AN EXTREME POINT THEOREM FOR ORDERED POLYMATROIDS ON CHAIN ORDERS

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ABSTRACT. We consider *Ordered Polymatroids* as a generalization of polymatroids and extend the extreme point characterization of polymatroids by the greedy algorithm to the ordered case.

It is proved that a feasible point of an Ordered Polymatroid is a vertex iff it is a Greedy-Vector with respect to an appropriate primal Greedy-Procedure.

1. INTRODUCTION AND NOTATIONS

In [2] Faigle and Kern considered Submodular Linear Programs of the type

$$
\begin{aligned}
(1) \qquad & max \quad \sum_{e \in E} c_e x_e \\
& \sum_{e \in A^+} x_e \quad \leq \quad f(A) \text{ for all } A \in \mathcal{A}
\end{aligned}
$$

where $P = (E, \leq)$ is a finite partially ordered set, $c: \quad E \longrightarrow \mathbb{R}$ a objective function and f a submodular function with respect to a distributive lattice of ideals in P called $\mathcal{A}$. An ideal of P is a subset $I \subseteq E$ which satisfies the property

$$e \in A \text{ and } f < e \quad \Longrightarrow \quad f \in A.$$

Submodularity of f refers to the property

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \text{ for all } A, B \in \mathcal{A}$$

as usually. The set of maximal elements of an ideal, say I, is denoted by I^+. This notation is used for

$$x(I^+) := \sum_{e \in I^+} x_e.$$

Later we easily write $x^+(I)$ instead of $x(I^+)$.

We keep restricted to the special case $\mathcal{A} = 2^{(E,\leq)}$ troughout this paper, i.e. $\mathcal{A}$ consists of all ideals with respect to $(E, \leq)$. Then the polyhedron

$$\mathbb{P}(f) = \{x : \quad x(A) \leq f(A) \text{ for all } A \in 2^{(E,\leq)}\}$$

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is called "Extended Ordered Polymatroid".

Notice that $\mathbb{P}(f)$ describes the set of all feasible vectors of (1).

The attribute "extended" will be left if all feasible x are required to be non-negative, i.e.

$$\mathbb{P}_+(f) := \mathbb{P}(f) \cap \mathbb{R}^+$$

defines an "Ordered Polymatroid". The name "Ordered Polymatroid" is reasonable because ordinary polymatroids can easily be recognized as a subclass if P is assumed to be the trivial order. The reader is referred to Chapter 10 of Grötschel, Lovász, Schrijver [4, pp. 305-329] for a introduction into Polymatroid-Theory and Submodular Functions.

Faigle and Kern developed a primal-dual greedy algorithm for the optimal solution of (1) (see [2]). Furthermore they proved total dual-integrality for pairs of Ordered Polymatroids on rooted forests, see [1]. The characterization of the extreme points of an Extended Ordered Polymatroid $\mathbb{P}(f)$ by primal Greedy-Vectors already follows from Faigle, Kern's greedy algorithm. Our proof method also works for the polytope $\mathbb{P}_+(f)$. Notice that the face lattice of the Ordered Polymatroid $\mathbb{P}_+(f)$ especially includes the faces

$$\mathbb{P}_+(f) \cap \{x_e = 0\}$$

for $e \in E$.

We mainly put our attention to chain orders as a special case of rooted forests here. The order $P = (E, \leq)$ is a chain order if it consists of k totally ordered disjoint subsets. The proof of the extreme point characterization can easily be generalized to rooted forests afterwards. The order $P = (E, \leq)$ is a rooted forest if each element has at most one upper neighbour (see [2, p. 6]).

2. Feasibility of Greedy-Vectors

Let us introduce Greedy-Vectors as the main object of our interest. Therefore let $\pi = (p_1, p_2, \ldots, p_n)$ be an arbitrary linear extension of P and $k \in \{1, \ldots, n\}$ arbitrary. Then the result of the procedure

PRIMAL GREEDY :

$$\begin{aligned}
x_{p_1} &:= f(p_1) \\
x_{p_2} &:= f(p_1 p_2) - x^+(p_1 \setminus I(p_2)) \\
&\vdots \\
x_{p_k} &:= f(p_1 p_2 \ldots p_k) - x^+(p_1 p_2 \ldots p_{k-1} \setminus I(p_k)) \\
x_{p_{k+1}} &= \ldots = x_{p_n} = 0
\end{aligned}$$

is called Greedy-Vector with respect to $\mathbb{P}_+(f)$.

We recall a basic result of Faigle, Kern [2, page 6] concerning feasibility of Greedy-Vectors and adapt the proof to the special case of chain orders.

Lemma 1. *The procedure PRIMAL GREEDY yields a feasible vector x with respect to $\mathbb{P}(f)$ for $k = n$.*

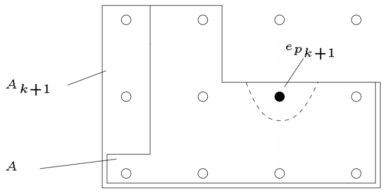

FIGURE 1. Illustration for the Proof.

Proof. We argue by induction on $\mid E \mid$ and assume that

$$\sum_{e \in A^+} x_e \le f(A) \text{ for all } A \in 2^{(E,\le)} \text{ and } A \subseteq A_k,$$

where

$$A_k := \{e_1, \dots, e_k\}.$$

Now, let A be an ideal with $A \subseteq A_{k+1}$ and $p_{k+1} \in A^+$. Furthermore, the set $A \setminus \{p_{k+1}\} = A \cup A_k$ is denoted by B.

We get

$$\begin{aligned} f(A) &\ge f(A \cap A_k) + f(A \cup A_k) - f(A_k) \\ &\ge f(B) + f(A_{k+1}) - f(A_k) \end{aligned}$$

by submodularity of f and

$$f(A_{k+1}) - f(A_k) = x_{p_k} - x_{p_{\bar{k}}},$$

where the index $p_{\bar{k}}$ corresponds to the lower neighbour $e_{p_{\bar{k}}}$ of e_{p_k} with respect to "$\le$" , by the procedure PRIMAL GREEDY.

Finally, we get

$$f(A) \ge x^+(B) + x_{p_k} - x_{p_{\bar{k}}} = x^+(A)$$

by $f(B) \ge x^+(B)$ which holds by assumption. ♢

Remark.

The Greedy-Vector is feasible in $\mathbb{P}_+(f)$ for arbitrary $k \in \{1, \dots, n\}$ if f satisfies monotonicity on antichains, i.e.

$$A^+ \subseteq B^+ \implies f(A) \le f(B) \text{ for } A, B \in 2^{(E,\le)}.$$

Then

$$\begin{aligned} x_{p_i} &= f(p_1 p_2 \dots p_k) - x^+(p_1 p_2 \dots p_{k-1} \setminus I(p_k)) \\ &\ge f(p_1 p_2 \dots p_k) - f(p_1 p_2 \dots p_{k-1} \setminus I(p_k)) \\ &\ge 0 \end{aligned}$$

holds for the i-th component of the Greedy-Vector ($i = 1, \dots, k$). ♢

3. Main Result

In order to state and proof the main result further notation is necessary to be introduced. From now on the symbols A_{iJ} and A_{Ij} are used to denote the j-th column and the i-th line of the matrix A. Furthermore the vector $v_{tr(i)}$ denotes the truncation of the vector v, where all components v_j of v with index $j > i$ are cut off. Now we are ready for

Theorem 1. *A feasible vector of an Ordered Polymatroid is a vertex if and only if it is a Greedy-Vector for a certain linear arrangement π w.r.t. P and a suitable $k \in \{1, \dots, n\}$.*

Proof. Only the case $k = n$ is outlined here. For $k < n$ see the remark in the end of the proof.

It is a well-known fact from Linear-Programming-Theory that there exist a linear function $c^T x$ for each vertex x^* of a polyhedron $\mathbb{P}$ such that the maximal value of $c^T x$ subject to $x \in \mathbb{P}$ is achieved by x^*.

Let $\pi = (p_1, p_2, \dots, p_n)$ a feasible permutation with respect to $(E, \leq)$ which additionally satisfies

$$c_{p_i} \geq c_{p_j} \quad \text{if} \quad p_i \parallel p_j \quad (j > i).$$

Then the recursion

$$\begin{aligned} y_{\{p_1,p_2,\dots,p_i,\dots,p_n\}} &:= c_{p_n} \\ y_{\{p_1,p_2,\dots,p_i\}} &:= c_{p_i} - \sum_{i^* > j > i} y_{p_j} \quad \text{for} \quad i = n-1, \dots, 2, 1 \end{aligned} \tag{2}$$

where

$$i^* = \begin{cases} n+1 & \text{if} \quad p_i \quad \text{is maximal in} \quad E \\ \text{index of the upper neighbour of} \quad p_i & \text{otherwise} \end{cases}$$

and $y_A = 0$ for all A with $A \neq \{p_1, \dots, p_i\}$ $(i \in \{1, \dots, n\})$.

yields a feasible vector y of the dual problem of (1) which is given by

$$\begin{aligned} \sum_{i=1}^{n} f(A) y_A &\longrightarrow min \\ \sum_{A^+ \ni j} y_A &\geq c_j \\ y_A &\geq 0\ . \end{aligned} \tag{3}$$

For reader's convenience we write $\hat{y_{p_i}}$ instead of $y_{\{p_1,p_2,\dots,p_i\}}$ and $\hat{f}(p_i)$ instead of $f(p_1, \dots, p_i)$. We show

$$\begin{aligned} &\sum_{i=1}^{n} \hat{f}(p_i) \left[c_{p_i} - \sum_{i^* \geq j \geq i} \hat{y_{p_j}} \right] = \\ &\qquad \sum_{i=1}^{n} c(p_i) \left[\hat{f}(p_i) - x^+(p_1 \dots p_{i-1} \setminus I(p_i)) \right], \end{aligned} \tag{4}$$

which is equal to

$$\sum_{i=1}^{n} \hat{f}(p_i) \sum_{i^* \geq j \geq i} \hat{y_{p_j}} = \sum_{i=1}^{n} c(p_i) \cdot x^+(p_1 \dots p_{i-1} \setminus I(p_i)). \tag{5}$$

Assume (4) holds, i.e. x an $\hat{y}$ have the same values with respect to the objective functions of (1) and it's dual. Thus x and $\hat{y}$ are optimal solutions of (1) and (3), respectively. Especially, we have $x = x^*$. Since x^* is arbitrary we know that each vertex of $\mathbb{P}(f)$ is a Greedy-vector.

In order to proof (4) the vectors $\jmath(p_i) \in \mathbb{R}^{n-i}$ and $\wp(p_i) \in \mathbb{R}^i$ are introduced for each index $p_i \in \{p_1, p_2, \dots, p_n\}$. Both vectors $\jmath(p_i)$ and $\wp(p_i)$ only consist of 0 and 1-components. The elements of $\jmath(p_i)$ correspond to the elements $p_{i+1}, p_{i+2}, \dots, p_n$ of the groundset E and the elements of $\wp(p_i)$ to the elements $p_1, p_2, \dots, p_i$. The vector $\jmath(p_i)$ is defined as

$$\jmath(p_i) = (1, 1, \dots, 1, 0, \dots, 0),$$

such that the first 0 corresponds to the index i^*. The vector $\wp(p_i)$ is defined by

$$\wp(p_i)_j = 1 \quad \Longleftrightarrow \quad p_j \text{ is maximal in the ideal } (p_1, p_2, \dots, p_i)\ .$$

Similarly let $(p_i, \dots, p_j)^+$ be the $1-0$-vector such that the 1-entries correspond to the maximal elements of the set $\{p_i, \dots, p_j\}$ which is not necessarily an ideal.

Using this notation (4) equivalently can be written as

(6)

$$(\hat{f}(p_1), \hat{f}(p_2), \dots, \hat{f}(p_n)) M_1 \begin{pmatrix} c_{p_1} \\ c_{p_2} \\ \vdots \\ c_{p_n} \end{pmatrix} = (c_{p_1}, c_{p_2}, \dots, c_{p_n}) \hat{M}_n \begin{pmatrix} \hat{f}(p_1) \\ \hat{f}(p_2) \\ \vdots \\ \hat{f}(p_n) \end{pmatrix}$$

where the matrices M_1 and $\hat{M}_n$ are defined by the following recursions :

$$\begin{array}{ll} \hat{M}_1 = 1; & M_n = 1; \\ \vdots & \vdots \\ \hat{M}_{i+1} = \begin{pmatrix} & 0 \\ \hat{M}_i & \vdots \\ & 0 \\ -\wp(p_i) \cdot \hat{M}_i & 1 \end{pmatrix} & M_j = \begin{pmatrix} 1 & -\jmath(p_i) \cdot M_{i+1} \\ 0 & \\ \vdots & M_{j+1} \\ 0 & \end{pmatrix} \\ i = 1, \dots, n-1\ . & j = n-1, \dots, 1 \end{array}$$

Obviously, equation (6) holds if $M_1 = \hat{M}_n^{\ T}$, which is shown by induction now. We proceed on the lower subdiagonals of M_1 and $\hat{M}_n^{\ T}$.

First notice that both matrices are lower triangle matrices which consist of 1-entries in their diagonals. Considering the elements $m_{ij} = (M_1)_{i-j}$ and

$\hat{m}_{ij} = (\hat{M}_n{}^T)_{ij}$ we assume that the first $i-j-1$ subdiagonals of M_1 and $\hat{M}_n{}^T$ are equal to each other. Then we get

$$
\begin{aligned}
\hat{m}_{ij} &= -\wp(p_{i-1}) \cdot \hat{M}_{i-1,J} \\
&= -(p_j, \dots, p_{i-1})^+ \cdot \begin{pmatrix} \hat{m}_{jj} \\ \hat{m}_{j+1,j} \\ \vdots \\ \hat{m}_{i-1,j} \end{pmatrix} = -(p_j, \dots, p_{i-1})^+ \cdot \begin{pmatrix} m_{jj} \\ m_{j+1,j} \\ \vdots \\ m_{i-1,j} \end{pmatrix} \\
&\quad \text{(use induction hypothesis)} \\
&= -(p_j, p_{j+1}, \dots, p_{i-1})^+ \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
&\quad -(p_{j+1}, p_{j+2}, \dots, p_{i-1})^+ \\
&\qquad \begin{pmatrix} 1 & & & \\ \hat{m}_{j+2,j+1} & 1 & \dots & 0 \\ & \vdots & & \\ \hat{m}_{i-1,j+1} & \hat{m}_{i-1,j+2} & & 1 \end{pmatrix} \cdot \jmath(p_j)|_{tr(i-1)} \\
&= -(p_j, p_{j+1}, p_{j+2}, \dots, p_{i-1})^+ \cdot \\
&\qquad \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ m_{j+2,j+1} & 1 & \dots & 0 & 0 \\ & \vdots & & & \\ m_{i-1,j+1} & m_{i-1,j+2} & \dots & 1 & 0 \end{pmatrix} \cdot \jmath(p_j)|_{tr(i)} \\
&\quad -(0, 0, \dots, 0, 1) \cdot \jmath(p_j)|_{tr(i)} \\
&= -(m_{i,j+1}, m_{i,j+2}, \dots, m_{ii}) \cdot \jmath(p_j)|_{tr(i)} \\
&= -(M_{j+1}^T)_{Ij} \cdot \jmath(p_j) \\
&= m_{ij} \ .
\end{aligned}
$$

by the recursive definitions of M_1 and $\hat{M}_n{}^T$.

For $k < n$ just start the recursion (2) with $y_{p_k} = c_{p_k}$ and set $c_{p_j} = 0$ for $j > k$. $\diamondsuit$

Theorem 1 and its proof were motivated by the Greedy-Algorithm-Characterization of ordinary polymatroids due to Edmonds (for example see Satz 1.3.1 Girlich/Kowaljow [3, Seite 39] or Schrijver [5, pp. 27-28, Theorem 2]).

The paper is finished with an important consequence of Theorem 1.

Corollary 1. *All vertices of $\mathbb{P}_+(f)$ are integral if the rank funktion f is integer-valued.*

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