Rothe's method for parabolic problems with nonlinear degenerating coefficient

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Abstract

In the paper we investigate local existence of a weak solution u to a degenerate parabolic boundary value problem. Degeneration occurs in the coefficient $g(x,t,u)$ in front of the time derivative which is not assumed to be bounded below by some positive constant. Moreover, this coefficient depends on the solution u.

We approximate the problem by semidiscretization in time (Rothe's method) and prove uniform convergence of the approximations to a Hölder continuous weak solution. The assumptions with respect to u for the nonlinear coefficient and the right-hand side are supposed only in a neighbourhood of the initial function, therefore we have to derive a L_∞-estimate for the approximates. This estimate that implies also Hölder continuity is obtained by a Moser technique.

1 Introduction

In this paper we prove existence of a weak solution to the degenerated parabolic initial boundary value problem

$$g(x,t,u)\,u_t + A(t)u = f(x,t,u) \qquad \text{in } Q_T, \tag{1.1}$$

$$u(x,t) = 0 \qquad \text{on } \Gamma, \tag{1.2}$$

$$u(x,0) = U_0(x) \qquad x \in G, \tag{1.3}$$

where

$$A(t)u = -\sum_{i,k=1}^{N} \frac{\partial}{\partial x_k}\left(a_{ik}(x,t)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{N} a_i(x,t)\frac{\partial u}{\partial x_i}\,,$$

by means of semidiscretization in time (Rothe's method). Here $G \subset \mathbb{R}^N$, $N \geq 2$, denotes a simply connected, bounded domain with boundary $\partial G \in C^1$, $I = [0,T]$, $Q_T = G \times I$, $\Gamma_T = \partial G \times I$. The nonlinear coefficient g may degenerate.

Problems with nonlinear degeneration of the form $b(u)_t$ were investigated by means of Rothe's method by J. Kačur [4], [3], where degeneration occurs on a certain level set of the solution u because of condition $g(u) = b'(u) \geq 0$. We regard, however, degeneration on a fixed subset of the domain Q_T characterized by $g(x,t,u) \geq 0$ and boundedness of $1/g(\cdot,t,u)$ in some $L_\sigma(G)$. Solution of equation (1.1) with regular nonlinear coefficient g by means of Rothe's method is given by Kačur [5]. However, the discretization of g used in [5] is more complicated compared with our simple linearization (2.1_j). In [5] existence of a solution and convergence in $C(I, L_2(G)) \cap L_2(I, W_2^1(G))$ is proved. Our result is stronger even in the non-degenerated case since we prove convergence in Hölder space $C^\alpha(\bar{Q}_T) \cap L_\infty(I, \overset{o}{W}{}_r^1(G))$,

$r > N$. The crucial point to get such convergence result is an estimate of the discrete time derivative uniformly in $t \in I$, which is proved by use of a nonlinear Gronwall lemma.

Moreover, we have only local assumptions for g and f w.r.t. u in a neighbourhood of the initial function U_0 and omit any growth condition. For this purpose we need L_∞-estimates of u which we derive by the help of a technique of Moser [7] and Alikakos [1]. There L_p-estimates for arbitrary $p \geq 2$ are proved by means of power-type test functions followed by a limit process $p \to \infty$. In order to avoid unlimited increase of the bounds depending on p we have to use a recursive method where L_p-estimates for arbitrary p are recured to L_{p_0}-estimate with fixed p_0.

The results of this paper may be carried over to the case of a quasilinear differential operator A (cf. [8] for non-degenerated quasilinear equation). The basics of our investigation of Rothe's method for degenerated equation were developed in [10] where the case $g = g(x,t)$ was treated.

2 Preliminaries

In the following $\|\cdot\|_p$ denotes the norm in $L_p(G)$ and $\langle\cdot,\cdot\rangle$ the duality between $L_p(G)$ and $L_{p'}(G), 1/p + 1/p' = 1$. $L_{p,\chi}(G)$ is the weighted Lebesgue space with weight $\chi = \chi(x)$ and finite norm $\|u\|_{p,\chi} = (\int_G \chi(x)|u(x)|^p\,dx)^{1/p}$. By $\|\cdot\|_0$ and $\|\cdot\|_{0,\alpha}$ we denote the norm in $C(\bar{G})$ and Hölder space $C^\alpha(\bar{G})$, respectively. $W_p^1(G), \overset{o}{W}{}_p^1(G)$ are the usual Sobolev spaces where $\overset{o}{W}{}_p^1(G)$ is normed by $\|u\|_{1,p} := \|\nabla u\|_p$. For $t \in I$ the operator $A(t)$ generates a bilinear form on $\overset{o}{W}{}_p^1(G) \times \overset{o}{W}{}_{p'}^1(G)$ denoted by $A_{(t)}(\cdot,\cdot)$. Moreover, we use standard functional spaces $C(I,V), C^{0,1}(I,V)$, $L_p(I,V)$, and $W_p^1(I,V)$ (see e.g. [14]). By c we denote a generic nonnegative constant being independent of the subdivision and of p if p is variable.

In order to solve the problem by semidiscretization in time (Rothe method) we subdivide the time interval I by points $t_j = jh, j = 0,\cdots,n$, and replace (1.1)–(1.3) by the time discretized problem (in weak formulation)

$$\langle g_j\,\delta u_j, v\rangle + A_j(u_j,v) = \langle f_j, v\rangle \qquad \forall v \in \overset{o}{W}{}_{p'}^1(G)\,, \tag{2.1$_j$}$$

$$u_j = 0 \qquad \text{on } \partial G\,, \tag{2.2$_j$}$$

$$u_0 = U_0\,, \tag{2.3$_0$}$$

$j = 1,\cdots,n$, where $\delta u_j := (u_j - u_{j-1})/h$, $g_j := g(x,t_j,u_{j-1})$, $f_j := f(x,t_j,u_{j-1})$, and $A_j(\cdot,\cdot) := A_{(t_j)}(\cdot,\cdot)$. This is a set of linear elliptic boundary value problems to determine the approximates u_j.

For given $U_0 \in C(\bar{G})$ and $R > 0$ we now define the set

$$\mathcal{M}_R(U_0) = \{(x,t,u) \in \mathbb{R}^{N+2} \;:\; x \in G,\, t \in I,\, |u - U_0(x)| \leq R\}$$

and the ball

$$\mathcal{B}_R(U_0) = \{u \in C(\bar{G}) \;:\; \|u - U_0\|_0 \leq R\}\,.$$

We will show below that the approximates u_j and the solution of our problem belong to $\mathcal{B}_R(U_0)$ for sufficiently small t. Therefore, we suppose the following local conditions.

Assumptions. *For given $R>0$ let g and f be Carathéodory functions defined on $\mathcal{M}_R(U_0)$. Let further r_1, r_2, r_3, μ_1, μ_2, ν_1, ν_2, σ, κ be real numbers fulfilling the relations $r_1 > N$, $r_2 > \frac{N(\kappa-1)}{2\kappa-N}$, $r_3 > \frac{N}{2}$, $\frac{N}{2} < \mu_1 \le \nu_1 \le \frac{\sigma}{\sigma+1}\kappa$, $\frac{N\kappa}{(2\kappa+N-2)} < \mu_2 \le \nu_2 < \frac{N\kappa}{N-2}$, $1<\sigma$, $2 \le \kappa < \infty$. Then we suppose*

(i) $U_0 \in \overset{o}{W}{}^1_{r_1}(G)\ ,\ A(0)U_0 \in L_1(G)$;

(ii) Let $g(\cdot,t,u): I \times \mathcal{B}_R(U_0) \to L_{r_2}(G)$ *be bounded in* $L_{r_2}(G)$ *and fulfil Lipschitz condition*

$$\|g(\cdot,t,u)-g(\cdot,t',u')\|_{\mu_1} \le l_1(|t-t'|+\|u-u'\|_{\nu_1}) \qquad \forall t,t' \in I,\ u,u' \in \mathcal{B}_R(U_0).$$

Furthermore, let $g(x,t,u) \ge 0\ \ \forall (x,t,u) \in \mathcal{M}_R(U_0)$ *and* $1/g(\cdot,t,u) : I \times \mathcal{B}_R(U_0) \to L_\sigma(G)$ *be bounded in* $L_\sigma(G)$.

(iii) Let $a_{ik} \in C^{0,1}(I, C(\bar{G})), a_i \in C^{0,1}(I, L_\infty(G))$ *and fulfil ellipticity condition*

$$\textstyle\sum_{i,k} a_{ik}(x,t)\xi_i\xi_k \ge a\xi^2\ ,\quad \forall (x,t) \in \bar{Q}_T\ ,\ \forall \xi \in \mathbb{R}^N\ ,\ a>0.$$

(iv) Let $f(\cdot,t,u): I \times \mathcal{B}_R(U_0) \to L_{r_3}(G)$ *be bounded in* $L_{r_3}(G)$ *and fulfil Lipschitz condition*

$$\|f(\cdot,t,u)-f(\cdot,t',u')\|_{\mu_2} \le l_2(|t-t'|+\|u-u'\|_{\nu_2}) \qquad \forall t,t' \in I,\ u,u' \in \mathcal{B}_R(U_0).$$

(v) Let hold compatibility condition

$$(f(\cdot,0,U_0)-A(0)U_0)/g(\cdot,0,U_0) \in L_{\kappa,g(\cdot,0,U_0)}(G)\ .$$

Since some results are formulated in terms of ν_1 without loss of generality we may set $\nu_1 = \sigma\kappa/(\sigma+1)$.

Remark. *Unlike in paper [10] here the coefficient g may have singularities. This overcoming of boundedness condition from [10] is due to Weber [12] where degenerated problems with nonlinear Neumann boundary condition are investigated. We also include the non-degenerated case $\sigma = r_2 = \infty$.*

In the following we identify f and g with its globally defined extensions

$$f^R = \begin{cases} f(x,t,u) & \text{for } (x,t,u) \in \mathcal{M}_R(U_0), \\ f(x,t,U_0(x)+R\,\mathrm{sign}(u-U_0(x)) & \text{else;} \end{cases}$$

g^R defined in the same way. Obviously, f^R and g^R then fulfil assumptions (iv) and (ii), respectively, for every $u \in C(\bar{G})$. In Theorem 3.1 we prove that the argument $u = u_{j-1}$ belongs to $\mathcal{B}_R(U_0)$ for sufficiently small $t \in \hat{I}$, and therefore $g^R = g$, $f^R = f$. For simplicity we drop the superscript R from the beginning.

We derive some auxiliary assertions. Let $g = g(\cdot,t,v)$, $t \in I$, $v \in \mathcal{B}_R(U_0)$ fulfil assumption (ii). Then by means of Hölder's inequality with $p = 1+\sigma$, $p' = (1+\sigma)/\sigma$ we get

$$\begin{aligned} \|u\|_q &\le \Big(\int_G g^{-\sigma/(1+\sigma)}\, g^{\sigma/(1+\sigma)}\, |u|^q\, dx\Big)^{1/q} \\ &\le \|1/g\|_\sigma^{\sigma/(1+\sigma)}\, \|u\|_{(1+\sigma)q/\sigma,g} \le c\, \|u\|_{\tilde{q},g} \end{aligned} \tag{2.4}$$

for all $u \in L_{\tilde{q},g}(G)$, $\tilde{q} \ge (1+\sigma)q/\sigma$. Particularly, it holds

$$\|u\|_{\nu_1} \le c\, \|u\|_{\kappa,g}. \tag{2.5}$$

An essential tool in our investigations is the Nirenberg–Gagliardo interpolation inequality (see [6, pp. 80–84]). Let $1 \le q \le p \le \tilde{s}$ and $\frac{1}{p} < \frac{1}{\tilde{s}} + \frac{1}{N}$. Then for all $u \in \overset{o}{W}{}^1_p(G)$ we have

$$\|u\|_{\tilde{s}} \le c\, \|u\|_{1,p}^\theta\, \|u\|_q^{1-\theta} \qquad \text{with} \quad \bar{\theta} := \frac{1/q - 1/\tilde{s}}{1/q - 1/p + 1/N} \le \theta \le 1\ , \tag{2.6}$$

if $q = 1$ then $\bar\theta < \theta \le 1$. Furthermore, for $u \in \overset{o}{W}{}^1_r(G)$ with $r > N$, $\alpha < 1 - N/r$, and some $0 < \theta < 1$ depending on r, $q \ge 1$, and α it holds

$$\|u\|_{0,\alpha} \le c\,\|u\|_{1,r}^{\theta}\,\|u\|_q^{1-\theta} \tag{2.7}$$

As a corollary of the first interpolation inequality we get

Lemma 2.1. *Let $2 \le \tilde s < \frac{2N}{N-2}$ and $g = g^R(\cdot,t,v)$, $t \in I$, $v \in C(\bar G)$ be a weight function fulfilling assumption (ii). Then for $\varepsilon > 0$, $\alpha > 0$, $1 \le q \le 2$, and $(1+\sigma)q/\sigma \le \tilde q$ holds*

$$\|u\|_{\tilde s}^{2\alpha} \le \varepsilon\,\|u\|_{1,2}^2 + c_\varepsilon \|u\|_q^{2\beta} \qquad \forall u \in \overset{o}{W}{}^1_2(G) \tag{2.8}$$

$$\le \varepsilon\,\|u\|_{1,2}^2 + c_\varepsilon \|u\|_{\tilde q,g}^{2\beta} \qquad \forall u \in \overset{o}{W}{}^1_2(G) \cap L_{\tilde q,g}(G)\,, \tag{2.9}$$

where

a) if $0 < \alpha < 1$ then $0 < \beta \le \bar\beta < \alpha$ and $c_\varepsilon \sim \varepsilon^{-\frac{\alpha-\beta}{1-\alpha}}$

b) if $\alpha = 1$ then $\beta = 1$ and $c_\varepsilon \sim \varepsilon^{-\gamma}$ with $\bar\gamma \le \gamma < \infty$

c) if $1 < \alpha < \bar\alpha$ then $\alpha < \bar\beta \le \beta < \infty$ and $c_\varepsilon \sim \varepsilon^{-\frac{\beta-\alpha}{\alpha-1}}$

with

$$\bar\alpha = \frac{1+\bar\gamma}{\bar\gamma} \quad , \quad \bar\beta = \frac{\alpha}{1+(1-\alpha)\bar\gamma} \quad , \quad \bar\gamma = \frac{2N(\tilde s - q)}{q[2N-(N-2)\tilde s]}\,.$$

If $q = 1$ then the choice $\beta = \bar\beta$, $\gamma = \bar\gamma$ is excluded.

Proof: The proof of formula (2.8) is given in [8]. Then (2.9) follows immediately with (2.4). □

We choose now $2 \le p < \infty$ and define the frequently used notation $w := |u|^{(p-2)/2}u$. To insert w into (2.9) set $s = \frac{p}{2}\tilde s$, $\tilde p = \frac{p}{2}\tilde q$ and choose an appropriate $1 \le q \le 2$. Then (2.9) yields

Corollary 2.1. *Let $1 \le s < \frac{Np}{N-2}$, $\frac{1+\sigma}{2\sigma}p \le \tilde p \le p$ and $\alpha, \beta, \varepsilon, c_\varepsilon$ such as given in Lemma 2.1. Then $\forall u \in \overset{o}{W}{}^1_2(G) \cap L_\infty(G)$ holds*

$$\|u\|_s^{\alpha p} \le \varepsilon\,\|w\|_{1,2}^2 + c_\varepsilon \|u\|_{\tilde p,g}^{\beta p}\,. \tag{2.10}$$

In the same way from (2.6) follows

$$\|u\|_s^p \le c\,\|w\|_{1,2}^{2\theta}\,\|u\|_{\tilde p,g}^{(1-\theta)p} \qquad \forall u \in \overset{o}{W}{}^1_2(G) \cap L_\infty(G). \tag{2.11}$$

Another consequence of (2.10) is

$$\left|\|u\|_{p,\tilde g}^p - \|u\|_{p,\tilde{\tilde g}}^p\right| \le \|\tilde g - \tilde{\tilde g}\|_{\mu_1}\left(\varepsilon\,\|w\|_{1,2}^2 + c_\varepsilon\,\|u\|_{\tilde p,g}\right)\;,\quad \frac{1+\sigma}{2\sigma}p \le \tilde p \le p\,, \tag{2.12}$$

where $s = \mu_1' p < \frac{Np}{N-2}$, $\alpha = \beta = 1$ are used and $g, \tilde g, \tilde{\tilde g}$ are weight functions fulfilling assumption (ii).

In order to obtain a priori estimates in $L_{p,g}$-spaces we use powers of u as test functions. The following lemma describes the properties of the bilinear form $A_{(t)}(\cdot,\cdot)$ if such powers are inserted.

Lemma 2.2. *Let $u \in \overset{o}{W}{}_2^1(G) \cap L_\infty(G)$, $v \in \overset{o}{W}{}_r^1(G)$, $r > N$, and $w := |u|^{(p-2)/2}u$ for $2 \le p < \infty$. Moreover, let $g = g^R(\cdot, t, \hat{u})$, $t \in I$, $\hat{u} \in C(\bar{G})$ be a given weight function fulfilling assumption (ii) and $A(t)$ fulfil assumption (iii).*

Then $|u|^{p-2}u \in \overset{o}{W}{}_2^1(G) \cap L_\infty(G)$ with $\nabla(|u|^{p-2}u) = (p-1)|u|^{p-2}\nabla u$, and it holds

$$\| \, |u|^{p-2}u \, \|_{p'} = \|u\|_p^{p-1} \;\; ; \;\; \|w\|_2^2 = \|u\|_p^p \, , \qquad 1/p + 1/p' = 1, \tag{2.13}$$

$$A_{(t)}(u, |u|^{p-2}u) \ge k_1 \, \|w\|_{1,2}^2 - k_2 \, \|u\|_p^p \tag{2.14}$$

$$\ge \tilde{k}_1 \, \|w\|_{1,2}^2 - \tilde{k}_2 \, \|u\|_{\tilde{p},g}^p \, , \tag{2.15}$$

$$|A_{(t)}(v, |u|^{p-2}u)| \le c \, \|v\|_{1,r} \, \|w\|_{1,2} \, \|u\|_s^{(p-2)/2} \tag{2.16}$$

$$\le \varepsilon \, \|w\|_{1,2}^2 + \|v\|_{1,r}^p + c\varepsilon^{-c} \, \|u\|_{\tilde{p},g}^p \, , \tag{2.17}$$

$$|(A_{(t')} - A_{(t'')})(v, |u|^{p-2}u)| \le |t' - t''|(\varepsilon \, \|w\|_{1,2}^2 + \|v\|_{1,r}^p + c\varepsilon^{-c} \, \|u\|_{\tilde{p},g}^p) \, , \tag{2.18}$$

where $\frac{1+\sigma}{2\sigma}p \le \tilde{p} \le p$, $s < \frac{Np}{N-2}$, and $k_1, \tilde{k}_1, k_2, \tilde{k}_2 = \mathcal{O}(\frac{1}{p})$.

Proof: The first assetion follows from the chain rule for weakly differentiable functions (cf. [2, Theorem 7.8]). (2.13) is obvious, (2.14) is proved in [9, Lemma 3], and (2.16) follows from [8, Lemma 2.1]. (2.15) is derived from (2.14) by application of (2.10) with $\alpha = \beta = 1$, $s = p$ to the last item as well as (2.17) follows from (2.16) by means of Young's inequality and (2.10). At last, (2.18) is an obvious modification of (2.17). □

For the proof of Lemma 3.3 we add a modification of formula (2.17). Applying Young's inequality in (2.16) only to the last two items one has to estimate the result with Corollary 2.1 and obtains for arbitrary $p > 2$

$$\begin{aligned} |A_{(t)}(v, |u|^{p-2}u)| &\le c \, \|v\|_{1,r} \, (\varepsilon \, \|w\|_{1,2}^2 + c/\varepsilon \, \|u\|_s^{p-2}) \\ &\le c \, \|v\|_{1,r} \, (\varepsilon \, \|w\|_{1,2}^2 + c_\varepsilon \, \|u\|_{\tilde{p},g}^{\beta(p)p}) \, , \end{aligned} \tag{2.19}$$

where $\frac{1+\sigma}{2\sigma}p \le \tilde{p} \le p$ and $c_\varepsilon \sim \varepsilon^{-(2\gamma+1)}$, $\gamma = (\alpha - \beta)/(1-\alpha)$ due to Lemma 2.1 a). Here $\alpha = (p-2)/p$ and

$$\beta = \beta(p) = \frac{p-2}{p + 2\bar{\gamma} + 1} < \bar{\beta} \qquad \text{since } \bar{\beta} = \frac{\alpha}{1 + (1-\alpha)\bar{\gamma}} = \frac{p-2}{p + 2\bar{\gamma}} \tag{2.20}$$

were chosen. Note that $\gamma = \gamma(p) = (\alpha - \beta)/(1-\alpha) = (p-2)(2\bar{\gamma}+1)/2(p+2\bar{\gamma}+1) \le \gamma_M$ remains bounded as $p \to \infty$.

We close our auxiliary assertions with a nonlinear Gronwall lemma (see Willett, Wong [13]) that we need for estimation of the discrete time derivative (Lemma 3.1). We use the following discrete version.

Lemma 2.3. *Let d_i, $i = 0, 1, 2, \ldots, n$, be non-negative real numbers and $c_1, c_2, h, \beta \ge 0$, $k_0, h_0 > 0$ constants with $\beta \ne 1$, $h \le h_0 < 1/c_1$. Then the inequality*

$$d_i \le k_0 + c_1 \sum_{j=0}^{i} h d_j + c_2 \sum_{j=0}^{i-1} h d_j^\beta \, , \qquad i = 0, 1, \ldots, n \, ,$$

implies that

$$e_i d_i \le \left[K_0^{1-\beta} + (1-\beta) C_2 \sum_{j=1}^{i} h e_j^{1-\beta} \right]^{1/(1-\beta)}$$

for those $i = 0, 1, \ldots, n^*$, $n^* \le n$, such that the expession in brackets remains positive. Here $K_0 = k_0/(1 - c_1 h_0)$, $C_k = c_k/(1 - c_1 h_0)$, $(k = 1, 2)$, and $e_i = (1 + C_1 h)^{-i}$.

PROOF: Since $c_1 h \le c_1 h_0 < 1$ the Gronwall inequality implies

$$d_i \le K_0 + C_1 \sum_{j=0}^{i-1} h d_j + C_2 \sum_{j=0}^{i-1} h d_j^{\beta} \ , \qquad i = 0, 1, \dots , n \ .$$

Then the assertion of the lemma is a specialization of Theorem 4 in [13] to $u(i+1) = d_i$, $i = 0, \dots , n$, $v(i) = c_1 h$, $w(i) = c_2 h$, $i = 1, \dots , n$. □

Since $ih = t_i$ and $e_i = (1 + C_1 h)^{-i} \ge e^{-C_1 ih} = e^{-C_1 t_i}$ we have the following corollary.

Corollary 2.2. *Suppose assumtions of Lemma 2.3 with $\beta > 1$. Then*

$$d_i \le \left[K_0^{-(\beta-1)} - (\beta - 1) C_2 t_i \, e^{C_1(\beta-1)t_i} \right]^{-1/(\beta-1)} e^{C_1 t_i} \qquad \forall t_i < t^* \ ,$$

where $t^* > 0$ is defined by the equation $K_0^{-(\beta-1)} = (\beta - 1) C_2 t^* e^{C_1(\beta-1)t^*}$.

Before we come to a priori estimates we have to solve the discretized problem (2.1_j), (2.2_j), (2.3_0). Moreover, it is necessary to ensure $u_j \in C(\bar{G})$ that we are able to prove $u_j \in \mathcal{B}_R(U_0)$. This property follows from embedding theorems since we prove existence of a weak solution $u_j \in \overset{o}{W}{}^1_\rho(G)$ with $\rho > N$ in the next lemma. Remember that f_j, g_j were replaced by f_j^R, g_j^R.

Lemma 2.4. *Let assumptions (i)-(iv) be fulfilled. Then there are $h_0 > 0$, $\rho > N$ such that for $0 < h \le h_0$ the problem (2.1_j), (2.2_j), (2.3_0) has a unique solution $u_j \in \overset{o}{W}{}^1_\rho(G)$, $j = 1, \cdots , n$.*

PROOF: We choose $p = \tilde{p} = 2$ in (2.15) and obtain

$$\begin{aligned} B^0(u,u) &:= \frac{1}{h}\langle g_j u, u\rangle + A_j(u,u) \ge \frac{1}{h}\|u\|^2_{2,g_j} + \tilde{k}_1\|u\|^2_{1,2} - \tilde{k}_2\|u\|^2_{2,g_j} \\ &\ge \tilde{k}_1 \, \|u\|^2_{1,2} \qquad\qquad \forall h \le h_0 = 1/\tilde{k}_2 \ . \end{aligned}$$

Moreover, let $u_{j-1} \in L_\infty(G)$, then $F_j := \frac{1}{h} g_j u_{j-1} + f_j$ belongs to $L_{\min\{r_2,r_3\}}(G) \subset \left(\overset{o}{W}{}^1_{p'}(G)\right)^*$ due to continouos embedding $\overset{o}{W}{}^1_{p'}(G) \subset L_{\tilde{r}'}(G)$ with $\tilde{r} = \min\{r_2, r_3\}$, $p = N\tilde{r}/(N - \tilde{r})$. Obviously, $p > N$ since $\tilde{r} > N/2$ by the assumptions. Let now for a moment g_j belong to $L_\infty(G)$. Then from Simader [11, Thms. 7.3 and 5.4] follows existence of a unique solution $u_j \in \overset{o}{W}{}^1_p(G)$ of our problem $B^0(u,v) = \langle F_j, v\rangle \quad \forall v \in \overset{o}{W}{}^1_{p'}(G)$ (cf. also [11, Corollary 7.4]). Actually, we have proved above that [11, formula (7.7)] holds with $C_2' = 0$, hence we can choose there $\lambda = \lambda_0 = 0$. Return to our assumption $g_j \in L_{r_2}(G)$. By [11, Remark 6.4] the cited theorems formulated for bounded coefficients may be extended to the case of coefficients belonging to some $L_\lambda(G)$. As it is calculated in that remark it is sufficient to assume $g_j = a_{00} \in L_\lambda(G)$ with $\lambda = Np/(p + N)$ to obtain a solution in $\overset{o}{W}{}^1_p(G)$ with $p > N$. Since λ is close to $N/2$ if p is sufficiently close to N and $r_2 > N/2$ we find a $p = \rho$ with $N < \rho \le N\tilde{r}/(N - \tilde{r})$ such that $N/2 < \lambda \le r_2$. *) □

*) Another approach is to approximate $g_j \in L_{r_2}(G)$ by bounded coefficients $g_j^k \to g_j$ in $L_{r_2}(G)$ and to prove $u_j^k \to u_j$ as $k \to \infty$. This regularization method is described by Weber [12, Theorem 1.55] for Neumann boundary condition and may be carried over to our case of Dirichlet boundary condition.

3 A priori estimates for the discretized problem

First we derive an estimate of $\|u_j\|_{1,r}$ which is a consequence of the above existence result. From Hölder inequality with $p = \kappa/(\kappa-\varrho)$, $p' = \kappa/\varrho$ we get

$$\|g_j\delta u_j\|_\varrho = \Big(\int_G g_j^{\varrho-\varrho/\kappa} g_j^{\varrho/\kappa}\,|\delta u_j|^\varrho\,dx\Big)^{1/\varrho} \le \|g_j\|^{(\kappa-1)/\kappa}_{\varrho(\kappa-1)/(\kappa-\varrho)}\|\delta u_j\|_{\kappa,g_j}\ .$$

Hence due to assumption (ii)

$$\|g_j\delta u_j\|_\varrho \le \|\delta u_j\|_{\kappa,g_j} \quad \text{for } \varrho \le r_2\kappa/(r_2+\kappa-1)\ . \tag{3.1}$$

Additionally, let $\varrho \le r_3$. By the assumptions $r_2 > N(\kappa-1)/(2\kappa-N)$ and $r_3 > N/2$ we can choose $\varrho > N/2$. Since u_j is a weak solution of the equation $A_ju_j = f_j - g_j\delta u_j$ from [11, Theorem 6.3] then follows with $r = \min\{r_1, \rho, N\varrho/(N-\varrho)\} > N$

$$\begin{aligned}\|u_j\|_{1,r} &\le c_1\,\|f_j - g_ju_j\|_\varrho + c_2\,\|u_j\|_1\\ &\le c(\|f_j\|_{r_3} + \|g_j\delta u_j\|_\varrho + \|U_0\|_1 + \sum_{i=1}^{j} h\|\delta u_i\|_1)\\ &\le c(1 + \|\delta u_j\|_{\kappa,g_j} + h\sum_{i=1}^{j}\|\delta u_i\|_{\kappa,g_i})\ ,\end{aligned} \tag{3.2}$$

where (2.4) is used to estimate the sum. The choise $r \le r_1$ is used only in order to include u_0 into the estimates derived from (3.2).

Now we proceed in our estimations to obtain uniform boundedness of the approximates. Since g depends on u we will need boundedness of δu_j in order to estimate u_j. Therefore, at first we have to estimate δu_j, which is possible for small t by the help of the nonlinear Gronwall Lemma 2.3.

Lemma 3.1. *There is a time interval $[0,\hat t]$, $\hat t \le T$, such that independent of the subdivision holds*

$$\|\delta u_j\|_{\kappa,g_j} \ \le\ M_1 \qquad \forall t_j \le \hat t\ .$$

PROOF: We take the difference $(2.1_j) - (2.1_{j-1})$, $j = 2,\cdots,n$, and test it with $v = |\delta u_j|^{\kappa-2}\delta u_j$. By the use of the notation $\omega_j := |\delta u_j|^{(\kappa-2)/2}\delta u_j$ and (2.15) we estimate

$$\begin{aligned}&\langle\, g_j\delta u_j\,,\,|\delta u_j|^{\kappa-2}\delta u_j\,\rangle - \langle\, g_{j-1}\delta u_{j-1}\,,\,|\delta u_j|^{\kappa-2}\delta u_j\,\rangle + \tilde k_1 h\,\|\omega_j\|^2_{1,2}\\ &\qquad \le \tilde k_2 h\,\|\delta u_j\|^\kappa_{\kappa,g_j} + h|\langle\,\delta f_j\,,\,|\delta u_j|^{\kappa-2}\delta u_j\,\rangle| + |(A_{j-1}-A_j)(u_j, |\delta u_j|^{\kappa-2}\delta u_j)|\ .\end{aligned}$$

We replace the weight g_{j-1} in the second item by g_j[†)] and obtain then by means of Lipschitz condition (iv) and (2.18)

$$\begin{aligned}&\|\delta u_j\|^\kappa_{\kappa,g_j} - \langle g_j\delta u_{j-1}\,,\,|\delta u_j|^{\kappa-2}\delta u_j\rangle + ch\,\|\omega_j\|^2_{1,2}\\ &\quad\le\ \ ch\,\|\delta u_j\|^\kappa_{\kappa,g_j} + ch\,(1+\|\delta u_{j-1}\|_{\nu_2})\|\delta u_j\|^{\kappa-1}_{(\kappa-1)\mu_2'}\\ &\qquad\quad + ch\,(\varepsilon\|\omega_j\|^2_{1,2} + \|u_j\|^\kappa_{1,r} + c_\varepsilon\|\delta u_j\|^\kappa_{\kappa,g_j}) + \int_G |g_{j-1}-g_j|\,|\delta u_{j-1}|\,|\delta u_j|^{\kappa-1}\,dx\ .\end{aligned}$$

[†)] These manipulations are to avoid an item $\|\delta u_j\|^{\beta\kappa}_{\kappa,g_j}$ with $\beta > 1$ since then the nonlinear Gronwall lemma would not be applicable below.

Because of the assumptions on μ_2, ν_2 we have $\nu_2, \mu_2'(\kappa-1) < N\kappa/(N-2)$, hence we obtain further with (2.10) and application of Hölder's inequality to the second item

$$\begin{aligned}\|\delta u_j\|^\kappa_{\kappa,g_j} - \|\delta u_{j-1}\|^\kappa_{\kappa,g_j} + ch\,\|\omega_j\|^2_{1,2} \\ \le\ & \varepsilon h\,(\|\omega_{j-1}\|^2_{1,2} + \|\omega_j\|^2_{1,2}) + c_\varepsilon h\,(1 + \|\delta u_{j-1}\|^\kappa_{\kappa,g_{j-1}} + \|\delta u_j\|^\kappa_{\kappa,g_j}) \\ & + ch\,\|u_j\|^\kappa_{1,r} + \kappa \int_G |g_{j-1} - g_j|\,|\delta u_{j-1}|\,|\delta u_j|^{\kappa-1}\,dx\ .\end{aligned}$$

The norm $\|u_j\|^\kappa_{1,r}$ is estimated by means of (3.2) observing inequality $(b_1 + \cdots + b_j)^\kappa \le j^{\kappa-1}(b_1^\kappa + \cdots + b_j^\kappa)$. In the second item we replace the weight g_j by g_{j-1} again,$^{\dagger)}$ hence

$$\begin{aligned}&\|\delta u_j\|^\kappa_{\kappa,g_j} - \|\delta u_{j-1}\|^\kappa_{\kappa,g_{j-1}} + ch\,\|\omega_j\|^2_{1,2} \\ &\le\ \varepsilon h\,(\|\omega_{j-1}\|^2_{1,2} + \|\omega_j\|^2_{1,2}) + c_\varepsilon h\,\Big(1 + \|\delta u_{j-1}\|^\kappa_{\kappa,g_{j-1}} + \|\delta u_j\|^\kappa_{\kappa,g_j} + t_j^{\kappa-1} h \sum_{\nu=1}^{j} \|\delta u_\nu\|^\kappa_{\kappa,g_\nu}\Big) \\ &\quad + \kappa \int_G |g_{j-1} - g_j|\,|\delta u_{j-1}|\,|\delta u_j|^{\kappa-1}\,dx + \int_G |g_{j-1} - g_j|\,|\delta u_{j-1}|^\kappa\,dx \\ &\le\ \varepsilon h\,(\|\omega_{j-1}\|^2_{1,2} + \|\omega_j\|^2_{1,2}) + c_\varepsilon h\,\Big(1 + \|\delta u_{j-1}\|^\kappa_{\kappa,g_{j-1}} + \|\delta u_j\|^\kappa_{\kappa,g_j} + h \sum_{\nu=1}^{j} \|\delta u_\nu\|^\kappa_{\kappa,g_\nu}\Big) \\ &\quad + c\,\|g_j - g_{j-1}\|_{\mu_1}\,(\|\delta u_{j-1}\|_s \|\delta u_j\|_s^{\kappa-1} + \|\delta u_{j-1}\|_s^\kappa)\ .\end{aligned}$$

Here we have applied Hölder's inequality with $p_1 = \mu_1$, $p_2 = s$, $p_3 = s/(\kappa-1)$ and $p_1 = \mu_1$, $p_2 = s/\kappa$, respectively. Since $\mu_1 > N/2$ and $\sum 1/p_i = 1$ the condition $s < N\kappa/(N-2)$ of Corollary 2.1 holds. For further estimation note that due to assumption (ii) and (2.5)

$$\|g_j - g_{j-1}\|_{\mu_1} \le ch\,(1 + \|\delta u_{j-1}\|_{\nu_1}) \le ch\,(1 + \|\delta u_{j-1}\|_{\kappa,g_{j-1}})\ .$$

Then

$$\begin{aligned}&\|\delta u_j\|^\kappa_{\kappa,g_j} - \|\delta u_{j-1}\|^\kappa_{\kappa,g_{j-1}} + ch\,\|\omega_j\|^2_{1,2} \\ &\le \varepsilon h\,(\|\omega_{j-1}\|^2_{1,2} + \|\omega_j\|^2_{1,2}) + c_\varepsilon h\,\Big(1 + \|\delta u_{j-1}\|^\kappa_{\kappa,g_{j-1}} + \|\delta u_j\|^\kappa_{\kappa,g_j} + h \sum_{\nu=1}^{j} \|\delta u_\nu\|^\kappa_{\kappa,g_\nu}\Big) \\ &\quad + ch\,\Big(\|\delta u_{j-1}\|^\kappa_s + \|\delta u_j\|^\kappa_s + \|\delta u_{j-1}\|_{\kappa,g_{j-1}}\,\|\delta u_{j-1}\|_s\,\|\delta u_j\|_s^{\kappa-1} + \|\delta u_{j-1}\|_{\kappa,g_{j-1}} \|\delta u_{j-1}\|^\kappa_s\Big).\end{aligned}$$

We apply Young's inequality with $p_1 = \beta_1\kappa$, $p_2 = \alpha\kappa$, $p_3 = \kappa/(\kappa-1)$ and $p_1 = \beta_2\kappa$, $p_2 = \alpha$, respectively, to the last two items. In order to apply then (2.10) we choose $\alpha > 1$ dependent on s such that the conditions of Lemma 2.1 c) are satisfied (remember that $\tilde{s} = \frac{2}{\kappa}s$, $q \le \frac{\sigma}{1+\sigma}\tilde{q} = \frac{\sigma}{1+\sigma}\frac{2}{\kappa}\tilde{p} = \frac{2\sigma}{1+\sigma}$). Thus β_1, β_2 are fixed since $\sum 1/p_i = 1$, and we obtain with (2.10)

$$\begin{aligned}\|\delta u_j\|^\kappa_{\kappa,g_j} - \|\delta u_{j-1}\|^\kappa_{\kappa,g_{j-1}} + ch\,\|\omega_j\|^2_{1,2} \\ \le\ & \varepsilon h\,(\|\omega_{j-1}\|^2_{1,2} + \|\omega_j\|^2_{1,2}) + c_\varepsilon h\,\Big(1 + \|\delta u_{j-1}\|^\kappa_{\kappa,g_{j-1}} + \|\delta u_j\|^\kappa_{\kappa,g_j} + h \sum_{\nu=1}^{j} \|\delta u_\nu\|^\kappa_{\kappa,g_\nu}\Big) \\ & + ch\,\Big(\|\delta u_{j-1}\|^\kappa_s + \|\delta u_j\|^\kappa_s + \|\delta u_{j-1}\|^{\alpha\kappa}_s + \|\delta u_{j-1}\|^{\beta_1\kappa}_{\kappa,g_{j-1}} + \|\delta u_{j-1}\|^{\beta_2\kappa}_{\kappa,g_{j-1}}\Big) \\ \le\ & \varepsilon h\,(\|\omega_{j-1}\|^2_{1,2} + \|\omega_j\|^2_{1,2}) + c_\varepsilon h\,\Big(1 + \|\delta u_j\|^\kappa_{\kappa,g_j} + \|\delta u_{j-1}\|^{\beta\kappa}_{\kappa,g_{j-1}} + h \sum_{\nu=1}^{j} \|\delta u_\nu\|^\kappa_{\kappa,g_\nu}\Big)\ ,\end{aligned}$$

where $\beta > 1$. Now we sum up these inequalities for $j = 2, \dots, i$,

$$\|\delta u_i\|_{\kappa,g_i}^{\kappa} - \|\delta u_1\|_{\kappa,g_1}^{\kappa} + ch \sum_{j=2}^{i} \|\omega_j\|_{1,2}^2$$
$$\leq 2\varepsilon h \sum_{j=1}^{i} \|\omega_j\|_{1,2}^2 + c_\varepsilon \Big(t_i + h \sum_{j=1}^{i} \|\delta u_j\|_{\kappa,g_j}^{\kappa} + h \sum_{j=1}^{i-1} \|\delta u_j\|_{\kappa,g_j}^{\beta\kappa} + h^2 \sum_{j=2}^{i} \sum_{\nu=1}^{j} \|\delta u_\nu\|_{\kappa,g_\nu}^{\kappa} \Big) ,$$

and obtain for sufficiently small $\varepsilon > 0$

$$\|\delta u_i\|_{\kappa,g_i}^{\kappa} + ch \sum_{j=2}^{i} \|\omega_j\|_{1,2}^2$$
$$\leq c + \|\delta u_1\|_{\kappa,g_1}^{\kappa} + ch\,\|\omega_1\|_{1,2}^2 + ch \sum_{j=1}^{i} \|\delta u_j\|_{\kappa,g_j}^{\kappa} + ch \sum_{j=1}^{i-1} \|\delta u_j\|_{\kappa,g_j}^{\beta\kappa} , \qquad (3.3)$$

$i = 2, \dots, n$. Before we can apply the nonlinear Gronwall lemma it is necessary to estimate the item $\|\delta u_1\|_{\kappa,g_1}^{\kappa} + ch\,\|\omega_1\|_{1,2}^2$ else. Therefore we regard $(2.1_1) - A_1(U_0, v)$, test it with $v = |\delta u_1|^{\kappa-2}\delta u_1$, and obtain

$$\begin{aligned}\|\delta u_1\|_{\kappa,g_1}^{\kappa} + \tilde{k}_1 h\,\|\omega_1\|_{1,2}^2 \leq{}& \tilde{k}_2 h\,\|\delta u_1\|_{\kappa,g_1}^{\kappa} + \|f_0 - f_1\|_{\mu_2} \|\delta u_1\|_{(\kappa-1)\mu_2'}^{\kappa-1} \\ &+ |(A_1 - A_0)(U_0, |\delta u_1|^{\kappa-2}\delta u_1)| \\ &+ \int_G |f(0,x,U_0) - A(0)U_0|\, g_0^{-(\kappa-1)/\kappa}\, g_0^{(\kappa-1)/\kappa} |\delta u_1|^{\kappa-1}\, dx \\ \leq{}& ch(1 + c_\varepsilon \|\delta u_1\|_{\kappa,g_1}^{\kappa} + \varepsilon\,\|\omega_1\|_{1,2}^2 + \|U_0\|_{1,r}^{\kappa}) \\ &+ \frac{1}{\kappa} \int_G |f_0 - A(0)U_0|^{\kappa} g_0^{-\kappa} g_0\, dx + (1 - \frac{1}{\kappa}) \Big| \|\delta u_1\|_{\kappa,g_1}^{\kappa} - \|\delta u_1\|_{\kappa,g_0}^{\kappa} \Big| .\end{aligned}$$

Here we have used Lipschitz condition (iv) as well as formulas (2.10) and (2.18). Moreover, $g_0 := g(x, 0, U_0)$. Application of (2.12) and (ii) to the last item for sufficiently small $\varepsilon > 0$ then yields

$$\|\delta u_1\|_{\kappa,g_1}^{\kappa} + ch\,\|\omega_1\|_{1,2}^2 \leq \|(f_0 - A(0)U_0)/g_0\|_{\kappa,g_0}^{\kappa} + c_1 h(1 + \|\delta u_1\|_{\kappa,g_1}^{\kappa}) .$$

If $h \leq h_0 < 1/c_1$ this yields boundedness

$$\|\delta u_1\|_{\kappa,g_1}^{\kappa} + ch\,\|\omega_1\|_{1,2}^2 \leq c .$$

Thus (3.3) passes over to the nonlinear Gronwall inequality

$$\|\delta u_i\|_{\kappa,g_i}^{\kappa} + ch \sum_{j=1}^{i} \|\omega_j\|_{1,2}^2 \leq c + ch \sum_{j=1}^{i} \|\delta u_j\|_{\kappa,g_j}^{\kappa} + ch \sum_{j=1}^{i-1} \|\delta u_j\|_{\kappa,g_j}^{\beta\kappa} , \qquad (3.4)$$

$i = 1, \dots, n$, with some $\beta > 1$. Omitting the second item on the left-hand side Corollary 2.2 yields

$$\|\delta u_i\|_{\kappa,g_i}^{\kappa} \leq M(t_i) \qquad \forall t_i < t^* ,$$

where $M(t_i)$ is the bound in Corollary 2.2. Hence the proof is complete if $\hat{t} < t^*$ is fixed. $\square$

From (3.2) and due to embedding $\overset{o}{W}{}_r^1(G) \subset C(\bar{G})$ for $r > N$ we obtain immediately the following

Corollary 3.1. *It holds independent of the subdivision*

$$\|u_j\|_{1,r} \le M_2 \quad , \quad \|u_j\|_0 \le M_3 \quad , \quad \|\delta u_j\|_{\nu_1} \le M_4 \quad , \qquad \forall t_j \le t,$$

and

$$\sum_{j=1}^{\hat n} h \, \|\omega_j\|_{1,2}^2 \le M_5 \ , \quad \textit{where } \omega_j = |\delta u_j|^{\kappa-2)/2}\delta u_j \ , \quad t_{\hat n} \le \hat t \ .$$

The last two estimates follow from (2.4) and (3.4), respectively.

Although Corollary 3.1 already yields a L_∞-bound for u_j, we have to prove yet that the functions u_j belong to the ball $\mathcal{B}_R(U_0)$ centered in the initial function U_0. To this aim we use a recursive procedure to estimate L_p-norm with arbitrary $p \ge p_0$ followed by a limit process $p \to \infty$. There, it is well-known that $\|u\|_p \to \|u\|_\infty$ as $p \to \infty$ $\forall u \in L_\infty(G)$. We formulate the corresponding property for the weighted norm.

Lemma 3.2. *Let $g \in L_1(G)$ and $g(x) > 0$ for a.e. $x \in G$. Then for $u \in L_\infty(G)$ holds* $\lim_{p\to\infty} \|u\|_{p,g} = \|u\|_\infty$.

Proof: The proof is similar to the proof for the unweighted norn. Obviously, it holds

$$\|u\|_{p,g} \ \le \ \|u\|_\infty \Big(\int_G g(x)\, dx \Big)^{1/p} \ \longrightarrow \ \|u\|_\infty \quad \text{as } p \to \infty \ .$$

To show that $\lim_{p\to\infty} \|u\|_{p,g} \ge \|u\|_\infty$ let $\varepsilon > 0$ be given and let $B = \{x \in G : |u(x)| > \|u\|_\infty - \varepsilon\}$. By definition of $\|u\|_\infty$, meas$(B) > 0$, hence $\int_B g(x)\, dx = b > 0$. Thus

$$\|u\|_{p,g} \ \ge \ \Big(\int_B g|u|^p \, dx \Big)^{1/p} \ \ge \ (\|u\|_\infty - \varepsilon)\, b^{1/p} \ \longrightarrow \ \|u\|_\infty - \varepsilon \quad \text{as } p \to \infty \ .$$

Since $\varepsilon > 0$ is arbitrary chosen the assertion is proved. □

We come to the time depending L_∞-estimate of the approximates. The technique to prove such estimates is based on an idea of Moser [7] which has been essentially developed by Alikakos (cf. e.g. [1]) to obtain L_∞-estimates by recursion from L_p-estimates with fixed p.

Lemma 3.3. *There is a $\gamma > 0$ such that for every $t \in [t_k, \hat t]$ holds*

$$\|u_j - u_k\|_0 \ \le \ c\,(t - t_k)^\gamma \qquad \forall t_j \in [t_k, t] \ .$$

Proof: We fix k and define $z_j := u_j - u_k$ for $k \le j \le n$. Then $z_j \in \overset{o}{W}{}^1_r(G)$ fulfils

$$\langle g_j \delta z_j, v\rangle + A_j(z_j, v) = \langle f_j, v\rangle - A_j(u_k, v) \qquad \forall v \in \overset{o}{W}{}^1_{r'}(G)$$

for $j = k+1, \dots, n$, where $z_k = 0$. We insert the test function $v = |z_j|^{p-2} z_j$ for arbitrary $p \ge p_0 > 2$, use the abbreviation $w_j = |z_j|^{(p-2)/2} z_j$ and obtain by means of Lemma 2.2

$$\begin{aligned} &\|z_j\|_{p,g_j}^p - \|z_{j-1}\|_{p,g_j} \|z_j\|_{p,g_j}^{p-1} + \tilde k_1 h \, \|w_j\|_{1,2}^2 \\ &\qquad \le \tilde k_2 h \, \|z_j\|_{\tilde p, g_{j+1}}^p + h\, \|f_j\|_{r_3} \|z_j\|_{r_3'(p-1)}^{p-1} + h\, |A_j(u_k, |z_j|^{p-2} z_j)| \ , \end{aligned}$$

which yields with Young's inequality applied to the second term and (2.19), owing to $k_1, k_2 = \mathcal{O}(1/p)$ and $\|f_j\|_{r_3} \le c$,

$$\begin{aligned} &\|z_j\|_{p,g_j}^p - \|z_{j-1}\|_{p,g_j}^p + ch\, \|w_j\|_{1,2}^2 \\ &\qquad \le ch\, \|z_j\|_{\tilde p, g_{j+1}}^p + chp\, \|z_j\|_{r_3'(p-1)}^{p-1} + chp\, \|u_k\|_{1,r} \left(\varepsilon \|w_j\|_{1,2}^2 + c\varepsilon^{-\gamma_1} \|z_j\|_{\tilde p, g_{j+1}}^{\beta(p)p} \right) \ . \end{aligned}$$

Here $\lambda p \le \tilde p \le p$ with $\lambda := (1+\sigma)/2\sigma$ and $s < Np/(N-2)$. Since later we intend to pass to the limit $p \to \infty$ we have to observe dependence of the estimates on p carefully. Note therefore that $\beta = \beta(p)$ is given by (2.20) and $\gamma_1 = \gamma_1(p)$ is bounded as $p \to \infty$. Moreover, since $s = r_3'(p-1) < Np/(N-2)$, we may apply (2.10) to $\|z_j\|^{p-1}_{r_3'(p-1)} = \|z_j\|_s^{\alpha p}$ with $\alpha = (p-1)/p$. The corresponding $\bar\beta = (p-1)/(p+\bar\gamma)$ is greater than that of (2.20), hence we can choose the same β as in (2.20) and obtain

$$\|z_j\|^p_{p,g_j} - \|z_{j-1}\|^p_{p,g_j} + ch\,\|w_j\|^2_{1,2} \;\le\; ch\,\|z_j\|^p_{\tilde p,g_{j+1}} + hp\left(\varepsilon\|w_j\|^2_{1,2} + c\varepsilon^{-c}\|z_j\|^{\beta(p)p}_{\tilde p,g_{j+1}}\right)\,.$$

Here we have used boundedness of $\|u_k\|_{1,r}$ due to Corollary 3.1. Furthermore, from (2.12) and boundedness of $\|\delta u_j\|_{\nu_1}$ follows

$$\begin{aligned}\left|\|z_j\|^p_{p,g_j} - \|z_j\|^p_{p,g_{j+1}}\right| \;&\le\; \|g_j - g_{j+1}\|_{\mu_1}\left(\varepsilon_1\|w_j\|^2_{1,2} + c_{\varepsilon_1}\|z_j\|^p_{\tilde p,g_{j+1}}\right)\\ &\le\; ch\left(\varepsilon_1\|w_j\|^2_{1,2} + c_{\varepsilon_1}\|z_j\|^p_{\tilde p,g_{j+1}}\right)\,.\end{aligned}$$

Now choose $\varepsilon_1 > 0$ independent of p sufficiently small and $\varepsilon = \varepsilon_1/p$, then we may continue by

$$\|z_j\|^p_{p,g_{j+1}} - \|z_{j-1}\|^p_{p,g_j} \;\le\; ch\left(\|z_j\|^p_{\tilde p,g_{j+1}} + p^c\,\|z_j\|^{\beta(p)p}_{\tilde p,g_{j+1}}\right)\,.$$

Because of Corollary 3.1 and assumption (ii) we have

$$\|z_j\|^p_{\tilde p,g_{j+1}} = \|z_j\|^{(1-\beta(p))p}_{\tilde p,g_{j+1}}\,\|z_j\|^{\beta(p)p}_{\tilde p,g_{j+1}} \le \left(\|z_j\|_0\|g_{j+1}\|_1\right)^{(1-\beta(p))p}\,\|z_j\|^{\beta(p)p}_{\tilde p,g_{j+1}} \le c\,\|z_j\|^{\beta(p)p}_{\tilde p,g_{j+1}}$$

since $(1-\beta(p))p$ is bounded due to (2.20). Hence we can join the last two items. Summing up over $k+1 \le j \le i$ this yields

$$\|z_i\|^p_{p,g_{i+1}} \;\le\; ch\sum_{j=k+1}^{i} p^c\,\|z_j\|^{\beta(p)p}_{\tilde p,g_{j+1}} \;\le\; cp^c\,(t_i - t_k)\,\max_{k\le j\le i}\|z_j\|^{\beta(p)p}_{\tilde p,g_{j+1}}\,.$$

Then also

$$\max_{t_k\le t_j\le t}\|z_j\|^p_{p,g_{j+1}} \;\le\; cp^c\,(t_i - t_k)\,\max_{t_k\le t_j\le t}\|z_j\|^{\beta(p)p}_{\tilde p,g_{j+1}} \tag{3.5}$$

$\forall p \ge p_0$, $\lambda p \le \tilde p \le p$, and $t_k \le t \le \hat t$. In order to pass to the limit $p \to \infty$ now we fix $p_0 > 2$, choose $\tilde p = \lambda p$ and a special sequence $p_\nu = \lambda^{-\nu}p_0$. Using notation

$$m_\nu := \max_{t_k\le t_j\le t}\|z_j\|_{p_\nu,g_{j+1}} \quad \text{and } \beta_\nu = \beta(p_\nu)$$

(3.5) turns into recursion

$$m_\nu \;\le\; \left[cp_\nu^c(t-t_k)\right]^{1/p_\nu}\cdot m_{\nu-1}^{\beta_\nu}\,, \qquad \nu = 1,2,\ldots\,.$$

A straightforward calculation (similar to the proof of Lemma 3.1 in [8]) using definition (2.20) of β_ν yields

$$m_\nu \;\le\; c(t-t_k)^{\alpha_\nu}\,m_0^{\prod_{i=1}^{\nu}\beta_i}\,, \qquad \nu = 1,2,\ldots\,.$$

We conclude from (2.20) that $\prod_{i=1}^\infty \beta_i = \prod_{i=1}^\infty\left(1 - \frac{2\bar\gamma+3}{p_i+2\bar\gamma+1}\right) = \beta > 0$ is convergent since $\sum_{i=1}^\infty \frac{2\bar\gamma+3}{p_i+2\bar\gamma+1} \le (2\bar\gamma+3)\sum_{i=1}^\infty \frac{1}{p_i}$ converges. Moreover, observe that $g_j > 0$ a.e. in G since $g_j \ge 0$ and $1/g_j \in L_\sigma(G)$. Hence, from Lemma 3.2 follows

$$\|z_j\|_\infty \;\le\; c(t-t_k)^\alpha\,m_0^\beta\,, \qquad \text{where } \alpha = \beta\sum_{i=1}^\infty \frac{1}{p_i}\,, \quad 0 < \beta < 1\,.$$

It remains to estimate $m_0 = \max_{t_k \le t_j \le t} \|z_j\|_{p_0, g_{j+1}}$. Choosing $p = \tilde{p} = p_0$ this follows immediately from (3.5),

$$m_0^{(1-\beta_0)p_0} \le c p_0^c (t - t_k) .$$

Because of $\|z_j\|_\infty = \|u_j - u_k\|_0$ this yields the assertion of the lemma with $\gamma = \alpha + 1/(1-\beta_0)p_0$. $\square$

Setting $k = 0$ in Lemma 3.3 we see that u_j is sufficiently close to U_0 if t is sufficiently small. This yields the main result of this section.

Theorem 3.1. *Suppose assumptions (i)–(v) with some $R > 0$. Then there is a time $\hat{T}$, $0 < \hat{T} \le T$, independent of the subdivision such that $u_j \in \mathcal{B}_R(U_0)$ for all $t_j \in \hat{I} := [0, \hat{T}]$.*

The assertion of Theorem 3.1 implies that the solutions u_j of the truncated problem with data f_j^R and g_j^R are solutions of the non-truncated original equations (2.1_j) for all $t_j \le \hat{T}$. In the following section we only regard this interval $\hat{I}$.

4 Convergence and existence result

First we introduce Rothe functions obtained from u_j by piecewise constant and piecewise linear interpolation w.r.t. time t, respectively,

$$\bar{u}^n(x,t) = \begin{cases} u_j(x) & \text{if } t \in (t_{j-1}, t_j] \\ u_0(x) & \text{if } t \le 0 \end{cases}$$

and

$$\tilde{u}^n(x,t) = \frac{t_j - t}{h} u_{j-1}(x) + \frac{t - t_{j-1}}{h} u_j(x) , \qquad t \in [t_{j-1}, t_j] .$$

Moreover we use the notation $\tau_h u(x,t) = u(x, t-h)$, then we write $\bar{f}^n = f(\cdot, \bar{t}^n, \tau_h \bar{u}^n)$, $\bar{g}^n = g(\cdot, \bar{t}^n, \tau_h \bar{u}^n)$, and $\bar{A}^n(\cdot,\cdot) = A_{(\bar{t}^n)}(\cdot,\cdot)$ with $\bar{t}^n = t_j^n$ if $t_{j-1}^n < t \le t_j^n$. Now piecewise constant interpolation of (2.1_j) over $\hat{I}$ yields

$$\int_{\hat{I}} \langle \bar{g}^n \tilde{u}_t^n \, , \, v \rangle \, dt + \int_{\hat{I}} \bar{A}^n(\bar{u}^n \, , \, v) \, dt \; = \; \int_{\hat{I}} \langle \bar{f}^n \, , \, v \rangle \, dt \qquad \forall v \in L_1(\hat{I}, \overset{\circ}{W}{}^1_{r'}(G) \cap L_{\mu'}(G)) , \tag{4.1^n}$$

$\mu = \min\{r_2, r_3\}$. The results of Section 3 carry over into the following obvious properties of the Rothe functions:[‡)]

$$\bar{u}^n(\cdot,t), \; \tilde{u}^n(\cdot,t) \in \mathcal{B}_R(U_0) , \tag{4.2}$$

$$\|\tilde{u}_t^n(\cdot,t)\|_{\nu_1} \le M_4 , \; \|\tilde{u}^n(\cdot,t) - \bar{u}^n(\cdot,t)\|_{\nu_1} \le M_4 h_n , \; \|\tilde{u}^n(\cdot,t) - \tau_h \bar{u}^n(\cdot,t)\|_{\nu_1} \le M_4 h_n , \tag{4.3}$$

$$\|\tilde{u}^n(\cdot,t)\|_{1,r} \le M_2 \quad , \quad \|\bar{u}^n(\cdot,t)\|_{1,r} \le M_2 , \tag{4.4}$$

$\forall t \in \hat{I}$.

‡) Exactly, the following estimates are not correct if t belongs to the last subinterval, i.e. if $t \le \hat{T}$ but $\bar{t}^n > \hat{T}$. This insignificant incorrectness may be overcome by a slightly different definition of $\tilde{u}^n$ and $\bar{u}^n$ for these t (see [12]). For simplicity we skip this fineness.

Before we are able to prove convergence we need two other estimates of discrete time derivative, else. Defining $\omega^n = |\tilde{u}_t^n|^{(\kappa-2)/2}\tilde{u}_t^n$ we estimate $\tilde{u}_t^n$ by means of (2.11),

$$\begin{aligned}\int_0^{\hat{T}} \|\tilde{u}_t^n\|_s^\kappa \, dt &\le c\int_0^{\hat{T}} \|\omega^n\|_{1,2}^{2\theta}\, \|\tilde{u}_t^n\|_{\kappa,\bar{g}^n}^{(1-\theta)\kappa}\, dt \\ &\le c\left(\int_0^{\hat{T}} \|\omega^n\|_{1,2}^2\, dt\right)^\theta \left(\int_0^{\hat{T}} \|\tilde{u}_t^n\|_{\kappa,\bar{g}^n}^\kappa\, dt\right)^{1-\theta} \\ &\le c\left(\sum_{j=1}^{\hat{n}} h\,\|\omega_j\|_{1,2}^2\right)^\theta \left(\hat{T}\max_j \|\delta u_j\|_{\kappa,g_j}^\kappa\right)^{1-\theta} ,\end{aligned}$$

where $\tilde{u}_t^n = \delta u_j$ for $t_{j-1}^n < t \le t_j^n$ is noted. Lemma 3.1 and Corollary 3.1 then imply boundedness

$$\|\tilde{u}_t^n\|_{L_\kappa(\hat{I},L_s(G))} \le c \qquad \text{for every fixed } s < N\kappa/(N-2)\ . \tag{4.5}$$

Moreover, from (3.1) and Lemma 3.1 follows

$$\|\bar{g}^n\tilde{u}_t^n(\cdot,t)\|_\varrho \le c \quad \forall t\in\hat{I} \qquad \text{with } \varrho = r_2\kappa/(r_2+\kappa-1)\ . \tag{4.6}$$

We prove now convergence of the Rothe appproximates $\tilde{u}^n$ to a weak solution $u \in L_\infty(\hat{I},\overset{o}{W}{}_r^1(G)) \cap W_\infty^1(\hat{I},L_{\nu_1}(G))$ with $u(\cdot,t)\in\mathcal{B}_R(U_0)$ for a.e. $t\in\hat{I}$ of problem (1.1)-(1.3) fulfilling the integral relation

$$\int_{\hat{I}}\langle g(\cdot,t,u)u_t\,,\,v\rangle\,dt + \int_{\hat{I}} A_{(t)}(u\,,\,v)\,dt = \int_{\hat{I}}\langle f(\cdot,t,u)\,,\,v\rangle\,dt \quad \forall v\in L_1(\hat{I},\overset{o}{W}{}_{r'}^1(G)\cap L_{\varrho'}(G))\ , \tag{4.7}$$

$\varrho' = \min\{r_2'\kappa', r_3'\}$, and initial condition (1.3).

Lemma 4.1. *There is a subsequence $\{\tilde{u}^{n_\nu}\}_{\nu=1,2,\dots}$ of Rothe functions converging in $C^\alpha(\bar{Q}_{\hat{T}})$ for some $\alpha>0$ to a solution u of (4.7).*

Proof: First we derive boundedness of $\{\tilde{u}^n\}$ in $C^\gamma(\hat{I},C(\bar{G})$ with γ defined in Lemma 3.3. Let t_j,t_k be grid points, then due to Lemma 3.3

$$\|\tilde{u}^n(\cdot,t_j) - \tilde{u}^n(\cdot,t_k)\|_0 \ \le\ c_1\,|t_j-t_k|^\gamma\ .$$

For arbitrary t',t'' belonging to the same subinterval I_j it holds

$$\|\tilde{u}^n(\cdot,t'') - \tilde{u}^n(\cdot,t')\|_0 \ =\ \frac{|t''-t'|}{h_n}\,\|u_j-u_{j-1}\|_0 \ \le\ c_1\,\frac{|t''-t'|}{h_n^{1-\gamma}} \ \le\ c_1\,|t''-t'|^\gamma\ .$$

Now, for $t_{k-1}<t'\le t_k\le t_{j-1}<t''\le t_j$ we obtain

$$\begin{aligned}&\|\tilde{u}^n(\cdot,t'') - \tilde{u}^n(\cdot,t')\|_0 \\ &\quad\le\ \|\tilde{u}^n(\cdot,t'') - \tilde{u}^n(\cdot,t_{j-1})\|_0 + \|\tilde{u}^n(\cdot,t_{j-1}) - \tilde{u}^n(\cdot,t_k)\|_0 + \|\tilde{u}^n(\cdot,t_k) - \tilde{u}^n(\cdot,t')\|_0 \\ &\quad\le\ c_1(|t''-t_{j-1}|^\gamma + |t_{j-1}-t_k|^\gamma + |t_k-t'|^\gamma)\ \le\ c_1 3^{1-\gamma}\,|t''-t'|^\gamma\end{aligned}$$

uniformly for all subdivisions. Moreover, from (4.4) and continuous embedding $\overset{o}{W}{}_r^1(G)\subset C^\lambda(\bar{G})$, $\lambda\le 1-N/r$, follows uniform boundedness of $\{\tilde{u}^n\}$ in $L_\infty(\hat{I},C^\lambda(\bar{G}))$. Together this yields

$$\|\tilde{u}^n\|_{C^{\alpha_1}(\bar{Q}_{\hat{T}})}\ \le\ c\ , \qquad \alpha_1 = \min\{\gamma, 1-N/r\}\ .$$

Thus compact embedding $C^{\alpha_1}(\bar{Q}_{\hat{T}}) \subset C^{\alpha}(\bar{Q}_{\hat{T}})$ for $\alpha < \alpha_1$ implies existence of a subsequence $\{\tilde{u}^{n_\nu}\}_{\nu=1,2,\dots}$ with

$$\tilde{u}^{n_\nu} \longrightarrow u \qquad \text{in } C^{\alpha}(\bar{Q}_{\hat{T}}) \quad \text{as } \nu \to \infty\,. \tag{4.8}$$

Because of $\tilde{u}^{n_\nu}(\cdot,t) \in \mathcal{B}_R(U_0)$ and uniform convergence we have $u(\cdot,t) \in \mathcal{B}_R(U_0)$ $\forall t \in \hat{I}$, too.

Furthermore, due to boundedness (4.3) and (4.4), we may select a subsubsequence $\{\tilde{u}^{n_{\nu_\mu}}\}$ (we write $\{\tilde{u}^{n_\nu}\}$ ever again if we choose subsubsequences) with

$$\tilde{u}_t^{n_\nu} \stackrel{*}{\rightharpoonup} u_t \qquad \text{in } L_\infty(\hat{I}, L_{\nu_1}(G)) \quad \text{as } \nu \to \infty \tag{4.9}$$

and

$$\tilde{u}^{n_\nu} \stackrel{*}{\rightharpoonup} u \qquad \text{in } L_\infty(\hat{I}, \overset{o}{W}{}_r^1(G)) \quad \text{as } \nu \to \infty\,. \tag{4.10}$$

Of course, u_t and u then belong to the corresponding banach spaces from (4.9) and (4.10), respectively.

In order to investigate convergence of $\bar{u}^n$ we state first that in view of (4.3) and (4.8) $\bar{u}^{n_\nu} \to u$, $\tau_h \bar{u}^{n_\nu} \to u$ in $L_\infty(\hat{I}, L_{\nu_1}(G))$ holds. Then interpolation inequality (2.6) and boundedness (4.4) imply

$$\bar{u}^{n_\nu}, \tau_h \bar{u}^{n_\nu} \longrightarrow u \qquad \text{in } L_\infty(\hat{I}, C^{\alpha}(\bar{G})) \quad \text{as } \nu \to \infty\,. \tag{4.11}$$

Obviously, convergence (4.10) holds for $\{\bar{u}^{n_\nu}\}$, too. Futhermore, by means of assumptions (ii) and (iv) convergence (4.11) implies

$$\bar{g}^{n_\nu} = g(x, \bar{t}^{n_\nu}, \tau_h \bar{u}^{n_\nu}) \longrightarrow g(x,t,u) \qquad \text{in } L_\infty(\hat{I}, L_{\mu_1}(G))\,, \tag{4.12}$$

$$\bar{f}^{n_\nu} = f(x, \bar{t}^{n_\nu}, \tau_h \bar{u}^{n_\nu}) \longrightarrow f(x,t,u) \qquad \text{in } L_\infty(\hat{I}, L_{\mu_2}(G))\,, \tag{4.13}$$

as $\nu \to \infty$. However, if we pass to the limit, from (4.9) and the assumptions on ν_1, r_2 it is not sure for small space dimensions ($N = 2,3$) that $g(x,t,u)u_t$ is integrable. Hence we conclude from (4.5) and (4.6) that

$$\tilde{u}_t^{n_\nu} \longrightarrow u_t \qquad \text{in } L_\kappa(\hat{I}, L_s(G))\,, \quad s < N\kappa/(N-2)\,, \tag{4.14}$$

and

$$\bar{g}^{n_\nu}\tilde{u}_t^{n_\nu} \stackrel{*}{\rightharpoonup} w \qquad \text{in } L_\infty(\hat{I}, L_\varrho(G))\,, \quad \varrho = r_2\kappa/(r_2+\kappa-1)\,. \tag{4.15}$$

Moreover, from (4.12) and (4.14) follows

$$\bar{g}^{n_\nu}\tilde{u}_t^{n_\nu} \rightharpoonup g u_t \quad \text{in } L_\kappa(\hat{I}, L_p(G)) \tag{4.16}$$

with $\frac{1}{p} = \frac{1}{s} + \frac{1}{\mu_1} < \frac{1}{s} + \frac{2}{N}$. Thus we have $p > 1$ if we fix $N/(N-2) < s < N\kappa/(N-2)$. This implies $w = g(\cdot,\cdot,u)u_t \in L_\infty(\hat{I}, L_\varrho(G))$ in (4.15).

Now we are ready to pass to the limit $n_\nu \to \infty$ in (4.1^{n_ν}). Observing (4.15), (4.10), and (4.13) then (4.1^{n_ν}) passes into (4.7). Finally, it remains to mention that u fulfils initial condition (1.3) because of $\tilde{u}^n(\cdot,0) = U_0$ and uniform convergence (4.8). $\square$

The solution u of Lemma 4.1 may not be uniquely determined. However, we can prove uniqueness even in a slightly more general class.

Lemma 4.2. *There is at most one solution of (4.7), (1.3) belonging to the space* $X = \{u \in L_2(\hat{I}, \overset{o}{W}{}_2^1(G) \cap W_\infty^1(\hat{I}, L_{\nu_1}(G)) \ :\ u(\cdot,t) \in \mathcal{B}_R(U_0)$ *for a.e.* $t \in \hat{I},\ g(\cdot,\cdot,u)u_t \in L_1(\bar{Q}_{\hat{T}})\}$

Proof: Let u' be one of the solutions of (4.7), (1.3) that exists due to Lemma 4.1 and $u \in X$ be another solution. Defining $z = u - u'$ we fix $t_0 \in \hat{I}$ and obtain from (4.7) with test function $v = |z|^{\kappa-2}z$ if $t \le t_0$, $v = 0$ else,

$$\int_0^{t_0} \langle g(\cdot,t,u)u_t - g(\cdot,t,u')u'_t\,,\,|z|^{\kappa-2}z\rangle\,dt + \int_0^{t_0} A_{(t)}(z\,,\,|z|^{\kappa-2}z)\,dt$$
$$= \int_0^{t_0} \langle f(\cdot,t,u) - f(\cdot,t,u')\,,\,|z|^{\kappa-2}z\rangle\,dt\,. \quad (4.17)$$

Because of $u_t \in L_\infty(\hat{I}, L_{\nu_1}(G))$ we have $u \in C^{0,1}(\hat{I}, L_{\nu_1}(G))$, therefore $g(\cdot,t,u(\cdot,t))$ is Lipschitz continuous w.r.t. t in $L_{\mu_1}(G)$ due to assumption (ii). Hence $g(\cdot,t,u(\cdot,t))$ has a weak derivative $\frac{dg}{dt} \in L_\infty(\hat{I}, L_{\mu_1}(G))$. Note moreover that $g(\cdot,\cdot,u)u'_t \in L_1(\bar{Q}_{\hat{T}})$ for every u belonging to the domain of g since $u'_t \in L_\kappa(\hat{I}, L_s(G))$ in view of (4.14). Thus we may estimate by the use of (4.17) and $z(\cdot,0) = 0$

$$\begin{aligned}
\|z(\cdot,t_0)\|^\kappa_{\kappa,g(\cdot,t_0,u(\cdot,t_0))} &= \int_0^{t_0} \frac{d}{dt}\|z(\cdot,t)\|^\kappa_{\kappa,g(\cdot,t,u(\cdot,t))}\,dt \\
&= \int_0^{t_0}\int_G \frac{dg}{dt}|z|^\kappa\,dxdt + \kappa\int_0^{t_0}\int_G g(x,t,u)z_t\,|z|^{\kappa-2}z\,dxdt \\
&\le \int_0^{t_0}\left\|\frac{dg}{dt}\right\|_{\mu_1}\|z\|^\kappa_{\mu'_1\kappa}\,dt + \kappa\int_0^{t_0}\langle g(\cdot,t,u)u_t - g(\cdot,t,u')u'_t\,,\,|z|^{\kappa-2}z\rangle\,dt \\
&\quad + \kappa\int_0^{t_0}\|g(\cdot,t,u) - g(\cdot,t,u')\|_{\mu_1}\|u'_t\|_s\|z\|_s^{\kappa-1}\,dt \\
&\le c\int_0^{t_0}\|z\|^\kappa_{\mu'_1\kappa}dt - \kappa\int_0^{t_0} A_{(t)}(z\,,\,|z|^{\kappa-2}z)\,dt \\
&\quad + \kappa\int_0^{t_0}\langle f(\cdot,t,u) - f(\cdot,t,u'), |z|^{\kappa-2}z\rangle dt + c\int_0^{t_0}\|z\|_{\nu_1}\|u'_t\|_s\|z\|_s^{\kappa-1}dt\,.
\end{aligned}$$

The index s follows from Hölder's inequality and may be chosen such that $s < N\kappa/(N-2)$ since $\mu_1 > N/2$. In the same way the assumptions imply that $\mu'_1\kappa, \mu'_2(\kappa-1), \nu_2 < N\kappa/(N-2)$. Let s be the maximum of this quantities. With the usual notation $w = |z|^{(\kappa-2)/2}z$ we continue estimation by means of (2.14), (iv), and (2.5),

$$\begin{aligned}
\|z(\cdot,t_0)\|^\kappa_{\kappa,g} &\le c\int_0^{t_0}\|z\|_s^\kappa\,dt - c\int_0^{t_0}\|w\|^2_{1,2}\,dt + c\int_0^{t_0}\|z\|^\kappa_\kappa\,dt \\
&\quad + c\int_0^{t_0}\|z\|_{\nu_2}\,\|z\|_s^{\kappa-1}\,dt + c\int_0^{t_0}\|z\|_{\kappa,g}\,\|u'_t\|_s\,\|z\|_s^{\kappa-1}\,dt \\
&\le c\int_0^{t_0}\|z\|_s^\kappa\,dt - c\int_0^{t_0}\|w\|^2_{1,2}\,dt + c\int_0^{t_0}\|z\|_{\kappa,g}\,\|u'_t\|_s\,\|z\|_s^{\kappa-1}dt\,. \qquad (4.18)
\end{aligned}$$

We estimate the last item separately. Since u' is a solution given by Lemma 4.1 we have $u'_t \in L_\kappa(\hat{I}, L_s(G))$ as mentioned above and obtain

$$\int_0^{t_0}\|z\|_{\kappa,g}\,\|u'_t\|_s\,\|z\|_s^{\kappa-1}dt \le c\left(\int_0^{t_0}\|u'_t\|_s^\kappa\,dt\right)^{1/\kappa}\left(\int_0^{t_0}\|z\|^{\kappa'}_{\kappa,g}\,\|z\|_s^\kappa\,dt\right)^{1/\kappa'}\,.$$

Now the interpolation inequality (2.11) is applied to $\|z\|_s$ and repeated application of Hölder's inequality and then Young's inequality imply

$$\begin{aligned}
\int_0^{t_0}\|z\|_{\kappa,g}\,\|u'_t\|_s\,\|z\|_s^{\kappa-1}dt &\le c\left(\int_0^{t_0}\|z\|^{\kappa'}_{\kappa,g}\,\|w\|^{2\theta}_{1,2}\,\|z\|^{\kappa(1-\theta)}_{\kappa,g}\,dt\right)^{1/\kappa'} \\
&\le c\left(\int_0^{t_0}\|w\|^2_{1,2}\,dt\right)^{\theta/\kappa'}\left(\int_0^{t_0}\|z\|^{\kappa+\kappa'/(1-\theta)}_{\kappa,g}\,dt\right)^{(1-\theta)/\kappa'} \\
&\le \varepsilon\int_0^{t_0}\|w\|^2_{1,2}\,dt + c_\varepsilon\left(\int_0^{t_0}\|z\|^{\kappa\gamma}_{\kappa,g}\,dt\right)^{1/\gamma}\,.
\end{aligned}$$

Here we used abbreviation $1 + \kappa'/\kappa(1-\theta) = (\kappa'-\theta)/(1-\theta) =: \gamma \geq 1$. Return to (4.18), then application of (2.11) to the first item and choice of a sufficiently small $\varepsilon > 0$ yields

$$\|z(\cdot,t_0)\|^{\kappa}_{\kappa,g(\cdot,t_0,u(\cdot,t_0))} \leq c\left(\int_0^{t_0} \|z(\cdot,t)\|^{\kappa\gamma}_{\kappa,g(\cdot,t,u(\cdot,t))}\,dt\right)^{1/\gamma},$$

where from Gronwall's lemma follows $\|z\|^{\kappa\gamma}_{\kappa,g} = 0\ \forall t \in \hat{I}$, i.e. $u = u'$. □

The above uniqueness assertion implies that all our convergence results for subsequences (4.8), (4.11), (4.9), (4.10), (4.14), (4.15) hold for the whole sequence of Rothe approximates corresponding to a subdivision with arbitrary $h_n \to 0$, too.

We summarize the results.

Theorem 4.1. *Suppose assumptions (i)–(v). Then the following assertions hold:*

a) There is an interval $\hat{I} = [0,\hat{T}]$ such that problem (1.1)-(1.3) has a unique weak solution $u \in L_\infty(\hat{I}, \overset{o}{W}{}^1_r(G)) \cap W^1_\infty(\hat{I}, L_{\nu_1}(G))$, $r > N$, in the sense of relation (4.7) with $u(\cdot,t) \in \mathcal{B}_R(U_0)$ for all $t \in \hat{I}$.

b) The solution u has the additional regularity properties $u \in C^\alpha(\bar{Q}_{\hat{T}})$ for some $\alpha > 0$, $u_t \in L_\kappa(\hat{I}, L_s(G))$ for every $s < \frac{N\kappa}{N-2}$, and $g(\cdot,\cdot,u)u_t \in L_\infty(\hat{I}, L_\varrho(G))$, $\varrho = \frac{r_2\kappa}{r_2+\kappa-1}$.

c) The solution u is pointwise uniformly approximated by the Rothe functions $\tilde{u}^n$ which have convergence properties

$$\begin{aligned}
\tilde{u}^n &\longrightarrow u && \text{in } C^\alpha(\bar{Q}_{\hat{T}})\\
\bar{u}^n &\longrightarrow u && \text{in } L_\infty(\hat{I}, C^\alpha(\bar{G}))\\
\tilde{u}^n, \bar{u}^n &\overset{*}{\rightharpoonup} u && \text{in } L_\infty(\hat{I}, \overset{o}{W}{}^1_r(G))\\
\tilde{u}^n_t &\overset{*}{\rightharpoonup} u_t && \text{in } L_\infty(\hat{I}, L_{\nu_1}(G))
\end{aligned}$$

as n tends to infinity.

Remark. *The use of compactness criterions in Lemma 4.1 doesn't yield any assertion on the error of the approximation. However, assumptions (i)-(v) allow to prove an error estimate using a similar but more delicated technique as in Lemma 4.2. One obtains*

$$\|\tilde{u}^n(\cdot,t) - u(\cdot,t)\|_{\nu_1} \leq c\,h_n^{1/2} \qquad \forall t \in \hat{I}\,.$$

Moreover, by interpolation this yields an estimate in Hölder space

$$\|\tilde{u}^n(\cdot,t) - u(\cdot,t)\|_{0,\alpha} \leq c\,h_n^{(1-\alpha-N/r)/2}\,, \quad 0 < \alpha < 1 - N/r\,.$$

References

[1] N. D. Alikakos. L_p-bounds of solutions of reaction-diffusion equations. *Comm. Part. Diff. Equ.*, **4**(8), 827–868, 1979.

[2] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer, Berlin–Heidelberg–New York, 1977.

[3] W. Jäger and J. Kačur. Solution of doubly nonlinear and degenerate parabolic problems by relaxation schemes. *Comenius University, Faculty of Mathematics and Physics, Preprint No.* **M2-94**, 1994.

[4] J. Kačur. On a Solution of Degenerate Elliptic-parabolic Systems in Orlicz-Sobolev Spaces I,II. *Math. Zeitschrift*, **203**, 153–171, 569–579, 1990.

[5] J. Kačur. Solution to strongly nonlinear parabolic problems by a linear approximation scheme. *Comenius University, Faculty of Mathematics and Physics, Preprint No.* **M1-94**, 1994.

[6] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and Quasilinear Equations of Parabolic Type.* Nauka, Moscow, 1967.

[7] J. Moser. A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. *Comm. Pure and Appl. Math.*, **13**(3), 457–468, 1960.

[8] V. Pluschke. Local solutions to quasilinear parabolic equations without growth restrictions. to appear in Z. Anal. Anwend.

[9] V. Pluschke. Local solution of parabolic equations with strongly increasing nonlinearity by the Rothe-method. *Czechoslovak Mathematical Journal*, **38(113)**, 642–654, 1988.

[10] V. Pluschke. Rothe's method for semilinear parabolic problems with degeneration. *Math. Nachr.*, **156**, 283–295, 1992.

[11] C. G. Simader. *On Dirichlet's Boundary Value Problem.* Lecture Notes in Math. 268. Springer, Berlin–Heidelberg–New York, 1972.

[12] F. Weber. *Die Anwendung der Rothe-Methode auf parabolische Probleme mit nichtlinearen Neumannschen Randbedingungen.* PhD thesis, Martin-Luther-University Halle-Wittenberg, 1995.

[13] D. Willett and J. Wong. On the Discrete Analogous of Some Generalizations of Gronwall's Inequality. *Monatshefte für Math.*, **69**, 362–367, 1965.

[14] E. Zeidler. *Nonlinear Functional Analysis and its Applications*, volume II/A: Linear Monotone Operators. Springer, New York–Berlin–Heidelberg–Tokyo, 1990.