Sensitivity analysis in integer linear programming: the right hand side case

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Abstract

An algorithm based on a special duality concept is used to solve an integer linear programming problem and to perform sensitivity analysis for changes of the right-hand side vector. An optimality criterion that easily can be checked is formulated. Dual prices are introduced in order to identify inactive restrictions.

1 Introduction

In general, methods for solving optimization problems assume that coefficients are known exactly. Almost all coefficients of practical mathematical problems are estimated with errors. A purpose of sensitivity analysis is to answer the question: How much can data be variied without destroying an optimal solution?
In linear programming (LP) exists a well-developed theory and practical application is possible, e.g., see Dinkelbach [1] or Gal [2]. The approaches are frequently based on an optimality condition which is necessary and sufficient.
In general, there doesn't exist such a simple optimality test in integer programming, unfortunately. Several efforts have been made in order to handle the problem, most of them in relation to specific algorithms, e.g., see Holm/Klein [4], Rountree/Gillett [6] or Schrage/Wolsey [7]. This paper deals with a new approach based on a special solution technique. First we describe a special duality approach and give an outline of a related algorithm. (For some basic results see Köhler [5].) Changes of the right- hand side will be considered and dual prices will be proposed. Using a simple criterion the numerical effort to prove optimality is not expensive and the dual prices are a by-product of the algorithm. A small example to illustrate the method is given.

2 A special duality approach and an algorithm

We consider the mixed-integer problem

$$(P) \quad \max \quad c^{1T}x^1 + c^{2T}x^2$$

$$A^1x^1 + A^2x^2 \leq b$$

$$x^2 \geq 0$$

$$x^1 \in \mathbb{N}^q$$

with $A^1 \in \mathbb{R}^{mxq}$, $A^2 \in \mathbb{R}^{mx(n-q)}$ and all other terms of conformable size. ($\mathbb{R}$ is the real number line, the set of nonnegative integers is denoted by $\mathbb{N}$.)

The problem

$$(D) \quad \max_{x^1} \quad \min_{u} \quad c^{1T}x^1 + u^T(b - A^1x^1)$$

$$A^{2T} \quad u \geq 0$$

$$u \geq 0$$

$$x^1 \in \mathbb{N}^q$$

will be called the dual of (P). There hold theorems of "strong duality" and "complementary slackness". It should be remarked that $A^{2T}u \geq c^2$ contains only continuous variables.
A two-step-algorithm for solving the primal problem (P) is developed in Köhler [5].

Algorithm:

Step 1: Solve the q linear programming problems (each one with a single integer parameter)

$$\tilde{z}_j = \max_{tj} \quad \min_{u} \quad u^T(b - a_jt_j) + c_jtj$$

$$A_j^T u - w = d$$

$$u, w \geq 0 \qquad (\mathrm{SP}^j)$$

$$t_j \in \mathbb{N},$$

where $A_j(d \quad resp.)$ is derived from $A(c)$ by deleting $j-th$ column (component) and $j = 1, \ldots, q$. Determine intervals I_j where the optimal value of x_j^1 will be found. The variables x^2 will be fixed by solving a LP problem.

Step 2: If $(x^1, x^2)^T$ is not optimal determine new

values for x_j^1 from interval $I_j (j = 1, \ldots, q)$ and

compute x^2!

The algorithm terminates because of the intervals I_j are finite. For more details see Köhler [5].

3 Changes of the right-hand side vector

We consider the pure integer programming problem with variable right-hand side

$$\begin{aligned} \max \quad & c^T x \\ Ax - \triangle b & \leq b \\ \triangle b & \leq s \\ \triangle b & \geq 0 \\ x & \in \mathbb{N}^n \end{aligned} \tag{VRHS}$$

(VRHS) is a special case of (P). Solving (VRHS) with the algorithm we determine intervals I_j which bounds are linear functions in s, denoted by

$$I_j = [f_j^1(s); f_j^2(s)].$$

Setting $s = 0$ (no changes of the right-hand side) we find a solution of the original problem. Now we can formulate conditions for saving the optimal solution if b varies.

THEOREM 1: An optimal solution x_0 of (VRHS) with $s = 0$ keeps optimality for all $s \geq 0$ with

$$\lceil f_j^1(0) \rceil - 1 < f_j^1(s) \leq x_{oj} \leq f_j^2(s) < \lfloor f_j^2(0) \rfloor + 1 \qquad (C)$$
$$(j = 1, \ldots, n).$$

Remark 1: The symbols $\lfloor \quad \rfloor$ and $\lceil \quad \rceil$ denote the integer parts "floor"and "ceiling" of a real number.

Remark 2: In the neighbourhood of I_j intervals containing no integer number for $s = 0$ can exist, which represent the optimal interval for some $s \neq 0$.

If step 1 of the algorithm is optimal we can strength the result of theorem 1. See example of part 5.

We can sharpen the theorem 1 in the following way:

THEOREM 2: Let x_0 be an optimal solution of (VRHS) with $x_{0j}^1 = \lceil f_j^1(0) \rceil$ for $c_j^1 < 0$ and $x_{0j}^1 = \lfloor f_j^2(0) \rfloor$ for $c_j^1 \geq 0$. The solution x_0 keeps optimality for $s \geq 0$ with $x_{0j} - 1 < f_j^1(s) \leq x_{oj} \leq f_j^2(s)$ for $c_j < 0$ and $f_j^1(s) \leq x_{0j} \leq f_j^2(s) < x_{0j} + 1$ otherwise.

4 Dual prices

Dual prices are an useful tool in linear programming. They are a measure for influence of a restriction to an optimal solution. Several author attached the problem of re-computing dual prices if the optimum is reached using Gomory's cutting plane algorith (see Gomory/Baumol [3], Uebe [8], et al.). Unfortunately, there are several difficulties as numerical effort or economical interpretation.
We present dual prices which are a by-product of the algorithm.

Definition: Let u_i^k be the $i-th$ component of an optimal solution to the $k-th$ subproblem (SP^k).

$u_i^* := \max\limits_{k=1,\ldots,q} u_i^k$ will be called dual price of the $i-th$ constraint.

These prices have two advantages:

1. They are unique because of values of the variables belonging to the optimal interval are unique.

2. If $u_i^* = 0$ holds small changes of the right-hand side of the $i-th$ constraint don't influence the optimal solution.

These prices are relative prices.

5 Numerical example

We consider the problem

$$
\begin{aligned}
\max \quad z = -x_1 + 4x_2 \\
2x_1 + 4x_2 - \Delta b_1 \leq 7 \\
10x_1 + 3x_2 - \Delta b_2 \leq 15 \\
\Delta b_1 \leq s_1 \\
\Delta b_2 \leq s_2 \\
\Delta b_1, \Delta b_2 \geq 0 \\
x_1, x_2 \in \mathbb{N}
\end{aligned}
$$

Using the algorithm we get the subproblems

$$
\max_{t_1} \quad \min_{u} u_1(7-2t_1) + u_2(15-10t_1) + u_3 s_1 + u_4 s_2 - t_1
$$

$$
\begin{array}{cccccccl}
4u_1 & +3u_2 & & & -w_1 & & & = 4 \\
-u_1 & & +u_3 & & & -w_2 & & = 0 \\
& -u_2 & & +u_4 & & & -w_3 & = 0 \\
& & & & & & u_1, u_2, u_3, u_4, w_1, w_2, w_3 & \geq 0 \\
& & & & & & t_1 & \in \mathbb{N}
\end{array} \tag{SP^1}
$$

with $I_1 = [0, (39 - 3s_1 + 4s_2)/34]$ and

$$
\max_{t_2} \quad \min_{u} u_1(7-4t_2) + u_2(15-3t_2) + u_3 s_1 + u_4 s_2 + 4t_2
$$

$$
\begin{array}{cccccccl}
2u_1 & +10u_2 & & & -w_1 & & & = -1 \\
-u_1 & & +u_3 & & & -w_2 & & = 0 \\
& -u_2 & & +u_4 & & & -w_3 & = 0 \\
& & & & & & u_1, u_2, u_3, u_4, w_1, w_2, w_3 & \geq 0 \\
& & & & & & t_2 & \in \mathbb{N}
\end{array} \tag{SP^2}
$$

with $I_2 = [0, 7/4]$.

The next interval is $I_3 = [7/4, (7 + s_1)/4]$. We have to assure that I_3 doesn't contain an integer number.
Applying theorem 2 we get

$$
39 - 3s_1 + 4s_2 \geq 0
$$

and

$$(7 + s_1)/4 < 2.$$

A change of the first right hand side from 7 to 8 destroys the optimal solution $(0, 1)^T$, but the second constraint hasn't any influence. Indeed, we find the dual prices $u_1^* = 1$ and $u_2^* = 0$.

6 Conclusion

We have presented a new approach to sensitivity analysis in integer linear programming. The advantages of the described method are the following:

1. The q subproblems are LP problems (with an integer parameter in the objective function).

2. The sufficient optimality conditions (C) can easily be proved for a given vector s.

3. The algorithm produces unique dual prices with value 0 if restrictions are inactive analogous LP.

4. Results from LP can be used for describing changes of objective function or matrix coefficients.

5. The results can also be extended to mixed integer programming and problems with hyperbolic objective.

References

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