A characterization of 3-local geometry of $M(24)$

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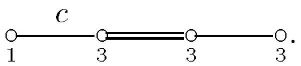
Abstract

The largest Fischer 3-transposition group $M(24)$ acts flag-transitively on a 3-local incidence geometry $\mathcal{G}(M(24))$ which is a c-extension of the dual polar space associated with the group $O_7(3)$. The action of the simple commutator subgroup $M(24)'$ is still flag-transitive. We show that $\mathcal{G}(M(24))$ is characterized by its diagram under the flag-transitivity assumption. The result implies in particular that $\mathcal{G}(M(24))$ is simply connected. The geometry $\mathcal{G}(M(24))$ appears as a subgeometry in the Buekenhout – Fischer 3-local geometry $\mathcal{G}(F_1)$ of the Monster group. The simple connectedness of $\mathcal{G}(M(24))$ has played a crucial role in the characterization of $\mathcal{G}(F_1)$, achieved recently. When determining the possible structure of the parabolic subgroups we have used an unpublished pushing-up result by U. Meierfrankenfeld.

1 Introduction

Let δ be the following diagram:

δ :

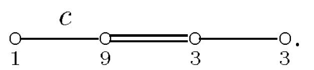


An (incidence) geometry $\mathcal{G}$ with diagram δ has rank 4; the types of its elements are 1, 2, 3 and 4 correspond to the nodes from the left to the right on the diagram. The residue of a flag of type $\{1,2\}$, $\{1,4\}$ or $\{3,4\}$ is, respectively, a projective plane of order 3, a generalized quadrangle of order $(3,3)$ or the geometry of vertices and edges of the complete graph on 5 vertices.

Suppose that $\mathcal{G}$ is a geometry whose diagram is δ and that it admits a flag-transitive automorphism group G. Let $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ be a flag in $\mathcal{G}$ where α_i is of type i. For a subset J of $I=\{1,2,3,4\}$ let G_J denote the stabilizer in G of the flag $F_J=\{\alpha_i \mid i\in J\}$. To simplify the notation we will write for instance 12 for the subset $\{1,2\}$. Let K_J denote the kernel of the action of G_J on the residue $\mathcal{G}_J$ of F_J in $\mathcal{G}$ and let $\bar{G}_J \cong G_J/K_J$ be the action induced on the residue. For an element $\beta\in\mathcal{G}$ by G_β we denote its stabilizer in G, so that G_{α_i} is another name for G_i.

Let $M(24)$ be the largest sporadic 3-transposition group discovered by B. Fischer. It is known [RSt] that $M(24)$ acts flag-transitively on a geometry $\mathcal{G}(M(24))$ whose diagram is δ and in which the residue of an element of type 1 is the dual polar space associated with the group $O_7(3)$. For the action of $G \cong M(24)$ on $\mathcal{G}(M(24))$ we have $\bar{G}_1 \cong O_7(3)$, $K_1 \cong 3^7$. Moreover, K_1 is the natural module for $\bar{G}_1$ and G_1 does not split over K_1. The parabolic G_4 is a split extension of $K_4 \cong 3_+^{1+10} : 2$ by $\bar{G}_4 \cong U_5(2) : 2$. Every element α in $\mathcal{G}(M(24))$ can be identified with certain elementary abelian 3-subgroup $V(\alpha)$ in $M(24)$ in such a way that the order of $V(\alpha)$ is 3, 3^2, 3^3 and 3^7 for α being of type 1, 2, 3 and 4, respectively; the stabilizer of α in G coincides with $N_G(V(\alpha))$ and α is incident to β if and only if either $V(\alpha) \leq V(\beta)$ or $V(\beta) \leq V(\alpha)$. The commutator subgroup $M(24)'$ also acts flag-transitively on $\mathcal{G}(M(24))$. With respect to the action of $G \cong M(24)'$ on the geometry we have $\bar{G}_1 \cong \Omega_7(3)$, $K_1 \cong 3^7$, $\bar{G}_4 \cong U_5(2) : 2$, $K_4 \cong 3_+^{1+10}$.

A triple cover $3 \cdot M(24)$ of $M(24)$ is the normalizer of a subgroup of order 3 in the Monster group F_1 and $\mathcal{G}(M(24))$ appears as a subgeometry in the Buekenhout – Fischer geometry $\mathcal{G}(F_1)$ of the Monster [BF] which has the following diagram:



Both $\mathcal{G}(M(24))$ and $\mathcal{G}(F_1)$ belong to the family of c-extended classical dual polar spaces and their characterization in terms of diagrams is very important for classification of all flag-transitive extended dual polar spaces [Yosh]. In the present paper we prove the following.

Theorem 1 *Let $\mathcal{G}$ be a geometry with the diagram δ admitting a flag-transitive automorphism group G. Then $\mathcal{G} \cong \mathcal{G}(M(24))$ and G is isomorphic either to $M(24)$ or to $M(24)'$.*

The geometry $\mathcal{G}(F_1)$ has been characterized in [IM] and Theorem 1 was crucial for this characterization.

2 The structure of residues

For the rest of the paper let $\mathcal{G}$ be a geometry with the diagram δ and G be a flag-transitive automorphism group of $\mathcal{G}$ for which we adopt the notation introduced in the previous section.

Consider the residue $\mathcal{G}_1$ of α_1 in $\mathcal{G}$. Then $\mathcal{G}_1$ is a flag-transitive C_3-geometry; its residual projective planes have order 3 and hence they are Desarguesian. The available results on flag-transitive C_3-geometries imply the following (cf. Section 14 in [Pas]).

Lemma 2.1 *$\mathcal{G}_1$ is the (dual) polar space associated with a 7-dimensional $GF(3)$-space equipped with a non-singular orthogonal form. $\bar{G}_1$ is either the special orthogonal group $O_7(3)$ or its index 2 simple commutator subgroup $\Omega_7(3)$.* $\square$

Let us recall some well known properties of the dual polar space $\mathcal{G}_1$. Let $V = V(\alpha_1)$ be a 7-dimensional $GF(3)$-space with a non-singular quadratic form ϕ on it. Then the elements of type 2, 3 and 4 in $\mathcal{G}_1$ are the 3-, 2- and 1-dimensional subspaces in V which are totally singular with respect to ϕ. Two subspaces are incident if one of them contains the other one. It is convenient to associate with $\mathcal{G}_1$ a graph Δ called the *dual polar space graph.* The vertices of Δ are maximal (which means 3-dimensional) totally isotropic subspaces of V with respect to ϕ with two such subspaces adjacent if their intersection is of dimension 2. In other terms the vertices of Δ are the elements of type 2 in $\mathcal{G}_1$ and two of them are adjacent if they are incident to a common element of type 3. For a vertex β of Δ and an integer i let $\Delta_i(\beta)$ denote the set of vertices in Δ which are at distance i from β in the natural metric of Δ. It is known [BCN] that Δ is distance-transitive on 1120 vertices with the intersection array $\{39, 36, 27; 1, 4, 13\}$. In particular the valency of Δ is 39 and the diameter is 3. Both $O_7(3)$ and $\Omega_7(3)$ act primitively on the vertex set of Δ.

The following two lemmas come from the standard information on the actions of $O_7(3)$ and $\Omega_7(3)$ on the associated dual polar space (compare [Atlas]).

Lemma 2.2 *$\bar{G}_{14} \cong O_5(3) \cong \Omega_5(3) : 2$ and $O_3(K_{14}/K_1) \cong 3^5$ is the irreducible orthogonal module for $\bar{G}_{14}$. If $\bar{G}_1 \cong \Omega_7(3)$ then $K_{14}/K_1 = O_3(K_{14}/K_1)$ and if $\bar{G}_1 \cong O_7(3)$ then K_{14}/K_1 is an extension of $O_3(K_{14}/K_1)$ by a fixed-point free involution.* □

Lemma 2.3 *$\bar{G}_{12} \cong L_3(3)$ and $O_3(K_{12}/K_1) \cong 3^{3+3}$. If $\bar{G}_1 \cong \Omega_7(3)$ then $K_{12}/K_1 = O_3(K_{12}/K_1)$ and if $\bar{G}_1 \cong O_7(3)$ then K_{12}/K_1 is an extension of $O_3(K_{12}/K_1)$ by an involution commuting with a subgroup of order 3^3 inside $O_3(K_{12}/K_1)$. In any case the elementwise stabilizer of $\Delta_1(\alpha_2)$ in K_{12}/K_1 is elementary abelian of order 3^3, it acts faithfully on $\Delta_2(\alpha_2)$ and on $\Delta_3(\alpha_2)$ and it is acted on irreducibly by $\bar{G}_{12}$.* □

The residue $\mathcal{G}_4$ of the element α_4 in $\mathcal{G}$ is a flag-transitive c-extension of a classical generalized quadrangle. All such geometries are known (cf. [Yosh] and references therein). There is only one flag-transitive c-extension of the generalized quadrangle $\mathcal{G}_{14}$ of the orthogonal group $O_5(3)$. This extension has 176 elements of type 1 and it possesses two flag-transitive automorphism groups: $U_5(2)$ and $U_5(2) : 2$. In the former group the stabilizer of an element of type 1 induces on the residual generalized quadrangle the simple group $\Omega_5(3)$ and this is in a contradiction with Lemma 2.2. So we have the following :

Lemma 2.4 *$\bar{G}_4 \cong U_5(2) : 2$.* □

Considering the action of $\bar{G}_4$ on the residual geometry $\mathcal{G}_4$ (compare [Atlas]) we obtain :

Lemma 2.5 *$G_{14}/K_4 \cong (3 \times \Omega_5(3)) : 2$ and K_{14}/K_4 is of order 3.* □

Now we are in a position to prove the following :

Lemma 2.6 *$K_1/(K_1 \cap K_4)$ has order 3, in particular $K_1 \neq 1$.*

Proof: By definition $K_1/(K_1 \cap K_4)$ is the action induced by K_1 on the residue $\mathcal{G}_4$ of α_4 and it is contained in the action K_{14}/K_4 induced by K_{14} on this residue. By Lemma 2.5 the latter action has order 3. Suppose that $K_1/(K_1 \cap K_4)$ is trivial. Then $O_3(K_{14}/K_1)$ must induce an order 3 action on $\mathcal{G}_4$ but since $O_3(K_{14}/K_1)$ has order 3^5 and it is irreducible under $\bar{G}_{14} \cong O_5(3)$ (cf. Lemma 2.2) this is impossible. □

Let Γ be the collinearity graph of $\mathcal{G}$ which means that the vertices of Γ are the elements of type 1 in $\mathcal{G}$ and two of them are adjacent if and only if they are incident to a common element of type 2.

It follows directly from the diagram δ of $\mathcal{G}$ that every element of type 2 is incident to exactly two elements of type 1. Hence an element of type 2 determines an edge of Γ. Since G_1 acts primitively on the set of elements of type 2 in $\mathcal{G}_1$, different elements of type 2 determine different edges of Γ. In particular there is a bijective mapping ψ of the vertex set of the dual polar space graph Δ associated with α_1 onto the set $\Gamma(\alpha_1)$ of vertices in Γ adjacent to α_1 and ψ commutes with the action of G_1.

Lemma 2.7 *Let β and γ be adjacent vertices of Δ. Then the images of β and γ under ψ are adjacent in Γ.*

Proof: Since β and γ are adjacent in Δ, they are incident to a common element of type 3 in $\mathcal{G}_1$, say to α_3. But the residue of the flag $\{\alpha_3, \alpha_4\}$ is formed by vertices and edges of a complete graph on 5 vertices so the result follows. □

In what follows we will identify the vertices and the edges of Γ with the elements in $\mathcal{G}$ of type 1 and 2, respectively.

Since the automorphism group of the residual geometry $\mathcal{G}_1$ acts faithfully on the set of elements of type 2 in the residue, we observe that K_1 coincides with the elementwise stabilizer in G_1 of the vertices from $\Gamma(\alpha_1)$.

Lemma 2.8 *K_1 is a non-trivial 3-group.*

Proof: $K_1 \neq 1$ by Lemma 2.6. As we observed in the above paragraph, K_1 coincides with the elementwise stabilizer in G of α_1 and of all the vertices adjacent to α_1 in Γ. On the other hand Γ is connected and G acts on it vertex-transitively. So to prove the lemma it is sufficient to show that K_1 induces a 3-group on $\Gamma(\beta)$ for every $\beta \in \Gamma(\alpha_1)$. In view of the obvious symmetry between α_1 and β, by Lemmas 2.3 and 2.7 the action induced by K_1 on $\Gamma(\beta)$ has order 3^3, so the result follows. □

In order to identify K_1 we apply the following result proved by U. Meierfrankenfeld using the pushing-up methods developed in his Ph D. Thesis [Meie1] and the classification [Meix] of the failure factorization modules for Lie type groups in odd characteristic.

Proposition 2.9 [Meie2] *Let L be a finite group and suppose that*

(i) *for $Q = O_3(L)$ we have $\bar{L} := L/Q \cong O_7(3)$ or $\Omega_7(3)$;*

(ii) *for V being the natural module for $\bar{L}$ and P being the full preimage of the stabilizer in $\bar{L}$ of a maximal totally isotropic subspace in V; if $N \leq Q$ is normal in L and characteristic in P, then $N = 1$.*

Then Q is either trivial or isomorphic to V, that is to the natural module of $\bar{L}$. □

Lemma 2.10 *$K_1 \cong V \cong 3^7$ is the natural module for $\bar{G}_1$.* □

Proof: We are going to apply Proposition 2.9 for $L = G_1$, $P = G_{12}$ and $Q = K_1$. Let N be a subgroup in K_1 which is normal in G_1 and characteristic in G_{12}. Since G_2 contains G_{12} as a normal subgroup (of index 2), N must be normal in G_2. Now, G_1 and G_2 generate G since both $\mathcal{G}$ and Γ are connected. So N fixes a vertex of Γ and it is normal in the group G which acts on Γ vertex-transitively. This means that N fixes every vertex of Γ. Since the action of G on Γ is faithful, we have $N = 1$ and the result follows. □

Lemma 2.11 *The following assertions hold:*

(i) *either $G_1 \cong 3^7.\Omega_7(3)$ or $G_1 \cong 3^7.O_7(3)$;*

(ii) *if $G_1 \cong 3^7.\Omega_7(3)$ then G_1 possesses an automorphism τ such that $\langle G_1, \tau \rangle \cong 3^7.O_7(3)$.*

Proof: The assertion (i) follows from Lemmas 2.1 and 2.10.

The assertion (ii) is trivial if G_1 splits over K_1. By [Küs] there is a unique isomorphism type of non-split extensions of $\Omega_7(3)$ by the natural module. This extension is contained in $M(24)'$ and the automorphism τ is realized in $M(24)$. □

Thus we have established that K_1 is canonically isomorphic to the 7-dimensional $GF(3)$-module V equipped with a non-singular quadratic form ϕ and the residual geometry $\mathcal{G}_1$ can be naturally defined in terms of the polar space (V, ϕ). In what follows we will identify V with K_1. For an element $\beta \in \mathcal{G}_1$ let $V(\beta)$ denote the subspace of V which corresponds to β. The dimension of $V(\beta)$ is 3, 2 and 1 for β being of type 2, 3 and 4, respectively. Then G_{1j} is the full normalizer of $V(\alpha_j)$ in G_1. For an arbitrary subspace U in V let $U^\perp$ denote the orthogonal complement of U with respect to ϕ.

Let us now specify K_4. We have that G_{14} is the normalizer in G_1 of the subspace $V(\alpha_4)$ which is 1-dimensional. It is a standard fact about orthogonal groups that under the action of G_{14}/K_1 the module V is uniserial, $V(\alpha_4)$ is its unique 1-dimensional submodule and $V(\alpha_4)^\perp$ is the unique submodule of codimension 1. This and Lemma 2.6 give :

Lemma 2.12 $K_1 \cap K_4 = V(\alpha_4)^\perp$. $\square$

The action induced by G_{14} on $\mathcal{G}_4$ is isomorphic to $(3 \times \Omega_5(3)) : 2$ (cf. Lemma 2.5) and K_1 induces an action of order 3. Hence $G_{14}/\langle K_1, K_4\rangle \cong O_5(3) \cong \Omega_5(3) : 2$. This and Lemma 2.2 imply :

Lemma 2.13 $\langle K_1, K_4\rangle = K_{14}$. $\square$

Lemma 2.14 $O_3(K_4) \cong 3_+^{1+10}$. *If* $G_1 \cong 3^7.\Omega_7(3)$ *then* $K_4 = O_3(K_4)$ *and if* $G_1 \cong 3^7.O_7(3)$ *then* K_4 *is an extension of* $O_3(K_4)$ *by an involution acting fixed-point freely on* $O_3(K_4)/Z(O_3(K_4))$.

Proof: By Lemmas 2.12 and 2.13 $O_3(K_4)$ has index 3 in $O_3(K_{14})$ and the latter is of order 3^{12}. Hence $|O_3(K_4)| = 3^{11}$. Clearly $V(\alpha_4)$ is in the center of K_4. We have that $O_3(K_{14}/K_1)$ acts faithfully on $V(\alpha_4)^\perp$ and centralizes $V(\alpha_4)^\perp/V(\alpha_4)$. Moreover, both $O_3(K_{14}/K_1)$ and $V(\alpha_4)^\perp/V(\alpha_4)$ are elementary abelian of order 3^5 and are acted on irreducibly by $\bar{G}_{14} \cong O_3(5)$. Hence $O_3(K_4)$ is extraspecial of plus type. Suppose that $G_1 \cong 3^7.O_7(3)$. Then there exists an involution τ in $K_{14} \setminus O_3(K_{14})$ which centralizes $V(\alpha_4)$ and acts fixed-point freely both on $O_3(K_{14}/K_4)$ and on $V(\alpha_4)^\perp/V(\alpha_4)$ so the result follows. $\square$

Suppose that $G_1 \cong 3^7.O_7(3)$ and let τ be an involution from $K_4 \setminus O_3(K_4)$. Then $C_{G_4}(\tau)$ is an extension of $\langle\tau\rangle \times V(\alpha_4)$ by $\bar{G}_4 \cong U_5(2) : 2$. Since the Schur multiplier of $U_5(2)$ is trivial we have $C_{G_4}(\tau) \cong \langle\tau\rangle \times V(\alpha_4) \times U_5(2) : 2$ and G_4 splits over $O_3(K_4)$.

Proposition 2.15 *Let* $\mathcal{G}$ *be a flag-transitive geometry with diagram* δ *and let* G *be the full automorphism group of* $\mathcal{G}$. *Then* $G_1 \cong 3^7.O_7(3)$ *and* $G_4 \cong 3_+^{1+10} : (2 \times U_5(2) : 2)$.

Proof: By Lemma 2.11 and the paragraph before the proposition all we have to show is that if $G_1 \cong 3^7.\Omega_7(3)$ then G is not the full automorphism group of $\mathcal{G}$. By a standard argument the elements of $\mathcal{G}$ can be identified with the right cosets of the subgroups G_i in G so that two elements are incident if and only if the intersection of the cosets is non-empty. So to prove the claim it is sufficient to show that G possesses a suitable automorphism τ which normalizes each G_i. By Lemma 2.11 (ii) there is an automorphism τ of G_1 such that $\langle G_1, \tau\rangle \cong 3^7.O_7(3)$. We can choose τ in such a way that it normalizes G_{14} and centralizes a complement F to $K_{14} = O_3(K_{14})$ in G_{14} (cf. Lemma 2.2). In this case τ acts fixed-points freely on $O_3(K_{14}/K_1)$ and on $(K_1 \cap K_4)/Z(K_4)$. Hence τ acts fixed-points freely on $K_4/Z(K_4)$. Then $C_{G_{14}}(\tau)/Z(K_4)$ is a complement to $O_3(K_4)/Z(K_4)$ in $G_{14}/Z(K_4)$. Since G_{14} contains a Sylow 3-subgroup of G_4, by Gaschütz's theorem and since the Schur multiplier of $U_5(2)$ is trivial, there is a complement H to $O_3(K_4)$ in G_4 which contains F. Then we can extend the action of τ on G_{14} to an automorphism of G_4 by declaring that τ commutes with H. The geometry $\mathcal{G}$ is connected, hence G_1 and G_4 generate G and we can extend τ to an automorphism of G. Since τ fixes the residue $\mathcal{G}_4$ elementwise, it clearly normalizes G_2 and G_3. $\square$

For the rest of the paper G is assumed to be the full automorphism group of $\mathcal{G}$. By the above proposition in this case $G_1 \cong 3^7.O_7(3)$ and $G_4 \cong 3_+^{1+10} : (2 \times U_5(2) : 2)$.

The group $U_5(2)$ has only one faithful 10-dimensional $GF(3)$-representation (compare [JLPW]). This representation is irreducible which means that there is a unique invariant quadratic form on the corresponding module. This observation and Proposition 2.15 specify G_4 up to isomorphism. By Proposition 2.15 and [Küs] the isomorphism type of G_1 depends only on whether or not it splits over K_1. On the other hand there is a unique way to specify G_{14} as a subgroup in G_4. Namely, let $U \cong U_5(2)$ be a complement to $O_3(G_4)$ in $O^2(G_4)$ (one can easily see that there is one class of such complements) and let λ be an element of order 3 in U such that $C_U(\lambda) \cong \langle\lambda\rangle \times \Omega_5(3)$. Then up to conjugacy in G_4 we have $K_1 = \langle \lambda, C_{O_3(G_4)}(\lambda)\rangle$ and $G_{14} = N_{G_4}(K_1)$. Since G_{14} contains a Sylow 3-subgroup of G_1, by Gaschütz's theorem G_1 splits over K_1 if and only if G_{14} does so. By the above argument the isomorphism type of G_{14} and the location of K_1 in G_{14} are uniquely specified and hence the isomorphism type of G_1 is also uniquely determined. Clearly, G_1 must be isomorphic to the parabolic subgroup associated with the action of $M(24)$ on $\mathcal{G}(M(24))$, that is to the non-split extension. So the amalgam consisting of the subgroups G_1 and G_4 is isomorphic to the one associated with the action of $M(24)$ on $\mathcal{G}(M(24))$.

Let $\beta \in \Gamma(\alpha_1)$ and $K(\beta)$ be the elementwise stabilizer in G of the vertices from $\Gamma(\beta)$ (which is a conjugate of K_1). By Lemmas 2.3 and 2.8 $K(\beta)$ induces on the dual polar space graph Δ an action of order 3^3 which is faithful on $\Delta_2(\beta)$ and on $\Delta_3(\beta)$. This implies the following two lemmas :

Lemma 2.16 *The intersection of $K(\beta)$ and K_1 has order 3^4 and it coincides with $V(\beta)^\perp$.* $\square$

Lemma 2.17 *Let ψ be the mapping of Δ onto $\Gamma(\alpha_1)$ which commutes with the action of G_1. Then $\beta, \gamma \in \Delta$ are adjacent in Δ if and only if their images under ψ are adjacent in Γ.* $\square$

By the above lemma locally Γ is the dual polar space graph Δ. This implies that the subgraph $\Xi = \Xi(\alpha_4)$ in Γ induced by the vertices incident to α_4 is strongly regular with parameters $v = 176$, $k = 40$, $\lambda = 12$, $\mu = 8$ and $\Xi \cap \Gamma(\alpha_1)$ are all the maximal totally isotropic subspaces in V which contain $V(\alpha_4)$.

3 Reconstruction of $\Sigma_3 \times D_4(3).\Sigma_3$

As above let G be the full automorphism group of a geometry $\mathcal{G}$ with diagram δ. To prove Theorem 1 we have to show that $G \cong M(24)$. Recall that $M(24)$ is generated by a conjugacy class of so-called 3-transpositions. If m is a 3-transposition then it is of order 2 and $C_{M(24)}(m) \cong \langle m\rangle \times M(23)$ where $M(23)$ is another 3-transposition group discovered by B. Fischer. Two distinct 3-transpositions either commute or generate an Σ_3-subgroup whose normalizer T in $M(24)$ is isomorphic to $\Sigma_3 \times D_4(3).\Sigma_3$. The aim of the present section is to reconstruct T inside G.

Under the action of $\bar{G}_1 \cong O_7(3)$ the set of 1-dimensional subspaces of V (the projective points) split into 3 orbits: S^0 (isotropic points), S^+ (plus points) and S^- (minus points). The orbit S^0 naturally corresponds to the elements of type 4 in $\mathcal{G}_1$. Let $W \in S^+$. Then W is non-singular, $V = W \oplus W^\perp$ and $W^\perp$ contains a maximal totally isotropic subspace of V. Let $M = C_{G_1}(W)$ and $\bar{M} = M/K_1$. Then then the action of $\bar{M} \cong O_6^+(3) \cong L_4(3).2_2$ on $W^\perp$ is faithful and coincides with the automorphism group of the restriction of ϕ to $W^\perp$. Further $N_{G_1}(W)$ is an extension of M by an involution m which centralizes $W^\perp$ and inverts W.

There are 80 maximal totally isotropic subspaces in $W^\perp$ which form a single M-orbit and under the action of $O^2(M) \cong 3^7.L_4(3)$ they split into two orbits of length 40 each. If the elements from different orbits are assumed to be of different types, then together with the set of 1-dimensional singular subspaces contained in $W^\perp$ they form the parabolic geometry of $O^2(M)/K_1 \cong L_4(3)$ with respect to the natural incidence relation. This is the geometry of type A_3 (or D_3) over $GF(3)$ and we will denote it by $\mathcal{H}(\alpha_1, W)$.

We assume that $V(\alpha_2)$ is contained in $W^\perp$. Then $V(\alpha_3)$ is also in $W^\perp$ and there is exactly one maximal totally isotropic subspace Z in $W^\perp$ such that $V(\alpha_2) \cap Z = V(\alpha_3)$. Moreover, $V(\alpha_2)$ and Z are in different $O^2(M)$-orbits and $\{V(\alpha_2), V(\alpha_4), Z\}$ is a maximal flag in the geometry $\mathcal{H}(\alpha_1, W)$.

Let β and γ be the vertices from $\Gamma(\alpha_1)$ such that $\{\alpha_1, \beta\} = \alpha_2$, $V(\{\alpha_1, \gamma\}) = Z$ (here $\{\alpha_1, \beta\}$ and $\{\alpha_1, \gamma\}$ are edges of Γ also considered as elements of type 2 in $\mathcal{G}$). Since dim $(V(\alpha_2) \cap Z) = 2$, α_1, β and γ form a triangle in Γ by Lemma 2.17.

Lemma 3.1 *The setwise stabilizer in $C_{G_4}(W)$ of the triangle $\{\alpha_1, \beta, \gamma\}$ induces the symmetric group Σ_3 on the vertices of the triangle.*

Proof: Let Θ be the set of vertices of Γ such that for $\lambda \in \Theta$, W is contained in $O_3(G_\lambda)$ and is of plus type with respect to the unique quadratic form on $O_3(G_\lambda)$ preserved by G_λ. Since G_1 acts transitively on S^+, $C_G(W)$ acts transitively on Θ. Since $O^2(M)$ acts flag-transitively on $\mathcal{H}(\alpha_1, W)$ and $M/O^2(M)$ induces a diagram automorphism of the geometry, the result follows. $\square$

Let $a_1 = \beta$, $a_2 = \alpha_4$, $a_3 = \gamma$, $a_4 = \alpha_1$. Let P_i be the intersection of the centralizers $C_{G_{a_j}}(W)$ for all $1 \leq j \leq 4$, $j \neq i$, in other words P_i is the largest subgroup in G which centralizes W and stabilizes a_j for every $1 \leq j \leq 4$, $j \neq i$. Put $\bar{P}_i = P_i/O_3(P_i)$ and $\bar{P}_{ij} = \langle P_i, P_j \rangle / O_3(\langle P_i, P_j \rangle)$, $1 \leq i, j \leq 4$.

Lemma 3.2 *$\bar{P}_i \cong GL_2(3)$ for $1 \leq i \leq 4$. If $i \neq 2$ and $j \neq 2$ then $\bar{P}_{ij} \cong (SL_2(3) * SL_2(3)).2$ and $\bar{P}_{2i} \cong L_3(3)$ for $i = 1$, 3 and 4.*

Proof: Let $1 \leq i < j \leq 4$. By Lemma 3.1 we can assume that $i \neq 4$ and $j \neq 4$. In this case $P_i, P_j \leq O^2(M)$ where $M = C_{G_1}(W)$ as above. Then P_i and P_j are minimal parabolic subgroups associated with the (unfaithful) action of $O^2(M)$ on $\mathcal{H}(\alpha_1, W)$ and the result follows from the standard facts about the action of $L_4(3)$ on its parabolic geometry (that is on the projective space $PG(3,3)$). □

Notice that for every $1 \leq i \leq 4$, P_i is contained either in $C_{G_1}(W)$ or in $C_{G_4}(W)$ or in both.

Lemma 3.3 *Let T be the subgroup in G generated by $N_{G_1}(W)$ and $N_{G_4}(W)$. Then $T \cong \Sigma_3 \times D_4(3).\Sigma_3$.*

Proof: Let $P = \langle P_i \mid 1 \leq i \leq 4\rangle$. It is easy to see that W is the largest subgroup which is contained and normal in every P_i for $1 \leq i \leq 4$. This and Lemma 3.2 imply that $\{P_i \mid 1 \leq i \leq 4\}$ is a parabolic system of type D_4 over $GF(3)$ and by [Tit] $P/W \cong D_4(3)$. Since the 3-part of the Schur multiplier of $D_4(3)$ is trivial, $P \cong 3 \times D_4(3)$. There is an element $\tau_{13} \in M \setminus O^2(M)$ which normalizes P_2 and P_4 and permutes P_1 and P_3. By Lemma 3.1 there are similar elements τ_{14} and τ_{34} and these three elements generate Σ_3 modulo P. This shows that the centralizers of W in G_1 and G_4 generate in G a subgroup $T_1 \cong 3 \times D_4(3).\Sigma_3$. Finally T contains T_1 with index 2 and since the outer automorphism group of $D_4(3)$ (isomorphic to Σ_4) does not contain $\Sigma_3 \times 2$, the result follows. □

4 Reconstruction of $2 \times M(23)$

In this section we reconstruct in G a subgroup Q isomorphic to the centralizer $2 \times M(23)$ of a 3-transposition in $M(24)$. The reconstruction is based on enumeration by a computer of the cosets of a subgroup in a group given in terms of generators and relations.

We start with the Steinberg presentation for $D \cong D_4(3)$ to obtain generators and relations for $T \cong \Sigma_3 \times D_4(3).\Sigma_3$.

We use the numbering for the fundamental system $\Pi = \{1, 2, 3, 4\}$ in the root system of type D_4 as in the previous section, so that 2 is the central node in the Dynkin diagram. Then the positive roots correspond to multisets from the following family (here, for instance, 12234 stays for the multiset $\{1, 2, 2, 3, 4\}$):

$$\mathcal{J} = \{1, 2, 3, 4, 12, 23, 24, 123, 124, 234, 1234, 12234\}.$$

With every $I \in \mathcal{J}$ a subgroup X_I of order 3 in $D \cong D_4(3)$ (a root subgroup) is associated so that such subgroups taken for all $I \in \mathcal{J}$ generate a Sylow 3-subgroup of D. The root subgroups satisfy the following commutator relations:

$$[X_I, X_J] = \begin{cases} X_{I \cup J}, & \text{if } I \cup J \in \mathcal{J}; \\ 1, & \text{otherwise.} \end{cases}$$

Here $\cup$ is understood as the union operation on multisets.

For every fundamental root $i \in \Pi$ we have a Weyl element w_i in D, which is of order 4 and together with X_i generate in D a subgroup isomorphic to $SL_2(3)$. In addition for $I \in \mathcal{J}$ and $I \neq \{i\}$ we have

$$X_I^{w_i} = \begin{cases} X_{I \cup \{i\}}, & \text{if } I \cup \{i\} \in \mathcal{J}; \\ X_{I \setminus \{i\}}, & \text{if } I \setminus \{i\} \in \mathcal{J}; \\ X_I, & \text{otherwise.} \end{cases}$$

The product of two Weyl elements w_i and w_j is an element of order 3 if i and j are adjacent in the Dynkin diagram of D_4 and they commute otherwise.

To obtain a concrete realization of the Steinberg presentation for $D_4(3)$, in each root subgroup X_I for $I \in \mathcal{J}$ we have to choose a generator x_I (a root element) and the concrete form of the relations depends on this choice. We have chosen the generators so that the non-trivial commutator relations among the root elements are determined by the following ones:

$$[x_1, x_2]x_{12}, \quad [x_3, x_2]x_{23}, \quad [x_1, x_{23}]x_{123}, \quad [x_1, x_{24}]x_{124}, \quad [x_1, x_{234}]x_{1234}, \quad [x_2, x_4]x_{24},$$

$$[x_2, x_{1234}]x_{12234}, \quad [x_3, x_{24}]x_{234}, \quad [x_3, x_{124}]x_{1234}, \quad [x_4, x_{23}]x_{234}^{-1}, \quad [x_4, x_{123}]x_{1234}^{-1},$$

$$[x_{234}, x_{12}]x_{12234}, \quad [x_{124}, x_{23}]x_{12234}, \quad [x_{123}, x_{23}]x_{12234}, \quad [x_{12}, x_4]x_{124} \quad , [x_{12}, x_3]x_{123}^{-1}.$$

For $1 \leq i \leq 4$ the $SL_2(3)$-subgroup generated by x_i and w_i is given by the following relations:

$$(x_i^{w_i} x_i^{-1})^4, \quad [(x_i^{w_i} x_i^{-1})^2, x_i], \quad (x_i^{w_i} x_i^{-1})^2 w_i^2, \quad w_i^{-1} x_i x_i^{w_i} x_i.$$

The center of $\langle x_i, w_i \rangle$ is generated by w_i^2.

For $1 \leq i \leq 4$ and $I \in \mathcal{J}$ the Weyl element w_i commutes with the root element x_I if $I \cup \{i\} \notin \mathcal{J}$ and $I \setminus \{i\} \notin \mathcal{J}$ and the non-trivial conjugation relations are determined by the following:

$$x_2^{w_1} x_{12}^{-1}, \quad x_{23}^{w_1} x_{123}^{-1}, \quad x_{24}^{w_1} x_{124}^{-1}, \quad x_{234}^{w_1} x_{1234}^{-1}, \quad x_1^{w_2} x_{12}, \quad x_3^{w_2} x_{23}, \quad x_{1234}^{w_2} x_{12234}^{-1}, \quad x_4^{w_2} x_{24}^{-1},$$

$$x_{12}^{w_3} x_{123}^{-1}, \quad x_2^{w_3} x_{23}^{-1}, \quad x_{24}^{w_3} x_{234}^{-1}, \quad x_{124}^{w_3} x_{1234}^{-1}, \quad x_2^{w_4} x_{24}, \quad x_{23}^{w_4} x_{234}, \quad x_{12}^{w_4} x_{124}, \quad x_{123}^{w_4} x_{1234}.$$

The above relations (where it is also assumed that $x_I^3 = w_i^4 = 1$ for $I \in \mathcal{J}$ and $1 \leq i \leq 4$) constitute the Steiberg presentation for the universal Lie group of type D_4 over $GF(3)$, that is of an extension of $D_4(3)$ by a center of order 4. In order to obtain a presentation for the simple group we put the additional relations

$$w_1^2 = w_3^2 = w_4^2.$$

Now to obtain a presentation for $T \cong \Sigma_3 \times D_4(3).\Sigma_3$ we have to add automorphisms of D inducing Σ_3 and elements generating another copy of Σ_3 as a direct factor. It is clear that $D = \langle x_i, \ w_j \mid 1 \leq i \leq 4, \ 1 \leq j \leq 4 \rangle$ so it is sufficient to define the automorphisms on the x_i and w_j. The automorphisms generating Σ_3 correspond to symmetries of the Dynkin diagram and can be generated by elements t and ν satisfying the relations:

$$t^2,\quad w_1^t w_3^{-1},\quad [t,w_2],\quad [t,w_4],\quad x_1^t x_3^{-1},\quad [t,x_2],\quad [t,x_4]$$

and

$$\nu^3,\quad (t\nu)^2,\quad w_1^\nu w_3^{-1},\quad w_3^\nu w_4,\quad x_1^\nu x_3^{-1},\quad x_3^\nu x_4,\quad x_2^\nu x_2^{-1},\quad w_2^\nu w_2^{-1}.$$

The Σ_3 direct factor comes by adjoining the generators m and ϱ satisfying the following obvious relations, where $1 \leq i \leq 4$ and $I \in \mathcal{J}$:

$$m^2,\quad [m,w_i],\quad [m,x_I],\quad [m,t],\quad [m,\nu];$$

$$\varrho^3,\quad (\varrho m)^2,\quad [\varrho,w_i],\quad [\varrho,x_I],\quad [\rho,t],\quad [\varrho,\nu].$$

If P_i is the minimal parabolic subgroup of $D \cong D_4(3)$ as in the previous section, then $O^2(P_i)$ is generated by the root elements x_I for all $I \in \mathcal{J}$ together with the Weyl element w_i, and the root elements other than x_i generate $O_3(P_i)$, $1 \leq i \leq 4$. The elements t and ν normalize P_2 and conjugate the parabolics P_1, P_3 and P_4 in the way they conjugate the corresponding Weyl elements.

We are going to produce $Q \cong 2 \times M(23)$ as the subgroup in G generated by $C_T(m)$ and $C_{G_4}(m)$. The former of the centralizers is isomorphic to $2 \times D_4(3).\Sigma_3$; it is generated by all the above generators of T except for ϱ subject to all the defining relations of T, except for the relations which involve ϱ.

Let us discuss $C_{G_4}(m)$. First of all, it follows directly from the construction of T in the previous section that $T \cap G_4$ contains the normalizer T_2 in T of the maximal parabolic subgroup of $D_4(3)$ generated by P_1, P_3 and P_4. Since T_2 is maximal in T we have $T_2 = T \cap G_4$. In order to modify the above presentation for T into a presentation for T_2 we have to drop the generator w_2 together with all the relations which involve w_2 and to introduce a new generator h (which corresponds to the square of w_2 in T) and the following set of relations:

$$h^2;\ (w_i h)^2 \text{ for } i = 1,3,4;\quad [h,t],\ [h,\nu],\ [h,m],\ [h,\rho];$$

$$(x_I h)^2, \text{ if } I \cup \{2\} \in \mathcal{J} \text{ or } I \setminus \{2\} \in \mathcal{J} \text{ and } [x_I, h], \text{ otherwise.}$$

Similarly by dropping the generator ϱ together with the relations which involve ϱ we obtain a presentation for $C_{T_2}(m)$.

One can see in [Atlas] that $C_{T_2}(m) \cong \langle m\rangle \times 3_+^{1+8} 2_-^{1+6} 3_+^{1+2}(2 \times \Sigma_3)$, in particular $O_3(C_{T_2}(m))$ is extraspecial of order 3^9. Let us locate m and $C_{T_2}(m)$ in G_4. Comparing the 3-parts in the orders of $C_{T_2}(m)$ and $U_5(2) : 2 \times 2 \cong G_4/O_3(G_4)$ we see that $O_3(C_{T_2}(m))$ is contained in $O_3(G_4)$ which means that m has 8-dimensional centralizer in $O_3(G_4)/Z(O_3(G_4)) \cong 3^{10}$. Comparing this with [JLPW], we conclude that m is the product of an involution from $O_{3,2}(G_4)$ (acting fixed-point freely on $O_3(G_4)/Z(O_3(G_4))$ and a central involution from $O^2(G_4) \cong 3_+^{1+10}.U_5(2)$. This shows (compare [Atlas]) that $C_{G_4}(m) \cong \langle m\rangle.3_+^{1+8} 2_-^{1+6} 3_+^{1+2} GL_2(3)$, which means that $C_{G_4}(m)$

contains $C_{T_2}(m)$ with index 4. In more details the situation is the following. Let $R = O_{3,2}(C_{G_4}(m)) \cong \langle m \rangle \times 3_+^{1+8}2_-^{1+6}$. Then $C_{G_4}(m)/R$ and $C_{T_2}(m)/R$ are, respectively, a maximal 3-local parabolic subgroup and the corresponding Borel subgroup in $\mathrm{Out}(R/O_3(R)) \cong Sp_4(3)$.

Thus $[C_{G_4}(m) : C_{T_2}(m)] = 4$ and $C_{G_4}(m) \cong \langle m \rangle.3_+^{1+8}2_-^{1+6}3_+^{1+2}GL_2(3)$ while $C_{T_2}(m) \cong \langle m \rangle \times 3_+^{1+8}2_-^{1+6}3_+^{1+2}(2 \times \Sigma_3)$. Let us introduce some subgroups in $C_{T_2}(m)$:

$$X_1 = \langle x_{12234}, x_{1234}, x_{123}, x_{124}, x_{234}, x_{23}, x_{12}, x_{24}, x_2 \rangle,$$

$$X_2 = \langle w_1, w_3, w_4, w_1^{x_1}, w_3^{x_3}, w_4^{x_4} \rangle,$$

$$X_3 = \langle x_1 x_3 x_4^{-1}, \nu, x_1 x_4 \rangle, \quad X_4 = \langle t, h, x_4 \rangle.$$

Then $X_1 = O_3(C_{T_2}(m)) \cong 3_+^{1+8}$, X_2 is a complement to $X_1 \times \langle m \rangle$ in $O_{3,2}(C_{T_2}(m))$, which is isomorphic to the central product of three copies of the quaternion group, that is to 2_-^{1+6}, $X_3 \cong 3_+^{1+2}$ and $X_4 \cong (2 \times \Sigma_3)$. So that $C_{T_2}(m) = \langle m \rangle \times X_1 X_2 X_3 X_4$. Our intention is to obtain a presentation for $C_{G_4}(m)$ by adjoining to $C_{T_2}(m)$ an element t_1 so that $\langle t_1, x_4, h \rangle (\langle m \rangle X_1 X_2 X_3)/(\langle m \rangle X_1 X_2 X_3) \cong GL_2(3)$. To do so we have to determine the action of t_1 on X_3, X_2 and X_1. After that, we will find a relation between t_1 and w_2. Eventually we obtain generators and relations for the subgroup Q in G generated by $C_T(m)$ and $C_{G_4}(m)$ and identify Q with $2 \times M(23)$ using coset enumeration.

The additional generator t_1 can be taken so that $\langle t_1, x_4, h \rangle \cong GL_2(3)$ and the following relations hold:

$$(t_1^{x_4})^2 t_1^2, \quad [t_1, t_1^{x_4}] t_1^2, \quad t_1^{x_4^{-1}} t_1 t_1^{x_4}, \quad [t_1, t], \quad t_1^h t_1.$$

Then t_1 is of order 4 and t_1^2 must be equal to t, tm, $w_1^2 t$, or $w_1^2 tm$. Since w_1^2 is in the center of $X_2 X_3$, the action of t_1 on $X_2 X_3$ is independent on the exact form of t_1^2. Since $\langle t_1, x_4, h \rangle \cong GL_3(2)$ is the full outer automorphism group of X_3, the action of t_1 on X_3 is uniquely determined and we get the following relations :

$$[x_1 x_3 x_4^{-1}, t_1], \quad (x_1 x_4)^{t_1} \nu^{-1} x_1 x_3 x_4^{-1}.$$

Now we determine the action of t_1 on X_2. One can see that $C_{X_2}(\langle t, h \rangle)/\langle w_1^2 \rangle = \langle w_1 w_3, w_1^2 \rangle / \langle w_1^2 \rangle$, which shows that $\langle t_1, t_1^{x_4} \rangle$ has to centralize $w_1 w_3$. In addition $[t_1, x_1 x_3 x_4^{-1}] = 1$, $t_1^h = t_1^{-1}$ and $t_1^2 \in \{t, tm, w_1^2 t, w_1^2 tm\}$ which imply the following relations :

$$[w_1 w_3, t_1], \quad [w_1^{x_1} w_3^{x_3}, t_1], \quad [w_1^2, t_1], \quad [t_1^{x_1 x_3}, w_1 w_3], \quad [t_1^{x_1 x_3}, w_1^{x_1} w_3^{x_3}],$$

$$[t_1^{x_1 x_3}, w_1^2], \quad (w_3^{t_1})^{-1} (w_1 w_3 w_3^{x_3} w_4^{-1} (w_4^{-1})^{x_4})^{x_1^{-1} x_3^{-1} x_4} w_1^2.$$

Now it is the right time to decide whether $t_1^2 = t$, tm, $w_1^2 t$, or $w_1^2 tm$. We can calculate the action of t and $w_1^2 t$ on $Y = X_1/\langle x_{12234} \rangle$. We get that $|C_Y(w_1^2 t)| = 3^2$ while $|C_Y(t)| = 3^6$. This shows that the images of w_1^2 and t in $U := O^2(G_4)/O_3(G_4) \cong U_5(2)$ are not conjugated. On the other hand we have the following :

Lemma 4.1 *In the above notation the images of t_1^2 and w_1^2 in U are conjugated.*

Proof: We identify t_1^2 and w_1^2 with their images in U and follow notation from [Atlas]. Then $w := w_1^2$ is a central involution (that is of type $2a$) in U and t_1 is an element of order 4 in $C_U(w)$ which commutes with the element $x := x_1x_3x_4^{-1}$ which is also contained in $C_U(w)$ and acts fixed-point freely on $O_2(C_U(w))/\langle w\rangle \cong 2^6$. Then $C_U(wx)$ has order 1 296 and hence wx is of type $6cd$. This means that x is of type $3cd$. The square of t_1^2x is an element of type $3cd$, hence t_1^2x is of type $6cd$, $6ef$ or $6jk$. In the former two cases the cube is of type $2a$ and in the latter case it is of type $2b$. But the elements of type $6jk$ are not squares in U. Hence t_1^2 is of type $2a$ and the result follows. □

Since $|C_Y(w_1^2)| = 1$, we see that w_1^2 must be conjugate to w_1^2t or w_1^2tm. As $|C_{O_3(G_4)/Z(O_3(G_4))}(w_1^2)| = 3^2$ and $[\rho, w_1^2t] = 1$, we see that $|C_{O_3(G_4)/Z(O_3(G_4))}(w_1^2t)| \geq 3^3$. This gives that w_1^2 is conjugate to w_1^2tm and then

$$t_1^2 = w_1^2tm.$$

Next we determine the action of t_1 on X_1. Set $Y_1 = [X_1, w_1^2t]$. Then

$$Y_1 = \langle x_{1234}, x_{123}, x_{124}x_{234}, x_{12}x_{23}, x_{24}, x_2\rangle.$$

Since $t_1^2 = w_1^2tm$, $h^{t_1} = hw_1^2tm$, $[x_1x_3x_4^{-1}, t_1] = 1$, and $t_1^{x_4^{-1}}t_1t_1^{x_4} = 1$ we get the relations:

$$x_{12234}^{t_1}x_{12234}^{-1},\quad x_{1234}^{t_1}x_{123}x_{234}x_{124},\quad (x_{124}x_{234})^{t_1}(x_{1234}(x_{12}x_{23})^{-1}x_{24}^{-1})^{-1},$$

$$(x_{12}x_{23}x_{24})^{t_1}x_{123},\quad x_{24}^{t_1}(x_{124}x_{234}x_2^{-1})^{-1},\quad x_2^{t_1}(x_{1234}x_{23}^{-1}x_{12}^{-1})^{-1}.$$

Now using the fact that $O_2(\langle t_1, x_4, h\rangle)$ acts trivially on $C_{X_1}(w_1^2tm)$, we see that

$$[t_1, x_{124}x_{234}^{-1}] = 1 = [t_1, x_{12}x_{23}^{-1}].$$

This completes the set of relations presenting $C_{G_4}(m)$.

To generate the subgroup $Q = \langle C_T(m), C_{G_4}(m)\rangle$ we need all the generators of T except for ϱ together with the element t_1. Nevertheless, the relations given above still define an infinite group, namely the free amalgamated product of $C_T(m)$ and $C_{G_4}(m)$. In fact, for each of the above relations, all the generators involved in the relation are contained either in T or in G_4 and none of the relations involve both t_1 (which is not in T) and w_2 (which is not in G_4). To find the missing relation we are going to analyse the parabolic subgroup G_3 associated with the action of G on $\mathcal{G}$.

First we are going to identify the intersections $G_i \cap T$ for $1 \leq i \leq 4$. It follows directly from the construction of T in Section 3, that $G_1 \cap T$ is the normalizer in T of the subgroup generated by the minimal parabolics P_1, P_2 and P_3. This means that $G_1 \cap T$ is generated by all the generators of T except for w_4 and ν. Moreover, $O_3(G_1)$ is contained in $G_1 \cap T$ and it is generated by the root elements x_I such that I contains

4, and ϱ. Similarly, directly from the construction of T we see that $G_2 \cap T$ is the normalizer in T of the subgroup generated by P_2 and P_3; $O_3(G_2) \cap T$ is generated by the root elements x_I such that I contains either 1 or 4 or both of them. This means that $O_3(G_2) \cap T$ is of order 3^9 and hence its index in $O_3(G_2)$ is 3^4. The element α_2 is identified with the edge $\{\alpha_1, \beta\}$ of the collinearity graph Γ of $\mathcal{G}$. Clearly $G_\beta \cap T$ is the normalizer in T of the subgroup generated by the minimal parabolics P_2, P_3 and P_4, so that $O_3(G_\beta)$ is generated by the root elements x_I for I containing 1. This and Lemma 2.16 imply that $E := \langle x_{124}, x_{1234}, x_{12234}, \varrho\rangle$ is the intersection of $O_3(G_1)$ and $O_3(G_\beta)$, hence $N_{G_i}(E) = G_i \cap G_2$ for $i = 1$, 3 and 4. Moreover it is easy to see, that in terms introduced in Section 2 we have $V(\alpha_2) = \langle x_{124}, x_{1234}, x_{12234}\rangle$.

We have already observed that $G_4 \cap T$ is the normalizer in T of the subgroup generated by P_1, P_2 and P_3, in particular $Z(O_3(G_4)) = \langle x_{12234}\rangle$.

Let us consider the intersection $G_3 \cap T$. We know that G_3 is the setwise stabilizer in G of the subgraph Φ in the collinearity graph Γ of $\mathcal{G}$, induced by the vertices (elements of type 1) incident to α_3. It follows directly from the diagram of $\mathcal{G}$ that Φ is a complete subgraph on 5 vertices and that it contains α_1 and β. Since G_{13} is the full preimage in G_1 of the stabilizer in $\bar{G}_1 \cong O_7(3)$ of an isotropic 2-dimensional subspace in V, one can see (compare [Atlas]) that $G_{13}/O_3(G_{13}) \cong \Sigma_4 \times GL_2(3)$ and that G_{13} induces the natural action of Σ_4 on $\Phi \setminus \{\alpha_1\}$. This implies that $G_3/O_3(G_3) \cong \Sigma_5 \times GL_2(3)$ and that G_3 induces on Ξ the natural action of Σ_5. By Lemma 2.17 Φ is a maximal clique in Γ and each triangle in Γ is contained in a unique clique of size 5 which is an image of Φ under G. So without loss of generality we assume that the triangle $\{\alpha_1, \beta, \gamma\}$ is contained in Φ (cf. the terms introduced before Lemma 3.1). Moreover this triangle is the intersection of Φ with the subgraph Θ defined in the proof of Lemma 3.1. This means that $G_3 \cap T$ is the setwise stabilizer in G_3 of the triangle $\{\alpha_1, \beta, \gamma\}$ and $(G_3 \cap T)O_3(G_3)/O_3(G_3) \cong GL_2(3) \times \Sigma_3 \times 2$. Similarly $G_3 \cap T$ is the setwise stabilizer of $\{\alpha_1, \beta, \gamma\}$ in T, which means that $G_3 \cap T$ is the normalizer of P_2 in T. This shows that $\langle x_2, w_2\rangle$ is contained in $G_3 \cap T$ and it maps isomorphically into the $GL_2(3)$-direct factor of $G_3/O_3(G_3)$. Notice that ν is not in $O_3(G_3)$ since it rotates the vertices α_1, β and γ. Thus $O_3(G_3) \cap T$ is generated by the root elements x_I for $I \in \mathcal{J} \setminus \{2\}$ together with ϱ. In particular $O_3(G_3) \cap T$ is of index 3^2 in $O_3(G_3)$.

Let $V(\alpha_3)$ be the subspace of dimension 2 in $V = O_3(G_1)$ as introduced in Section 2, such that $G_1 \cap G_3 = N_{G_1}(V(\alpha_3))$. Since $O_3(G_3)$ is contained in G_1, we can calculate in the latter group that $V(\alpha_3)$ is the center of $O_3(G_3)$. Using the relations for T one can see that $\langle x_{12234}, x_{1234}\rangle$ is the center of $O_3(G_3) \cap T$. This shows that

$$Z_3 := Z(O_3(G_3)) = V(\alpha_3) = \langle x_{12234}, x_{1234}\rangle$$

and that $\langle x_2, w_2\rangle$ acts faithfully on Z_3.

Put $S = C_{O_3(G_3)}(\langle x_2, w_2\rangle)Z_3$. One can calculate in G_1 that S is of order 3^6 and that $S/Z_3 = Z(O_3(G_3)/Z_3)$. This and direct calculations with the relations for T, show that $S = \langle x_{12234}, x_{1234}, x_{234}, x_{123}, x_{124}, \varrho\rangle$ and that $[S, O_3(G_3)] = Z_3$.

We assume that $\Phi = \{\alpha_1, \beta, \gamma, \delta, \epsilon\}$. Notice that every pair of vertices from Φ form an edge of Γ and hence can be considered as an element of type 2 in $\mathcal{G}$ incident to α_3. Clearly an automorphism from $O_3(G_2) \setminus O_3(G_3)$ stabilizes α_1 and β, permuting γ, δ and ϵ. This means that $\langle \nu \rangle O_3(G_3)/O_3(G_3)$ and $O_3(G_2)O_3(G_3)/O_3(G_3)$ both are Sylow 3-subgroups in the Σ_5-direct factor of $G_3/O_3(G_3) \cong \Sigma_5 \times GL_2(3)$.

Let $W := C_{S/Z_3}(\langle \nu \rangle)Z_3$. Then direct calculations with the relations for T show that

$$W = C_{S/Z_3}(\langle \nu \rangle)Z_3 = \langle x_{12234}, x_{1234}, x_{234}x_{123}x_{124}, x_{124}^{-1}x_{234} \rangle.$$

As above let $E = O_3(G_1) \cap O_3(G_\beta)$. Then E is contained S, contains Z_3 and $O_3(G_3)$ centralizes E/Z_3. This and order reason give

$$E = C_{S/Z_3}(O_3(G_2))Z_3 = \langle x_{124}, x_{1234}, x_{12234}, \varrho \rangle.$$

Since $\langle \nu \rangle O_3(G_3)/O_3(G_3)$ and $O_3(G_2)O_3(G_3)/O_3(G_3)$ are Sylow 3-subgroups in the Σ_5-direct factor of $G_3/O_3(G_3)$, there is an element $y \in G_3 \cap G_4$ which conjugates W to E and by Frattini argument we can choose y so that $[y, w_2] = 1$. Since $[t_1, W] \subseteq W$ and $G_4 \cap G_2 = N_{G_4}(E)$ we see $t_1^y \in G_2$ and $o(t_1^y w_2) = o(t_1 w_2)$.

The relations for T and $C_{G_4}(m)$ describe the actions of w_2 and t_1 on W. Comparing these actions we observe that the product $w_2 t_1$ induces on W an action of order 3 which means that $(w_2t_1)^3$ centralizes W. Hence $(w_2t_1^y)^3$ is in $C_{G_2}(E)$. The structure of G_2 shows that $C_{G_2}(E)$ is contained in $O_3(G_2)$ and has order 3^{10}. On the other hand the relations for T show that the subgroup

$$U_1 = \langle x_{12234}, x_{1234}, x_{234}, \varrho, x_{124}, x_{123}, x_{24}, x_{23}, x_3, x_4 \rangle$$

has order 3^{10} and centralizes W. Hence $U_1 = C_{G_2^{y^{-1}}}(W)$ and so $(w_2t_1)^3 \in U_1$. We know that $\langle w_1, w_2 \rangle \cong \Sigma_4$ and so $w_1^{2^{w_2}} = w_1^2 w_2^2 = w_1^2 h$. Further $(w_2t_1)^h = w_2 t_1 w_1^2 tm$. As t and m both centralize w_2 and t_1, we see $((w_2t_1)^3)^h = (w_2t_1w_1^2)^3 tm$. Now

$$w_2t_1w_1^2w_2t_1w_1^2w_2t_1w_1^2 = w_2t_1w_2w_1^2ht_1w_1^2w_2t_1w_1^2 =$$
$$w_2t_1w_2t_1w_1^2tmhw_2t_1w_1^2 = (w_2t_1)^3tm.$$

So we have that $[h, (w_2t_1)^3] = 1$. Hence $(w_2t_1)^3 \in C_{U_1}(h) = \langle x_{234}, x_{123}, x_{124}, \varrho \rangle$. As both w_2 and t_1 commute with m we see that $(w_2t_1)^3 \in \langle x_{234}, x_{123}, x_{124} \rangle$. As w_2t_1 centralizes $(w_2t_1)^3$, we have $(w_2t_1)^3 \in \langle x_{124}^{-1}x_{234} \rangle$. But t centralizes w_2t_1 and inverts $x_{124}^{-1}x_{234}$, so we finally get

$$(w_2t_1)^3 = 1.$$

Now the subgroup $Q = \langle C_T(m), C_{G_4}(m) \rangle$ is generated by t_1 and all the generators of T except for ϱ. These generators satisfy the relations given above. The coset enumeration with a computer program implementing the Todd-Coxeter algorithm has shown that the presentation defines a group whose order is equal to the order of $2 \times M(23)$ (we enumerated the cosets of $C_T(m)$ in Q and have obtained the index 137 632). Since the relations are uniquely determined by the conditions on the geometry and since $M(24)$ acts as the full automorphism group of a geometry of the considered type, we have the following result.

Lemma 4.2 *Let Q be the subgroup in G generated by the subgroups $C_T(m)$ and $C_{G_4}(m)$. Them $Q \cong 2 \times M(23)$.* $\square$

5 The transposition graph

In this section we complete the proof of Theorem 1 by showing that the full automorphism group G of the geometry $\mathcal{G}$ under consideration is isomorphic to the Fischer group $M(24)$. We will achieve this by considering a graph Λ on the set of cosets in G of the subgroup $Q \cong 2 \times M(23)$. We will establish the isomorphism between Λ and the graph on the set of 3-transpositions of $M(24)$ in which two transpositions are adjacent if they do not commute (i.e. if their product is of order 3).

As above let $W = \langle \rho \rangle$ be the subgroup of order 3 in $V = O_3(G_1)$ which is of plus type with respect to the orthogonal form on V preserved by G_1/V and $T = \langle N_{G_1}(W), N_{G_4}(W) \rangle$. Then $T \cong \Sigma_3 \times D_4(3).\Sigma_3$ by Lemma 3.3. Let m be an involution in T which commutes with the subgroup $D \cong D_4(3)$ in T and let $Q = \langle C_T(m), C_{G_4}(m) \rangle$. Then $Q \cong 2 \times M(23)$ by Lemma 4.2 and it is straightforward that $Q \cap T = C_T(m) \cong 2 \times D_4(3).\Sigma_3$, so that $[T : T \cap Q] = 3$.

Let $R = O_3(G_4)C_{G_4}(m)$. Then directly from the construction of Q in the previous section we have the following :

Lemma 5.1 *The following assertions hold.*

(i) $R \cap T \cong \Sigma_3 \times 3_+^{1+8}2_-^{1+6}3_+^{1+2}(2 \times \Sigma_3)$ *and* $[R : R \cap T] = 12$;

(ii) $R \cap Q \cong 2 \times 3_+^{1+8}2_-^{1+6}3_+^{1+2}GL_2(3)$ *and* $[R : R \cap Q] = 9$;

(iii) $W \leq O_3(G_4)$ *and* $|W \cap Q| = 1$. □

Define Λ to be a graph of the set of (right) cosets of Q in G in which two cosets are adjacent if and only if they intersect a common coset of T in G. Let $\lambda \in \Lambda$ be the coset containing the identity element so that $G_\lambda = Q$. Let Θ be the orbit of T on the vertex set of Λ which contains λ. Since $[T : T \cap Q] = 3$, Θ contains besides λ exactly two vertices, say μ and ν. Then $G_\lambda \cap G_\mu$ contains the subgroup $D \cong D_4(3)$ from T. As one can check with [Atlas], every proper subgroup in Q containing D is contained in $T \cap Q$. This shows that $G_\lambda \cap G_\mu \leq T$ and we have the following.

Lemma 5.2 (i) *The valency of* Λ *is* $275\,264 = 2 \cdot [Q : Q \cap T]$;

(ii) Θ *induces a triangle in* Λ *and* T *is its full stabilizer in* G;

(iii) $\langle W, m \rangle \cong \Sigma_3$ *acts faithfully on* Θ. □

Let Ω be the orbit of R on the vertex set of Λ which contains λ. By Lemma 5.1 (iii) and Lemma 5.2 (iii), Ω contains Θ and by Lemma 5.1 (ii) $|\Omega| = 9$. Let L be the kernel of the action of R on Ω. By Lemma 5.1 (ii) we see that $O_3(R)/(L \cap O_3(R))$ is elementary abelian of order 9 and acts regularly in Ω. Hence $R/O_3(R)L$ is a subgroup in $GL_2(3) \cong Aut(O_3(R)/(L \cap O_3(R)))$. On the other hand L is contained and normal in both $R \cap T$ and $R \cap Q$, by Lemma 5.1 this gives the following :

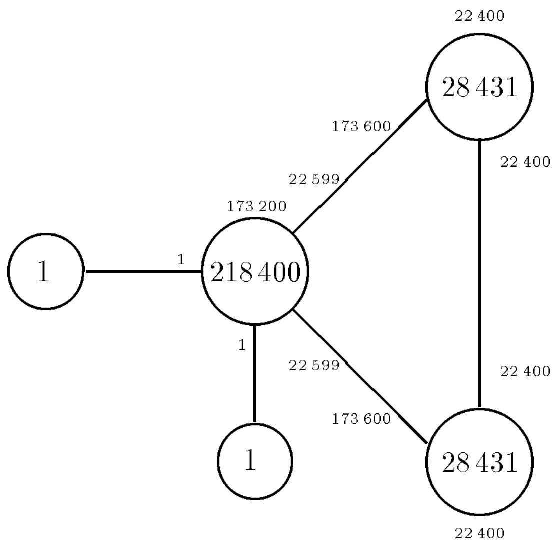


Figure 1: Distribution diagram of $M(23)$ acting of $D_4(3).3$

Lemma 5.3 *Let $N = R/O_3(R)L$. Then $N \cong GL_2(3)$ and $\langle m \rangle$ maps onto the center of N.* □

Let Q' be the commutator subgroup of Q, so that $Q' \cong M(23)$ and $Q = \langle m \rangle \times Q'$. As a direct consequence of the above lemma we have the following :

Lemma 5.4 (i) *The subgraph in Λ induced by Ω is complete and R induces a doubly transitive action on Ω;*

(ii) *both $Q \cap R$ and $Q' \cap R$ induce $GL_2(3)$ on $\Omega \setminus \{\lambda\}$;*

(iii) *Q' acts transitively on the set $\Lambda(\lambda)$ of vertices adjacent to λ and the stabilizer in Q' of $\mu \in \Lambda(\lambda)$ is isomorphic to $D_4(3).3$;*

(iv) *if $\sigma \in \Omega \setminus \Theta$, then the order of $Q'_\mu \cap Q'_\sigma$ is divisible by $3^{12} \cdot 2^7$.* □

Lemma 5.5 *$G_\lambda \cap G_\mu$ acting on $\Lambda(\lambda) \setminus \{\lambda\}$ has exactly 4 orbits: $\Lambda^1(\lambda, \mu) = \{\nu\}$, $\Lambda^2(\lambda, \mu)$, $\Lambda^3(\lambda, \mu)$ and $\Lambda^4(\lambda, \mu)$ with lengths 1, 218 400, 28 431 and 28 431, respectively. The intersection diagram of the graph on $\Lambda(\lambda)$ associated with $\Lambda^2(\lambda, \mu)$ is given on Figure 1 and $\Omega \setminus \Theta \subseteq \Lambda^2(\lambda, \mu)$.*

Proof: By Lemma 5.2 (i) $G_\lambda = Q$ acts on $\Lambda(\lambda)$ as it acts on the cosets of an index 2 subgroup in its maximal subgroup isomorphic to $2 \times D_4(3).\Sigma_3$ and by Lemma 5.4 (ii) Q' acts transitively on $\Lambda(\lambda)$. This specifies the action under consideration up to similarity. The subdegrees of this action and the intersection diagrams can be found in [PS]. It is clear that ν is stabilized by $G_\lambda \cap G_\mu$. If $\sigma \in \Omega \setminus \Theta$ then Lemma 5.4 (iv) and straightforward congruences show that $\sigma \in \Lambda^2(\lambda, \mu)$. □

Lemma 5.6 *Let Ξ be the graph on the set of 3-transpositions of $M(24)$ in which two transpositions are adjacent if they are different and commute. Then Λ is isomorphic to the complement of Ξ and $G \cong M(24)$.*

Proof: For $\alpha \in \Lambda$ let m_α be the unique non-trivial element in the center of $G_\alpha \cong 2 \times M(23)$, so that $m = m_\lambda$. If $\beta \in \Lambda(\alpha)$ then the product of m_α and m_β has order 3. On the other hand it is clear that Q contains elements conjugated to m and different from m. Hence m_α and m_β commute for some vertices α and β of Λ and the diameter of Λ is at least 2. Let α be a vertex adjacent to μ and at distance 2 from λ and let Φ the orbit of Q containing α. Then by Lemma 5.5 Φ contains $\Lambda^3(\lambda, \mu)$ or $\Lambda^4(\lambda, \mu)$ or both. The intersection diagram of the graph on $\Lambda(\lambda)$ associated with $\Lambda^2(\lambda, \mu)$ shows that α is adjacent to 173 600 vertices from $\Lambda^2(\mu, \lambda) \subseteq \Lambda(\lambda)$. Hence Φ contains at most $275\,264 \cdot (28\,431 + 28\,431)/173\,600 < 90\,162$. It is clear that Q induces a non-trivial action on Φ. Comparing this with the indices of maximal subgroups in $Q' \cong M(23)$ [Atlas] we conclude that Q' acting on Φ has one or two orbits of length 31 671 and the action on each of the orbits is similar to the action of $M(23)$ on the cosets of its subgroup $2 \cdot M(22)$. Suppose that $|\Phi| = 2 \cdot 31\,671$. Since α is adjacent to at least 173 600 vertices from $\Lambda(\lambda)$, in this case $\Lambda^3(\mu, \lambda) \cup \Lambda^4(\mu, \lambda) \subseteq \Phi$ and α is adjacent to 247 104 vertices from $\Lambda(\lambda)$. We may assume that $\alpha \in \Lambda^3(\mu, \lambda)$. Then the intersection diagram shows that α is adjacent to 22 400 vertices from $\Lambda^3(\mu, \lambda)$ and to 22 400 vertices from $\Lambda^4(\mu, \lambda)$. Since $247\,104 + 2 \cdot 22\,400 = 291\,904$ is larger than the valency of Λ, which is 275 264, this is impossible. Hence $|\Phi| = 31\,671$ and Q' acts transitively on Φ, $\Lambda(\mu) \cap \Phi = \Lambda^3(\mu, \lambda)$ and α is adjacent to 247 104 vertices from $\Lambda(\lambda)$. Comparing this with the valency of Λ we see that α is adjacent to 28 160 vertices outside $\Lambda(\lambda)$. The subdegrees of the action of $M(23)$ on the cosets of $2 \cdot M(22)$ are well known to be 1, 3 510 and 28 160. Since α is adjacent to 22 400 vertices from $\Lambda^3(\mu, \lambda)$ we see that $\Lambda = \{\lambda\} \cup \Lambda(\lambda) \cup \Phi$. Now the complement of Λ has valency 31 671 and locally it is the commuting graph of transpositions in $M(23)$. Also for any two vertices α and β in Λ the product $m_\alpha m_\beta$ has order at most 3, which means that $\{m_\alpha \mid \alpha \in \Lambda\}$ is a class of 3-transpositions in G and to complete the proof we can apply one of the available characterizations of $M(24)$, for instance [BW]. □

References

[BCN] A. Brouwer, A. Cohen and A. Neumaier, Distance Regular Graphs, Springer, Berlin 1989.

[BF] F. Buekenhout and B. Fischer, A locally dual polar space for the Monster, Unpublished manuscript.

[BW] J. van Bon and R. Weiss, A characterization of the groups Fi_{22}, Fi_{23} and Fi_{24}, Forum Math. **4** (1992), 425–432.

[Atlas] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, Alas of Finite Groups, Clarendon Press, Oxford, 1985.

[IM] A.A. Ivanov and U. Meierfrankenfeld, A characterization of the 3-local geometry of the Monster, Preprint, 1995.

[JLPW] Ch. Jansen, K. Lux, R. Parker and R. Wilson, An Atlas of Brauer Characters, Clarendon Press, Oxford, 1995.

[Küs] A.S. Küsefoglu, The second cohomology of finite orthogonal groups, I and II, J.Algebra, **56** (1979), 207–220 and **67** (1980), 88–109.

[Meie1] U. Meierfrankenfeld, Eine Lösung des Pushing-up Problems für eine Klasse endlicher Gruppen, Ph. D. Thesis, Bielefeld, 1986.

[Meie2] U. Meierfrankenfeld, Privite communication, 1994.

[Meix] T. Meixner, Failure of factorization modules for Lie type groups in odd characteristic, Comm. Algebra, **19** (1991), 3193–3223.

[Pas] A. Pasini, Geometries and Diagrams, Clarendon Press, Oxford, 1994.

[PS] C.E. Praeger and L.H. Soicher, Low rank representations and graphs for sporadic groups, Preprint 1994.

[RSt] M.A. Ronan and G. Stroth, Minimal parabolic geometries for the sporadic groups, Europ. J. Combin. **5** (1984), 59–91.

[Tit] J. Tits, A local approach to buildings, In: The Geometric Vein (Coxeter - Festschrift), pp. 519–547, Springer Verlag, Berlin, 1982.

[Yosh] S. Yoshiara, On some extended dual polar spaces, I, Europ. J. Combin. **15** (1994), 73–86.

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