\Omega)$. Combining (3.8) with Theorem 2.1 of [Cl-Li], we see that given $u_0 \in U$, where $U \subset (D(A_p), X)_{\frac{1}{p},p}$ is a nonempty set of initial values for some $p \in (1,\infty)$, there exists $T_1 \in (0, T_0]$ and a unique function

$$u \in W^{1,p}((0,T_1); X) \cap L^p((0,T_1); D(A_p)) \cap C([0,T_1]; (D(A_p), X)_{\frac{1}{p},p})$$

satisfying the quasilinear problem

$$\begin{aligned} u'(t) + A(u(t))u(t) &= f(t, u(t)) + g(t) \qquad t \in (0, T_1) \\ u(0) &= u_0. \end{aligned} \tag{3.9}$$

provided

a) $A \in C^{1-}(U; \mathcal{L}(D(A_p), X))$
b) $f \in C^{1-,1-}([0,T_0] \times U; X)$
c) $g \in L^p((0,T_0); X)$
d) $-A(u_0)$ is the generator of an analytic semigroup on $L^p(\Omega)$ satisfying the assumption of Theorem 3.1.

3.2 Examples

A) Elliptic operators in non-divergence form

1) Elliptic operators on $L^p(\mathbb{R}^n)$ with Hölder continuous coefficients

Let $A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$, $\rho \in (0,1)$, $a_\alpha \in BUC^\rho(\mathbb{R}^n, \mathbb{C})$ for $|\alpha| = m$ and $a_\alpha \in L^\infty(\mathbb{R}^n, \mathbb{C})$ for $|\alpha| \leq m$. Suppose that there exists a constant $c > 0$ such that

$$\sup_{|\xi|=1} Re a_m(x, \xi) < -c < 0$$

for all $x \in \mathbb{R}^n$, where $a_m(\cdot,\cdot)$ denotes the symbol of the prinicipal part of $A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$. Given $p \in (1,\infty)$ we define the L^p-realization $\mathcal{A}_p$ of A by

$$\begin{aligned} D(\mathcal{A}_p) &:= W_p^m(\mathbb{R}^n) \\ \mathcal{A}_p f &:= Af \qquad \text{for all} \quad f \in D(\mathcal{A}_p). \end{aligned}$$

It then follows from [A-H-S;Cor.9.5] and [Fr;Thm.9.4.2] that for $w > 0$ sufficiently large, $\mathcal{A}_p - w$ generates an analytic C_0-semigroup T_p on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, whose kernel satisfies a Poisson bound of order m with p defined by $p(r) = a \exp\{-br^{\frac{1}{m-1}}\}$ for suitable constants $a, b > 0$.

2) Second order elliptic operators on $L^p(\Omega)$ satisfying the Lopatinskii-Shapiro condition.

Let Ω be a bounded domain in $\mathbb{R}^n$ such that $\partial\Omega \in C^{2+\rho}$ for some $\rho \in (0,1)$. Consider a differential operator A of the form

$$A(x, \partial) := -\sum_{1 \leq i,j \leq N} a_{ij}(x) \partial_i \partial_j + \sum_{1 \leq i \leq N} a_i(x) \partial_i + a_0(x)$$

where $a_{ij}, a_i, a_0 \in BUC^\rho(\Omega)$ and

$$\sum_{1 \leq i,j \leq N} a_{ij}(x) \xi_i \xi_j \geq c|\xi|^2$$

for all $x \in \mathbb{R}^N$, $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ and some constant $c > 0$. Let $B(x, \partial) := b(x) \cdot \nabla + b_0(x)$ be boundary operators such that $b = (b_1, \ldots, b_n), b_i, b_0 \in C^{1+\rho}(\Omega)$ and $b(x) \cdot \nu(x) \geq c_0 > 0$, where $\nu(x)$ is the unit outward normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$. Given $p \in (1, \infty)$, the operator

$$D(\mathcal{A}_p) := \{u \in W_p^2(\Omega); Bu = 0\} \qquad \mathcal{A}_p u := Au$$

is called the L^p-realization of the boundary value problem (A, B). Then $-\mathcal{A}_p$ generates an analytic semigroup on $L^p(\Omega), 1 < p < \infty$. Furthermore, it is shown in [Iv] and [Sol] that the semigroup T_p generated by $-\mathcal{A}_p - w$ for sufficiently large $w > 0$ satisfies a Poisson bound of order $m = 2$ for $p(r) = ae^{-br}$.

B) Elliptic operators in divergence form

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let V be either $H^1(\Omega)$ or $H_0^1(\Omega)$. Consider a form $a : V \times V \to \mathbb{C}$ defined by

$$a(u, v) := \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) D_i u D_j v$$

where the coefficients a_{ij} are assumed to belong to $L^\infty(\Omega)$. Suppose that there exists a constant $c > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and $a.a. x \in \Omega$. Let A be the operator associated with a and let T be the analytic semigroup on $L^2(\Omega)$ generated by $A - w$ for $w > 0$. Let T_p be a consistent semigroups on $L^p(\Omega)$ with generator A_p. It was shown by Davies [Da;Cor.3.2.8, Thm.3.2.9] that T_p is respresented by a kernel satisfying a bound of order $m = 2$ and $\delta > 0$. This assertion remains even true for more general type of boundary conditions (see [A-tE]).

C) The Laplace Beltrami operator on manifolds

It was shown by Davies [Da;Thm.5.5.6] that the kernel of the semigroup associated with the Laplace-Beltrami operator on a complete Riemannian manifold with non-negative Ricci curvature satisfies a bound of order 2 with $p(r) = ae^{-br}$.

D) Schrödinger operators

Let $A = \Delta - V$ on $L^2(\mathbb{R}^n)$, where $V : \mathbb{R}^n \to \mathbb{R}$ is measurable and such that $V_+ \in L^1_{loc}(\mathbb{R}^n)$ and V_- belongs to Kato's class (cf. [Si]). Then A, equipped with a suitable domain, is a self-adjoint operator which generates a C_0-semigroup T on $L^2(\mathbb{R}^n)$ satisfying a bound of order $m = 2$ and $\delta > 0$ with $p(r) = ae^{-br}$.

E) Fractional powers

Let A be the generator of a bounded analytic semigroup T on $L^2(\Omega)$ which may be represented by a kernel satisfying a Poisson bound of order $m > 0$. For $\alpha \in (0, 1)$, the fractional powers $-(-A)^\alpha$ of A generate analytic semigroup T_α with kernels K_α. Let $1 < p, q < \infty$. The assumptions of Theorem 3.1 are satisfied for K_α, hence $A_p^\alpha \in MR(p, \mathrm{L}^q(\Omega))$.

4 Kernel bounds

In this section we collect and prove certain properties of the kernel K and the dominating function P. We start with an estimate which involves the Hardy-Littlewood maximal function defined by

$$(Mf)(x) := \sup_{\rho>0} |B(x,\rho)|^{-1} \int_{B(x,r)} |f(y)|dy$$

for $f \in L^p(\Omega)$ and $p \in [1,\infty]$. It is well known that the mapping

$$f \mapsto Mf$$

defines a bounded linear operator on $L^p(\Omega,\mu,d)$, $1 < p \leq \infty$, provided (Ω,μ,d) is a space of homogeneous type with respect to a pseudo-metric d satisfying the doubling property (3.1). The proof given in [St;Ch.1.3] carries over to our situation (see also [D-R]).

All of the three results which will be listed in the following Proposition 4.1 are due to Duong and Robinson [D-R].

4.1 Proposition *[D-R;Prop.2.1;Prop.4.1;Prop.2.4]. Let (Ω,μ,d) be a topological space satisfying (3.1) and (3.3). Assume that the kernel K of T satisfies a Poisson bound of order $m > 0$. Let δ be defined as in (3.7).*

a) For each $\gamma \in [0,\delta)$ there exists a constant $C > 0$ such that

$$\text{(4.1)} \qquad esssup_{x\in\Omega} \int_{d(x,y)>\rho} P(t,x,y)dy \leq C(1+\frac{\rho^m}{t})^{-\gamma}$$

for all $\rho \geq 0$ and all $t > 0$.

b) Assume in addition that T extends to an analytic semigroup of angle φ on $L^2(\Omega)$. Let $\psi \in (0,\varphi)$. Then for each $\varepsilon \in (0,1]$ and $\theta \in (0,\varepsilon\psi)$ there exists a constant $C > 0$ such that the kernel $K(z,\cdot,\cdot)$ of $T(z)$ satisfies

$$\text{(4.2)} \qquad |K(z,x,y)| \leq C|B(x,(Rez)^{1/m})|^{-1} p(\frac{d(x,y)^m}{|z|})^{1-\varepsilon}$$

for all $x,y \in \Omega$ and all $z \in [|\arg z| < \theta]$.

c) Let $p \in [1,\infty]$. Then there exists a constant $C > 0$ such that

$$\text{(4.3)} \qquad |(T(t)f)(x)| \leq \int_\Omega P(t,x,y)|f(y)|dy \leq C(Mf)(x) \qquad (x-a.e.)$$

for all $f \in L^p(\Omega)$ and all $t > 0$.

An estimate of the form (4.1) implies that the semigroup T, initially defined on $L^2(\Omega)$, extends to a uniformly bounded C_0-semigroup on $L^p(\Omega)$ for all $p \in [1,\infty)$. For a proof of this fact we refer to [A-tE;Lemma 2.1] or [D-R;Prop. 2.3].

Consider now the topological space $\mathbb{R}_+ \times \Omega$ equipped with the pseudo-metric d^p defined by

$$\text{(4.4)} \qquad d^p((t,x),(s,y)) := (d(x,y)^m + |t-s|)^{1/m} \qquad (x,y\in\Omega, t,s\in\mathbb{R}_+).$$

Note that d^p is a metric provided $m \geq 1$. If $m \in (0,1)$ then there exists a constant c_m such that

$$\text{(4.5)} \qquad d^p((t,x),(s,y)) \leq c_m(d^p((t,x),(r,z)) + d^p((r,z),(s,y)))$$

for all $x, y, z \in \Omega, r, s, t \geq 0$. For $\rho > 0$ and $(t, x) \in \mathbb{R}_+ \times \Omega$ we set

$$B_\rho^p(t,x) := \{(s,y) \in \mathbb{R}_+ \times \Omega; d^p((t,x),(s,y)) < \rho\}. \tag{4.6}$$

In the following we make use of an Harnack-type estimate. For $x, y \in \Omega, t, s \geq 0, \tau > 0, r \in \mathbb{R}$ we define the real-valued functions e_τ and D by

$$\begin{aligned} e_\tau(r) &:= \frac{\exp\{-\frac{|r|}{\tau}\}}{2\tau} \\ D(\tau,t,x,s,y) &:= P(\tau,x,y)e_\tau(t-s). \end{aligned} \tag{4.7}$$

Then the following holds.

4.2. Lemma. *For each $a > 0$ there exists a constant $C > 0$ such that*

$$\sup_{(s,y)\in B_\rho^p(r,z)} D(\tau,t,x,s,y) \leq C \inf_{(s,y)\in B_\rho^p(r,z)} D(2^m\tau,t,x,s,y)$$

for all $x, z \in \Omega, t, r \geq 0$ and all $\tau, \rho > 0$ such that $\rho^m \leq a\tau$.

Proof. Let $a > 0$. Consider first the case where $d(x,z) > 3\rho$. Let $(s_1,y_1),(s_2,y_2) \in B_\rho^p(r,z)$. Then

$$d(x,y_1) \leq d(x,y_2) + d(y_2,y_1) \leq d(x,y_2) + 2\rho \leq 2d(x,y_2).$$

Hence

$$\begin{aligned} \frac{1}{|B(x,\tau^{1/m})|} \; p(\frac{d(x,y_2)^m}{\tau}) &\leq \frac{1}{|B(x,\tau^{1/m})|}\frac{|B(x,2\tau^{1/m})|}{|B(x,2\tau^{1/m})|} \; p(\frac{d(x,y_1)^m}{2^m\tau}) \\ &\leq c_2\frac{1}{|B(x,2\tau^{1/m})|} \; p(\frac{d(x,y_1)^m}{2^m\tau}) \end{aligned} \tag{4.8}$$

for all $\tau > 0$. Moreover, the assumption $\rho^m \leq a\tau$ implies that

$$\frac{\exp\{\frac{-|t-s_2|}{\tau}\}}{\tau} \leq e^a 2^m \frac{\exp\{\frac{-|t-s_1|}{2^m\tau}\}}{2^m\tau} \tag{4.9}$$

for all $t \geq 0, \tau > 0$. Combining (4.8) with (4.9) we obtain the assertion.

Consider next the case where $d(x,z) \leq 3\rho$. Let $(s,y) \in B_\rho^p(r,z)$. Then $d(x,y) \leq 4\rho$ and $p(\frac{(4\rho)^m}{\tau}) \leq p(\frac{d(x,y)^m}{\tau})$ for $\tau > 0$. Since by assumption p is a bounded, decreasing, strictly positive function there exists a constant $C > 0$ such that

$$\begin{aligned} \frac{p(\frac{d(x,y)^m}{\tau})}{|B(x,\tau^{1/m})|} &\leq C\frac{1}{|B(x,\tau^{1/m})|}\frac{p(\frac{d(x,y)^m}{\tau})}{p(\frac{(4\rho)^m}{\tau})} \\ &\leq Cc_2\frac{1}{|B(x,2\tau^{1/m})|}\frac{p(\frac{d(x,y)^m}{2^m\tau})}{p(4^m a)} \end{aligned} \tag{4.10}$$

for all $x \in \Omega$ and all $\tau > 0$. By combinig (4.9) with (4.10), the proof is complete. $\square$

Our next estimate involves a certain version of the strong maximal function M_S defined by

$$(M_S f)(t,x) := \sup_{r>0,a>0} |Q|^{-1} \int_Q |f(s,y)| dsdy$$

for $f \in L^p(\mathbb{R}_+ \times \Omega)$ and $p \in [1,\infty]$, where Q is defined by $Q := [t-a, t+a] \times B(x,r)$. It is well known (see e.g. [St;Ch.II.5.20], [G-R;p.461]) that $f \mapsto M_S f$ defines a bounded linear operator on $L^p(\mathbb{R}^{n+1})$, $1 < p \leq \infty$, if in the above definition Q is replaced by $\prod_{i=1}^{n+1}[a_i, b_i]$, where $a_i, b_i \in \mathbb{R}$. In step II of the proof of Theorem 3.1 we make use of the fact that the classical proof carries over to our situation; i.e. that

$$f \mapsto M_S f \tag{4.11}$$

defines a bounded linear operator on $L^p(\mathbb{R}_+ \times \Omega)$, $1 < p \leq \infty$.

4.3 Lemma. *Let $p \in [1,\infty]$. Then there exists a constant $C > 0$ such that*

$$\int_0^\infty \int_\Omega D(\tau,t,x,s,y)|f(s,y)| dyds \leq C(M_S f)(t,x) \qquad (x - a.e., t \geq 0)$$

for all $f \in L^p(\mathbb{R}_+ \times \Omega)$ and all $\tau > 0$.

Proof. Let $\tau > 0$. Let $(g_l) = (\sum_{i=1}^{N(l)} k_i^l \chi_{[0,a_i^l]})$ be a sequence of step functions converging uniformly to the function e_τ defined in (4.7) such that $0 \leq g_l \leq e_\tau$ for all $l \in \mathbb{N}$. Then

$$\begin{aligned}\int_0^\infty \int_\Omega g_l(t-s) P(\tau,x,y)|f(s,y)| dyds &= \sum_i k_i^l \int_{t-a_i^l}^{t+a_i^l} \int_\Omega P(\tau,x,y)|f(s,y)| dyds \\ &= \sum_i k_i^l \int_\Omega P(\tau,x,y) \int_{t-a_i^l}^{t+a_i^l} |f(s,y)| dsdy.\end{aligned}$$

Proposition 4.1.c) implies the existence of a constant $C > 0$ such that the latter term above can be estimated by

$$\begin{aligned}&C \sum_i k_i^l (M \int_{t-a_i^l}^{t+a_i^l} |f(s,y)| ds)(x) \\ &\leq C \sum_i a_i^l k_i^l \sup_{r>0,a>0} |B(x,r) \times [t-a_i^l, t+a_i^l]|^{-1} \int_{t-a_i^l}^{t+a_i^l} \int_{B(x,r)} |f(s,y)| dyds \\ &\leq C(M_S f)(t,x)\end{aligned}$$

uniformly for $\tau > 0$ and $l \in \mathbb{N}$. The proof is complete. □

5 Proof of the main theorem

Let (Ω, μ, d) be a topological space satisfying the properties (3.1) and (3.3). Let d^p and B_ρ^p we defined as in (4.4) and (4.6), respectively. Consider now the Calderón-Zygmund decomposition of an integrable function. More precisely, let $f \in L^1(\mathbb{R}_+ \times \Omega)$ and fix $\alpha > 0$. Then there exist a function g, a sequence of functions (b_i), a sequence of balls $B_i^p := B_{\rho_i}^p(t_i, x_i)$ and a constant $C > 0$ such that (a1) $f = g + \sum_i b_i$

(a2) $|g(x)| \le C\alpha$ for a.a. x

(a3) $supp b_i \subset B_i^p$ and $x \in \Omega$ is contained in at most finetely many balls B_i^p

(a4) $||b_i||_1 \le C\alpha |B_i^p|$ for all i

(a5) $\sum_i |B_i^p| \le \frac{C}{\alpha} ||f||_1$

For a proof of this fact we refer to [St;Ch.I.4] or [C-W;Ch.III.2]. Observe that the above conditions (a4) and (a5) imply that

$$(5.1) \qquad || \sum_i b_i ||_1 \le C ||f||_1 \qquad \text{and} \qquad ||g||_1 \le (1+C)||f||_1.$$

We now decompose the function $b := \sum_i b_i$ as the sum of two functions, i.e. we set

$$b := h_1 + h_2,$$

where $h_1 := \sum_i g_i$ and $h_2 := \sum_i (b_i - g_i)$ and g_i is defined for $\tau_i > 0$ by

$$g_i(t,x) := \int_\Omega \int_0^\infty K(\tau_i, x, y) e_{\tau_i}(t-s) b_i(s,y) ds dy, \qquad (t,x) \in \mathbb{R}_+ \times \Omega.$$

In the following we prove that the operator $\mathcal{R}_{max}$ defined in (2.2) by

$$\mathcal{R}_{max} f = A \int_0^t T(t-s) f(s) ds$$

acts as a bounded operator from $L^1(\mathbb{R}_+; L^1(\Omega)) = L^1(\mathbb{R}_+ \times \Omega)$ into $L^1_w(\mathbb{R}_+ \times \Omega)$. To this end, we set

$$v := \mathcal{R}_{max} g, \qquad w_1 := \mathcal{R}_{max} h_1, \qquad w_2 := \mathcal{R}_{max} h_2.$$

For $\alpha > 0$ we obtain

$$\begin{aligned} \mu\{(t,x) \in \mathbb{R}_+ \times \Omega; |\mathcal{R}_{max} f(t,x)| > \alpha\} &\le \mu\{(t,x) \in \mathbb{R}_+ \times \Omega; |v(t,x)| > \alpha/3\} \\ &\le \mu\{(t,x) \in \mathbb{R}_+ \times \Omega; |w_1(t,x)| > \alpha/3\} \\ &\le \mu\{(t,x) \in \mathbb{R}_+ \times \Omega; |w_2(t,x)| > \alpha/3\}. \end{aligned}$$

We now show that each of the three terms above is bounded by a term of the form $\frac{C}{\alpha}||f||_1$. We subdivide the proof into three steps.

Step I : The function v

The Calderón-Zygmund decomposition of f implies that $g \in L^2(\mathbb{R}_+; L^2(\Omega))$. Hence it follows from (2.3)(a2) and (5.1) that

$$\mu\{(t,x) \in \mathbb{R}_+ \times \Omega; |v(t,x)| > \alpha\} \le \frac{1}{\alpha^2} ||v||_2^2 \le \frac{C}{\alpha^2} ||g||_2^2 \le \frac{C}{\alpha} ||g||_1 \le \frac{C}{\alpha} ||f||_1.$$

Step II : The function w_1

We choose $\tau_i = \rho_i^m$ for all i. The Harnack-type estimate in Lemma 4.2 and (a4) imply that there exists a constant $C > 0$ such that

$$
\begin{aligned}
|g_i(t,x)| &\le \int_\Omega \int_0^\infty D(\tau_i, t, x, s, y)|b_i(s,y)|dsdy \\
&\le ||b_i||_{L^1(B_i^p)} \sup_{(s,y)\in B_i^p} D(\tau_i, t, x, s, y) \\
&\le C\alpha |B_i^p| \inf_{(s,y)\in B_i^p} D(2^m\tau_i, t, x, s, y) \\
&\le C\alpha \int_0^\infty \int_\Omega D(2^m\tau_i, t, x, s, y)\chi_i(s,y)dyds \qquad (x-a.e., t\ge 0)
\end{aligned}
\tag{5.2}
$$

uniformly for all i. Here χ_i denotes the characteristic function of the ball $B_i^p = B_{\rho_i}^p(t_i, x_i)$.

Let $h \in L^2(\mathbb{R}_+ \times \Omega)$. Taking into account (5.2) and Lemma 4.3 we have

$$|(h, g_i)| \le C\alpha(M_S h, \chi_i).$$

Hence

$$||h_1||_2 = \sup_{||h||_2\le 1}\{|(h,h_1)|\} \le \sup_{||h||_2\le 1}\{\sum_i |(h,g_i)|\} \le C\alpha \sup_{||h||_2\le 1}\{\sum_i (M_S h, \chi_i)|\} \le C\alpha \sup_{||h||_2\le 1}\{(M_S h, \sum_i \chi_i)\}.$$

Since $h \mapsto M_S h$ is a bounded operator on $L^2(\mathbb{R}_+ \times \Omega)$ (see 4.11), it follows that $||h_1||_2 \le C\alpha||\sum_i \chi_i||_2$. Using (a3) and (a5) it follows that

$$||\sum_i \chi_i||_2 \le (\sum_i |B_i^p|)^{1/2} \le \frac{C}{\alpha^{1/2}}||f||_1^{1/2}.$$

Therefore

$$||h_1||_2^2 \le C\alpha||f||_1. \tag{5.3}$$

Finally, by (2.3) and (5.3) we have

$$\mu\{(t,x)\in \mathbb{R}_+\times\Omega; |w_1(t,x)| > \alpha\} \le \frac{1}{\alpha^2}||w_1||_2^2 \le \frac{C}{\alpha^2}||h_1||_2^2 \le \frac{C}{\alpha}||f||_1.$$

Step III : The function w_2

Recall that τ_i is defined by $\tau_i = \rho_i^m$ for all i. For $m > 0$ we set

$$B_{2,i}^p := \begin{cases} \{(s,y)\in\mathbb{R}_+\times\Omega; d((s,y),(t_i,x_i)) < 2\rho_i\} & , m \ge 1 \\ \{(s,y)\in\mathbb{R}_+\times\Omega; d((s,y),(t_i,x_i)) < 2c_m\rho_i\} & , m < 1 \end{cases},$$

where c_m is defined as in (4.5). Then we have

$$\mu\{(t,x)\in\mathbb{R}_+\times\Omega; |w_2(t,x)| > \alpha\} \le \sum_i |B_{2,i}^p| + \mu\{(t,x)\in(\mathbb{R}_+\times\Omega)\setminus\cup_i B_{2,i}^p; |w_2(t,x)| > \alpha\}. \tag{5.4}$$

The doubling property and (a5) imply that

$$\sum_i |B^p_{2,i}| \leq \frac{C}{\alpha}||f||_1.$$

It therefore remains to estimate the second term on the right hand side of (5.4). Set $(w_2)_i := b_i - g_i$. By definition of $\mathcal{R}_{max}$ we have for $t > 0$

$$(w_2)_i(t) = \int_0^t AT(t-s)[b_i(s) - \int_0^\infty e_{\tau_i}(s-r)T(\tau_i)b_i(r)dr]ds.$$

We claim that there exists a constant $C > 0$ such that

$$||(w_2)_i(1-\chi_{B^p_{2,i}})||_{L^1(\mathbb{R}_+\times\Omega)} \leq C||b_i||_{L^1(B^p_i)}$$

holds for all i.

Observe that $(w_2)_i$ can be represented as a convolution of b_i with an operator-valued function $G : \mathbb{R} \to \mathcal{L}(L^2)$. More precisely, we have

$$(w_2)_i(t) = \int_0^t G(t-s)b_i(s)ds,$$

where

$$G(t-s) := \begin{cases} AT(t-s) - \int_{-\infty}^t AT(t-r)e_{\tau_i}(r-s)T(\tau_i)dr, & t \geq s \\ -\int_{-\infty}^t AT(t-r)e_{\tau_i}(r-s)T(\tau_i)dr, & t < s. \end{cases}$$

By assumption, A generates an analytic semigroup of angle φ on $L^2(\Omega)$ of negative exponential type. Hence $\sigma(A) \subset [\pi/2 + \varphi \leq |argz| \leq \pi]$, where $\sigma(A)$ denotes the spectrum of A. Let $\varepsilon \in (0, \frac{\delta}{\delta+n})$ and choose $\theta \in (\pi/2, \pi/2 + \varepsilon\varphi)$. Define $\Gamma_1 = \Gamma_- + \Gamma_+$ to be the path given by $\Gamma_\pm = \{re^{\pm i\theta}; r \geq 0\}$. For $\mu \in \Gamma_+$ we have

$$(\mu - A)^{-1} = \int_{\Gamma_2} e^{-\mu z}T(z)dz,$$

where $\Gamma_2 = \{re^{-i\varepsilon\varphi}; r \geq 0\}$. We now represent the function G as $G = G_+ + G_-$, where for $t \geq s$ we have

$$G_+(t-s) = \frac{1}{2\pi i}\int_{\Gamma_+}\int_{\Gamma_2} \mu e^{-\mu z}e^{\mu(t-s)}[1 - \int_{-\infty}^t e^{\mu(s-r)}\frac{e^{-|r-s|/\tau_i}}{2\tau_i}e^{\mu\tau_i}]T(z)dzd\mu.$$

The two components G_- and G_+ are estimated in a similar manner. Consider G_+. Then

$$\begin{aligned} G_+(t-s) &= \frac{1}{2\pi i}\int_{\Gamma_+}\int_{\Gamma_2} \mu e^{-\mu z}e^{\mu(t-s)}e^{\mu\tau_i}[e^{-\mu\tau_i} - \int_0^\infty e^{\mu r}\frac{e^{-r/\tau_i}}{2\tau_i}dr - \int_s^t e^{\mu(s-r)}\frac{e^{-(r-s)/\tau_i}}{2\tau_i}dr]T(z)dzd\mu \\ &= \frac{1}{2\pi i}\int_{\Gamma_+}\int_{\Gamma_2} \mu e^{-\mu z}e^{\mu\tau_i}\{e^{\mu(t-s)}[e^{-\mu\tau_i} - \frac{1}{2(1-\mu\tau_i)} - \frac{1}{2(1+\mu\tau_i)}] + \frac{e^{-(t-s)/\tau_i}}{2(1+\mu\tau_i)}\}T(z)dzd\mu. \end{aligned}$$

For $t < s$ we represent G_+ as

$$\begin{aligned} G_+(t-s) &= -\frac{1}{2\pi i}\int_{\Gamma_+}\int_{\Gamma_2} \mu e^{-\mu z}\int_{-\infty}^t e^{\mu(t-r)}\frac{e^{-(s-r)/\tau_i}}{2\tau_i}e^{\mu\tau_i}T(z)dzd\mu \\ &= -\frac{1}{2\pi i}\int_{\Gamma_+}\int_{\Gamma_2} \mu e^{-\mu z}\frac{e^{-(s-t)/\tau_i}}{2(1-\mu\tau_i)}e^{\mu\tau_i}T(z)dzd\mu. \end{aligned}$$

Observe next that

$$||(w_2)_i(1-\chi_{B^p_{2,i}})||_{L^1} = \int\int_{B_i^p}\int\int_{B^p_{2,i}} |K_G(t-s,x,y)dxdt||b_i(s,y)|dydsdxdt,$$

where

$$K_G(t-s,x,y) = \frac{1}{2\pi i}\int_{\Gamma_+}\int_{\Gamma_2} \mu e^{-\mu z}e^{\mu\tau_i}u_{\tau_i}(t-s,\mu)K(z,x,y)dzd\mu$$

and u_{τ_i} is defined for $\mu \in \Gamma_+$ by

$$u_{\tau_i}(t-s,\mu) = \begin{cases} e^{\mu(t-s)}[e^{-\mu\tau_i} - \frac{1}{2(1-\mu\tau_i)} - \frac{1}{2(1+\mu\tau_i)}] + \frac{e^{-(t-s)/\tau_i}}{2(1+\mu\tau_i)} & , t \geq s \\ \frac{e^{-(s-t)/\tau_i}}{2(1-\mu\tau_i)} & , t < s. \end{cases}$$

By the kernel estimates for complex times z given in Proposition 4.1.b) there exist constants $C, \delta_1 > 0$ such that

$$\begin{aligned} &||(w_2)_i(1-\chi_{B^p_{2,i}})||_{L^1} \\ &\leq ||b_i||_{L^1}\sup_{y\in\Omega}\int_0^\infty \int_{d(x,y)^m+\tau\geq\tau_i} |K_G(\tau,x,y)|dxd\tau \\ &\leq C||b_i||_{L^1}\sup_{y\in\Omega}\int_0^\infty \int_{d(x,y)^m+\tau\geq\tau_i} |B(x,(Rez)^{1/m})|^{-1}p(\frac{d(x,y)^m}{|z|})^{1-\varepsilon}dx\int_{\Gamma_+}\int_{\Gamma_2}|\mu|e^{-\delta_1|\mu||z|}|u_{\tau_i}(\tau,\mu)|d|z|d|\mu|d\tau. \end{aligned}$$

We now estimate the sup on the right hand side of the above term by

$$\begin{aligned} &\sup_{y\in\Omega}\int_{\tau_i}^\infty \int_\Omega |B(x,(Rez)^{1/m})|^{-1}p(\frac{d(x,y)^m}{|z|})^{1-\varepsilon}dx\int_{\Gamma_+}\int_{\Gamma_2}|\mu|e^{-\delta_1|\mu||z|}|u_{\tau_i}(\tau,\mu)|d|z|d|\mu|d\tau \quad + \\ &\sup_{y\in\Omega}\int_0^{\tau_i} \int_{d(x,y)^m\geq\tau_i-\tau} |B(x,(Rez)^{1/m})|^{-1}|p(\frac{d(x,y)^m}{|z|})^{1-\varepsilon}dx\int_{\Gamma_+}\int_{\Gamma_2}|\mu|e^{-\delta_1|\mu||z|}|u_{\tau_i}(\tau,\mu)|d|z|d|\mu|d\tau \\ &=: I + II. \end{aligned}$$

Recall that $\varepsilon < \frac{\delta}{n+\delta}$. The estimate in Proposition 4.1.a) combined with (3.2),(3.3) and the estimate

$$|u_{\tau_i}(\tau,\mu)| \leq C[e^{-\delta_2|\mu|\tau}\frac{|\mu\tau_i|}{1+|\mu\tau_i|} + \frac{e^{-\delta_2|\mu|\tau_i}e^{-\tau/\tau_i}}{1+|\mu\tau_i|}]$$

which holds for some $\delta_2 > 0$ uniformly in $\tau > 0$ and $\mu \in \Gamma_2$ implies that

$$\begin{aligned} I &\leq C\int_{\tau_i}^\infty \int_{\Gamma_+}\int_{\Gamma_2}|\mu|e^{-\delta_1|\mu||z|}|u_{\tau_i}(\tau,\mu)|d|z|d|\mu|d\tau \\ &\leq C\int_{\Gamma_+}\int_{\Gamma_2}|\mu|e^{-\delta_1|\mu||z|}\tau_i e^{-\delta_2|\mu|\tau_i}d|z|d|\mu| \leq C < \infty. \end{aligned}$$

Consider finally the term II. Since $\varepsilon < \frac{\delta}{\delta+n}$ we find by (3.2), (3.3) and Proposition 4.1.a) a constant $\gamma > 0$ such that

$$\begin{aligned} II &\leq \int_0^{\tau_i}\int_{\Gamma_+}\int_{\Gamma_2}|\mu|e^{-\delta_1|\mu||z|}(e^{-\delta_2|\mu|\tau} + e^{-\delta_2|\mu|\tau_i})(1+\frac{\tau_i-\tau}{|z|})^{-\gamma}d|\mu|d|z|d\tau \\ &\leq C\int_{\Gamma_2}\int_0^{\tau_i}(\frac{1}{(\delta_1|z|+\delta_2\tau)^2} + \frac{1}{(\delta_1|z|+\delta_2\tau_i)^2})(1+\frac{\tau_i-\tau}{|z|})^{-\gamma}d\tau d|z| \\ &\leq C\int_0^\infty\int_0^1 \frac{1}{(\delta_1 r+\delta_2 s)^2}(1+\frac{1-s}{r})^{-\gamma}dsdr \leq C < \infty. \end{aligned}$$

Summing up, it follows that

$$||(w_2)_i(1-\chi_{B^p_{2,i}})||_{L^1(\mathbb{R}_+\times\Omega)} \le C||b_i||_{L^1(B^p_i)} \tag{5.5}$$

for all i. Since similar estimates are valid for G_- we deduce from (5.5) and (5.1) that

$$\begin{aligned}\mu\{(t,x)\in(\mathbb{R}_+\times\Omega)\setminus\cup_i B^p_{2,i}; |w_2(t,x)|>\alpha\} &\le \alpha^{-1}\int\int_{(\mathbb{R}_+\times\Omega)\setminus\cup_i B^p_{2,i}} |\sum_i (w_2)_i(t,x)|dxdt\\ &\le \alpha^{-1}\sum_i\int\int_{(\mathbb{R}_+\times\Omega)\setminus\cup_i B^p_{2,i}} |(w_2)_i(t,x)|dxdt\\ &\le C\alpha^{-1}\sum_i ||b_i||_1 \le C\alpha^{-1}||f||_1.\end{aligned}$$

Alltogether, we proved that the operator $\mathcal{R}_{max}$ maps $L^1(\mathbb{R}_+\times\Omega)$ into $L^1_w(\mathbb{R}_+\times\Omega)$. The Marcinkiewicz interpolation theorem and a duality argument show that $\mathcal{R}_{max}$ acts boundely on $L^p([0,\infty),L^p(\Omega))$ for all $p\in(1,\infty)$. The mixed L^p-L^q estimates follow finally from Proposition 2.4. The proof is complete. $\square$

Acknowledgements

The first named author would like to thank Michael Korey for several discussions on the maximal function.

References

[Am] H. Amann, *Linear and Quasilinear Parabolic Problems.* Birkhäuser Verlag, Basel, (1995).

[A-H-S] H. Amann, M. Hieber, G. Simonett, *Bounded H_∞-calculus for elliptic operators.* Differential Integral Equations 7, (1994), 613-653.

[A-tE] W. Arendt, A.F.M. ter Elst, *Gaussian estimates for second order elliptic operators with boundary conditions.* Preprint (1995).

[B-C-P] A. Benedek, A.P. Calderón, R. Panzone, *Convolution operators on Banach space valued functions.* Proc. Nat. Acad. Sci. USA 48, (1962), 356-365.

[B-L] J. Bergh, J. Löfström, *Interpolation Spaces. An Intoduction.* Springer Verlag, Berlin, (1976).

[C-V] P. Cannarsa, V. Vespri, *On maximal L^p regularity for the abstract Cauchy problem.* Boll. Un. Mat. It. B 5 (1986), 165-175.

[Cl-Li] Ph. Clément, S. Li, *Abstract parabolic quasilinear equations and applications to a groundwater flow problem.* Adv. Math. Sci. Appl. 3, (1993/94), 17-32.

[C-W] R.R. Coifman, G. Weiss, *Analyse Harmonique Non-Commutative sur Certain Espaces Homogenes.* Springer LN Math. 242, Berlin, (1971).

[Co-La] T. Coulhon, D. Lamberton, *Régularité L^p pour les équations d'évolution.* In: B. Beauzaumy, B. Maurey, G. Pisier (eds.) Seminaire d'Analyse Fonctionelle 1984/85; Publ. Math. Univ. Paris VII 26, (1986), 155-165.

[D-G] G. Da Prato, P. Grisvard, *Sommes d'opérateurs linéaires et équations différentielles opérationnelles.* J. Math. Pure Appl. 54 (1975), 305-387.

[Da] E.B. Davies, *Heat kernels and spectral theory.* Cambridge University Press, Vol. 92, Cambridge, (1989).

[deS] L. de Simon, *Un applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine.* Rend. Sem. Mat. Univ. Padova 34 (1964), 547-558.

[D-V] G. Dore, A. Venni, *On the closedness of the sum of two operators.* Math. Z. 196 (1987), 189-201.

[D-R] X.T. Duong, D.W. Robinson, *Semigroup kernels, Poisson bounds and holomorphic functional calculus.* Math. Research Report, The Australian National University, Canberra, Australia, (1995).

[Fr] A. Friedman, *Partial Differential Equations of Parabolic Type.* Prentice Hall, New Jersey, (1964).

[G-R] J. Garcia-Cuerva, J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics.* North-Holland Math. Studies 116, (1985).

[Iv] S.D. Ivasisien, *Green's matrices of boundary value problems for Petrovskii parabolic systems of general form. I.* Math. USSR Sbornik 42, (1982), 93-144.

[La] D. Lamberton, *Equations d'évolution linéaires associées á des semi-groupes de contractions dans les espaces L^p.* J. Func. Anal. 72 (1987), 252-262.

[L-S-U] O.A. Layzenskaya, V.A. Solonnikov, N.N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type.* Amer. Math. Soc., Providence, Rhode Island, (1968).

[Lu] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Equations.* Birkhäuser, Basel, (1995).

[Pi] G. Pisier, *unpublished manuscript.*

[Pr] J. Prüss, *Evolutionary Integral Equations and Applications.* Birkhäuser, Basel, (1993).

[Si] B. Simon, *Schrödinger semigroups*, Bull. Am. Math. Soc. 7, (1982), 447-526.

[Sob] P.E. Sobolevskii, *Coerciveness inequalities for abstract parabolic equations.* Soviet Math. Dokl. 5 (1964), 894-897.

[Sol] V.A. Solonnikov, *On boundary value problems for linear parabolic systems of differential equations of general form* . Trudy Mat. Inst. Steklov 83 (1965); English transl., Proc. Steklov Inst. Math. 83 (1965).

[St] E.M. Stein, *Harmonic Analysis: Real-Variables Methods, Orthogonality and Oscillatory Integrals.* Princeton University Press, Princeton, (1993).

[vW] W. von Wahl, *The equation $u'(t) + Au(t) = f$ in a Hilbert space and L^p estimates for parabolic equations.* J. London Math. Soc. 25 (1982), 483-497.