A Finite Element Method for an Ill-Posed Problem

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Abstract

For an ill-posed problem which has its origin in several applications (e.g. electrocardiology) a weak formulation is given over a Hilbert space without any constraints. This is achieved by means of Lagrangian multipliers. Beside theoretical questions (e.g. existence and uniqueness of a solutuion) a finite element approximation is considered. Error estimates, an investigation of the condition number of the corresponding finite dimensional system and a numerical example are given.

1. Introduction

This work has its origin in problems appearing in several applications, e. g. in electrocardiology [1,2,3]. In this field an electric potential U is measured on a subset γ_1 of the surface Γ_1 ($\gamma_1 \subset \Gamma_1$) of the human body. However what is needed in practice is the potential on the surface Γ_0 of the heart (see Figur 1 which shows a simplified situation). In most cases U can not be measured on Γ_0. In a classical formulation a mathematical steady state model is

Problem 1:

$$div(A\, grad\, U) = 0 \quad in \quad \Omega \subset R^n, \quad n = 2 \ or \ 3 \tag{1}$$

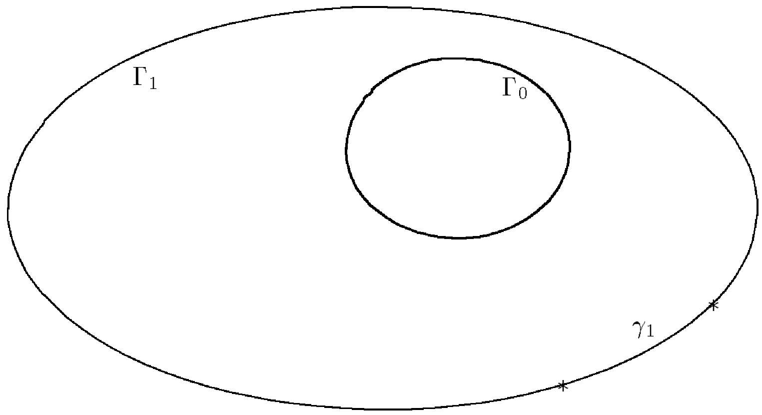
$$U(x) = D(x)\,, \quad x \in \gamma_1 \subset \Gamma_1 \tag{2}$$

$$A\frac{\partial U}{\partial n}(x) = 0\,, \quad x \in \Gamma_1. \tag{3}$$

Ω is the domain between the boundaries Γ_0 and $\Gamma_1$which later are assumed to be sufficiently smooth (e.g. Lipschitz continuous).$\frac{\partial}{\partial n}$ means differentiation with respect to the outer normal. A is the electrical conductivity which is bounded and positive: $0 < A_0 \leq A(x) \leq A_1 < \infty$ for $x \in \overline{\Omega}$. D on Γ_0 is assumed to be known. The boundary condition (3) means that there is no electric current going from the body on Ω into the surrounding medium.

It is well known that Problem 1 is ill-posed [2,3]. In the following we first consider a weak formulation using a representation with Lagrange multipliers. It is shown that there exists a unique solution in a suitable space. Then for a numerical solution error estimates, the condition number of the finite system and a finite element example are considered.

Fig. 1



2. Weak formulation with Lagrange multipliers

For convenience we transform Problem 1 supposing there is a smooth enough function ψ_0 which is defined on $\overline{\Omega}$ safisying $\psi_0 = D(x)$, $x \in \gamma_1$. Then $u = U - \psi_0$ must be a solution of

Problem 2:

$$- \, div(A\, grad\, u) + b_0 u \;=\; f \quad in\, \Omega \tag{4}$$

$$u \;=\; 0 \quad on \quad \gamma_1 \tag{5}$$

$$g \;+\; A\frac{\partial u}{\partial n} \;=\; 0 \quad on \quad \Gamma_1 \tag{6}$$

with the abbreviations $f = div(A\, grad\, \psi_0)$ and $g = A\frac{\partial \psi_0}{\partial n}$. For generality we have added in eq. (4) the term $b_0 u$ where b_0 may be a given smooth function defined on Ω. For $b_0 = 0$ Problem 2 is equivalent to Problem 1.
In order to get a weak formulation we introduce the space
$V = H^1(\Omega) \times H^{-1/2}(\Gamma)$, $\Gamma = \Gamma_0 \cup \Gamma_1$, with norm

$$\| \,(v,\mu)\, \|_V = \left(\| \, v \, \|^2{}_{H^1(\Omega)} + \| \, \mu \, \|^2{}_{H^{-1/2}(\Gamma)} \right)^{1/2} .$$

where standard notation has been used, see e.g.[4]. Furthermore we define the bilinear resp. linear forms

$$a(u,v) \;=\; \int_\Omega \left(A \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} \;+\; b_0 u\, v \right) dx\, , \quad dx = dx_1 dx_2 ... dx_n, \tag{7}$$

$$B(u,\lambda; v,\mu) \;=\; a(u,v) \;+\; \int_{\Gamma_0} \lambda v \, ds \;+\; \int_{\gamma_1} u\, \mu \, ds \tag{8}$$

$$F(v,\mu) \;\equiv\; F(v) \;=\; \int_\Omega f\, v\, dx \;-\; \int_{\Gamma_1} g\, v\, ds \tag{9}$$

(in the following we suppress the volumelement dx and the surface element ds) to get

Problem 3: Find $(u,\lambda) \in V$ such that

$$B(u,\lambda;v,\mu) \;=\; F(v,\mu) \qquad \forall\,(v,\mu)\;\in\; V. \tag{10}$$

This is the weak formulation of Problem 2. It reproduces the boundary conditions (5), (6) and in addition

$$A\frac{\partial u}{\partial n} \;+\; \lambda \;=\; 0 \quad on\ \Gamma_0. \tag{11}$$

Thus the Lagrange multiplier λ gives essentially the normal derivative of u on Γ_0 where no boundary condition is known a priori. Problems of this type were investigated in early times by Babuška [5].
Note that the underlying space V does not contain any constraints. The bilinear form $B(.,.)$ is in general not symmetric what is caused by the surface terms in (8).
In the next section the existence and uniqueness of a solution of Problem 3 is considered.

3. Existence and uniqueness

Since we did not succeed to prove the V-ellipticity of the bilinear form B the Lax-Milgram Lemma can not be applied. To prove existence and uniqueness we therefore use a theorem due to Babuška [6]:

Theorem 1: Let H_1 and H_2 be two Hilbert spaces with scalar products $(.,.)_{H_1}$ and $(.,.)_{H_2}$ respectively. Further let $b(u,v)$ be a bilinear form on $H_1 \times H_2$, $u \in H_1$, $v \in H_2$ such that

$$\mid b(u,v) \mid \;\;\leq c_1 \parallel u \parallel_{H_1} \parallel v \parallel_{H_2}, \quad c_1 \;<\; \infty \tag{12}$$

$$\sup_{\|u\|_{H_1}\leq 1} \;\mid b(u,v) \mid \;\;\geq\;\;\; c_2 \parallel v \parallel_{H_2}, \quad c_2 > 0 \tag{13}$$

$$\sup_{\|v\|_{H_2}\leq 1} \;\mid b(u,v) \mid \;\;\geq\; c_3 \parallel u \parallel_{H_1}, \quad c_3 > 0. \tag{14}$$

Further let $F \in H_2'$ be a linear form on H_2. Then there exists exactly one element $u_0 \in H_1$ such that

$$b(u_0,v) \;=\; F(v) \quad \forall v \;\in\; H_2$$

and

$$\parallel u_0 \parallel_{H_1} \leq \frac{1}{c_3} \parallel F \parallel_{H_2'} .$$

Herewith, H_2' ist the dual space of H_2.
The task is to verify the assumptions (12), (13), (14) for $b = B$ and $H_1 \;=\; H_2 \;=\; V$. We do this by a modification of the method presented in [5].
Some auxiliary results are necessary.

Lemma 1: Suppose $\lambda \in H^{-1/2}(\Gamma)$ and $0 < \underline{b}_0 \leq b_0 \leq \overline{b}_0$ on Ω. Let $w_{N,\lambda} \in H^1(\Omega)$ be the weak solution of $-\Delta w_{N,\lambda} + w_{N,\lambda} = 0$ in Ω, $\frac{\partial w_{N,\lambda}}{\partial n} = -\lambda$ on Γ, and let $\overline{w}_{N,\lambda} \in H^1(\Omega)$ be the (weak) solution of

$$a(u,v) = -\int_\Gamma \lambda\, v \quad \forall v \in H^1(\Omega).$$

Then there exist positive constants $\overline{c}_1$, $\overline{c}_2$ $(\overline{c}_1 \leq \overline{c}_2)$ which are independent of λ such that

$$\overline{c}_1 \parallel \overline{w}_{N,\lambda} \parallel_{H^1(\Omega)} \leq \parallel w_{N,\lambda} \parallel_{H^1(\Omega)} \leq \overline{c}_2 \parallel \overline{w}_{N,\lambda} \parallel_{H^1(\Omega)} . \tag{15}$$

Proof: The proof follows immediately from the definition of the weak solutions $w_{N,\lambda}$ and $\overline{w}_{N,\lambda}$. Since this proof is elementary we omit it here.

Theorem 2: Under the assumptions of Lemma 1 there are constants
$c_1, c_2, \quad 0 < c_1 \leq c_2 < \infty$ such
that

$$-c_1 \int_\Gamma \lambda \overline{w}_{N,\lambda} \quad \leq \quad \parallel \lambda \parallel^2_{H^{-1/2}(\Gamma)} \quad \leq \quad -c_2 \int_\Gamma \lambda \overline{w}_{N,\lambda} \tag{16}$$

and

$$\min_\Omega[A, b_0] \parallel \overline{w}_{N,\lambda} \parallel^2_{H^1(\Omega)} \leq -\int_\Gamma \lambda\, \overline{w}_{N,\lambda} \leq \max_\Omega[A, b_0] \parallel \overline{w}_{N,\lambda} \parallel^2_{H^1(\Omega)} . \tag{17}$$

Proof: Because the proof is a modification of the proof of Theorem 2.7 in [5] we only sketch it here.The inequalities (17) follow immediately from the variational equation (we suppress λ in $\overline{w}_{N,\lambda}$ and $w_{N,\lambda}$) $a(\overline{w}_N, v) = -\int_\Gamma \lambda v \quad \forall v \in H^1(\Omega)$ by setting $v = \overline{w}_N$. In order to prove (16) we estimate

$$\mid \int_\Gamma \lambda v \mid \leq \parallel \lambda \parallel_{H^{-1/2}(\Gamma)} \parallel v \parallel_{H^{1/2}(\Gamma)}$$

and take into account the norm definition

$$\parallel \lambda \parallel_{H^{-1/2}(\Gamma)} = \sup_{v \in H^{1/2}(\Gamma)} \frac{\mid \int_\Gamma \lambda v \mid}{\parallel v \parallel_{H^{1/2}(\Gamma)}}.$$

Now we use [7,p.90] $\parallel \lambda \parallel_{H^{-1/2}(\Gamma)} = \parallel w_N \parallel_{H^1(\Omega)}$ .Combining these relations and applying Lemma 1 we get the upper bound $\parallel \lambda \parallel^2_{H^{-1/2}(\Gamma)} \leq -c_2 \int_\Gamma \lambda \overline{w}_N$, $c_2 > 0$. To gain the lower bound we make use of the inquality (17) and Lemma 1 giving

$$-\overline{c} \int_\Gamma \lambda \overline{w}_N \leq \parallel w_N \parallel_{H^1(\Omega)}, \quad \overline{c} > 0.$$

Since furthermore [8,Chapter 8, Section 7.3]

$$\parallel w_N \parallel_{H^1(\Omega)} \leq c \parallel \lambda \parallel_{H^{-1/2}(\Gamma)}$$

we obtain the final lower bound. This proves the Theorem.

Lemma 2: Let $\mu \in H^{-1/2}(\Gamma)$ and $u = v - \overline{w}_{N,\mu} \in H^1(\Omega)$. Then there is a constant $c_0 > 0$ such that

$$\parallel u \parallel_{H^1(\Omega)} \leq c_0 \parallel (v, \mu) \parallel_V .$$

Proof: The following estimates are valid:

$$\| u \|_{H^1(\Omega)} \leq \| v \|_{H1(\Omega)} + \| \overline{w}_{N,\mu} \|_{H^1(\Omega)}$$

$$\leq \| v \|_{H^1(\Omega)} + \frac{1}{\sqrt{\min_\Omega[A,b_0]}} \sqrt{\int_\Gamma (-\mu)\overline{w}_{N,\mu}}$$

$$\leq \| v \|_{H^1(\Omega)} + \frac{1}{\sqrt{c_1 \min_\Omega[A,b_0]}} \| \mu \|_{H^{-1/2}(\Gamma)} .$$

The last two inequalities follow from ineqs. (17) and (16). From this and the definition of the norm $\| \ . \ \|_V$ Lemma 2 follows.

Now we can prove existence and uniqueness of Problem 3.

Theorem 3: (I) Assume for given $(v,\mu) \in V,\ v \neq 0$, there exists $\lambda \in H^{-1/2}(\Gamma)$ such that

$$I_1 \ := \int_\Gamma \mu v + \int_{\Gamma_0} \lambda v + \int_{\gamma_1} \mu v + \int_{\Gamma-\gamma_1} \mu \overline{w}_{N,\mu} \ \geq \ 0. \tag{18}$$

Then the bilinear form $B(.,.)$ satisfies assumption (13) of Theorem 1.
(II) Let $(u,\lambda) \in V, u \neq 0$, be given and $\mu \in H^{-1/2}(\Gamma)$ such that

$$I_2 \ := \ 2\int_{\Gamma_0} \lambda u + \int_{\Gamma_1} \lambda(u + \overline{w}_{N,\lambda}) + \int_{\gamma_1} u\mu \ \geq \ 0. \tag{19}$$

Then the bilinear form B(.,.) satisfies assumption (14) of Theorem 1.

Before we prove this Theorem we note that the conditions (18) and (19) can be fulfilled. For example, consider (18) and suppose any element $(v,\mu) \in V,\ v \neq 0,$ is given. Then one can compute the sum

$$c_0 := \int_\Gamma \mu v + \int_{\gamma_1} \mu v + \int_{\Gamma-\gamma_1} \mu \overline{w}_{N,\mu}.$$

Suppose $c_0 \geq 0$. Then one can take, e.g., $\lambda = 0$. If $c_0 < 0$, choose an arbitrary $\lambda_1 \in H^{-1/2}(\Gamma)$ and compute $c_1 := \int_{\Gamma_0} \lambda_1 v$. If $c_0 + c_1 \geq 0$, (18) is true. If $c_0 + c_1 < 0$ and $c_1 > 0$, replace λ_1 by $c\lambda_1$ and choose the constant $c > 0$ such that $c_0 + cc_1 \geq 0$. If $c_1 < 0$, replace λ_1 by $c\lambda_1$, $c < 0$, and choose c such that $c_0 + cc_1 \geq 0$. In every case (18) can be fulfilled. The other cases can be discussed in an analogous manner.

Proof of Theorem 3: (I) We take $u = v - \overline{w}_{N,\mu} \in H^1(\Omega)$ and obtain according to (8)

$$B(v - \overline{w}_{N,\mu}, \lambda; v, \mu) = a(v,v) - a(\overline{w}_{N,\mu}, v) + \int_{\Gamma_0} \lambda v + \int_{\gamma_1} (v - \overline{w}_{N,\mu})\mu.$$

Now $a(v,v) \geq \min_{\overline{\Omega}}[A,b_0] \ \| \ v \ \|^2_{H^1(\Omega)}$ and $a(\overline{w}_{N,\mu}, v) = -\int_\Gamma \mu v$. Theorem 2 gives

$$-c_2 \int_\Gamma \mu \overline{w}_{N,\mu} = -c_2 \left(\int_{\gamma_1} \mu \overline{w}_{N,\mu} + \int_{\Gamma-\gamma_1} \mu \overline{w}_{N,\mu} \right) \geq \| \ \mu \ \|^2_{H^{-1/2}(\Gamma).}$$

With these relations we can estimate

$$B(v - \overline{w}_{N,\mu}, \lambda; v, \mu) \geq \min_{\overline{\Omega}}[A,b_0] \ \| \ v \ \|^2_{H^1(\Omega)} + \frac{1}{c_2} \ \| \ \mu \ \|^2_{H^{-1/2}(\Gamma)} + I_1$$

where I_1 is given in (18). The right hand side (RHS) can be estimated further with the help of Lemma 2:

$$RHS \geq \overline{c}_0 \parallel (v,\mu) \parallel_V^2 + I_1 \geq \frac{\overline{c}_0}{c_0^{1/2}} \parallel (v,\mu) \parallel_V \parallel u \parallel_{H^1(\Omega)} + I_1$$

where $\overline{c}_0$ is a positive constant. According to Theorem 1, ineq. (13), we need

$$\sup_{\|(u,\lambda)\|_V \leq 1} \mid B(u,\lambda;v,\mu) \mid$$

which can be replaced by

$$\sup_{\|(u,\lambda)\|_V \neq 0} \frac{\mid B(u,\lambda;v,\mu) \mid}{\parallel (u,\lambda) \parallel_V}.$$

Assumption (18) implies

$$\sup_{\|(u,\lambda)\|_V \neq 0} \frac{\mid B(u,\lambda;v,\mu) \mid}{\parallel (u,\lambda) \parallel_V} \geq \sup_{\|(u,\lambda)\|_V \neq 0} \left(\frac{\overline{c}_0}{c_o^{1/2}} \parallel (v,\mu) \parallel \frac{\parallel u \parallel_{H^1(\Omega)}}{\parallel (u,\lambda) \parallel_V} \right)$$

and this RHS can be estimated by means of a positive constant c, RHS $\geq c \parallel (u,\lambda) \parallel_V$, which proves part (I) of the Theorem. The second part (II) can be verified in the same manner.

The first assumption of Theorem 1 (continuity of $B(.,.)$) is easy to prove. Hence Theorem 3 yields in conjunction with Theorem 1 the unique weak solution of Problem 3.

4. Finite dimensional Approximation (FEM)

For numerical approximations we now consider on the basis of Problem 3 a finite dimensional

Problem 4: Find $(u,\lambda)_h \in V_h$ such that

$$B((u,\lambda)_h;(v,\mu)_h) = F((v,\mu)_h) \quad \forall (v,\mu) \in V_h \tag{20}$$

where V_h is a finite dimensional subspace of V. First we are interested in error estimates and the basic result is (as always) a typical Cea-Lemma which is under certain conditions also valid in our case:

Lemma 3: Let $(u,\lambda) \in V$ and $(u,\lambda)_h \in V_h$ be the solution of Problem 3 and Problem 4 resp. Suppose the constants c_1 and c_3 in Theorem 1 are such that $c_1 < c_3$. Then

$$\parallel (u,\lambda) - (u,\lambda)_h \parallel_V \leq c \inf_{(v,\mu)_h \in V_h} \parallel (u,\lambda) - (v,\mu)_h \parallel_V$$

where

$$c = \frac{c_3 + c_1}{c_3 - c_1} \tag{21}$$

Proof: The proof starts from Theorem 1, ineq.(14), and uses the triangle inequality

$$c_3 \parallel (u,\lambda)_h - (v,\mu)_h \parallel_V \leq \sup_{\|(w,\nu)\|_V \leq 1} \mid B((u,\lambda)_h - (v,\mu)_h; (w,\nu)) \mid$$

$$= \sup_{\|(w,\nu)\|_V \leq 1} \mid B((u,\lambda)_h - (u,\lambda); (w,\nu)) + B((u,\lambda) - (v,\mu)_h; (w,\nu)) \mid .$$

The RHS can be estimated further by ineq. (12) and the triangle inequality again giving

$$(c_3 - c_1) \parallel (u,\lambda) - (u,\lambda)_h \parallel_V \leq (c_1 + c_3) \parallel (u,\lambda) - (v,\mu)_h \parallel_V$$

This completes the proof of Lemma 3.

Note that c_1 and c_3 in Theorem 1 must be such that $c_1 < c_3$.

Lemma 4: Let $(u,\lambda) \in V$ and $(u,\lambda)_h \in V_h$ be the solution of Problem 3 and Problem 4 resp. Let further s_h be the projection of (u,λ) onto V_h. Then

$$\parallel (u,\lambda) - (u,\lambda)_h \parallel_V \leq (1 + c_1/c_3) \parallel (u,\lambda) - s_h(u,\lambda) \parallel_V \tag{22}$$

where c_1, $c_3 > 0$ are the constants in Theorem 1.

Proof: The proof may be omitted here since it is similar to the proof of Lemma 3. Note that the parameters c_1, c_3 appear explicitly again.

Next we consider a finite element approximation of Problem 3. For that we suppose that $\overline{\Omega}$ is a polyhedron. This allows us to cover the sets $\overline{\Omega}$ resp. Γ with polygonal n-dimensional resp. $(n-1)$-dimensional elements ($n \geq 2$). In this way we construct a triangulation $\mathcal{T}_h$ of Ω which satisfies the usual assumptions [4,p.38]. On this triangulation two finite elements (K, P_K, Σ_K) and (K, P_{Kb}, Σ_{Kb}), $K \in \cup_h \mathcal{T}_h$ can be defined. (We use standard theory as developed, e.g., in [4, Chapters 2,3].) As usual the spaces P_K and P_{Kb} consist of polynomials. We suppose that both of the finite elements satisfy Ciarlets assumptions H_1, H_2, H_3 [4,p.132], i.e.
(H_1) $\mathcal{T}_h$ is a regular family of triangulations;
(H_2) both of the finite elements are affine families of finite elements;
(H_3) all finite elements are of class $\mathcal{C}^0$.
Since V is defined by $V = H^1(\Omega) \times H^{-1/2}(\Gamma)$ beside $H^1(\Omega)$ we need $H^{-1/2}(\Gamma)$ which is the dual space of the Sobolow space $H^{1/2}(\Gamma)$ and this space is generated by the trace operator $\gamma_{p0} : H^1(\Omega) \rightarrow L^2(\Gamma)$ such that $\gamma_{p0}v$ is the trace of $v \in H^1(\Omega)$ on Γ for every smooth v, see e.g. [7,8].
Let $H_h^1(\Omega)$ be a finite dimensional subspace of $H^1(\Omega)$. In our case the space

$$P_{Kb} = [v_h \mid_K;\ v_h \in H_h^1(\Omega)], \quad K \in \cup_h \mathcal{T}_h,$$

is chosen to generate a function the $\gamma_{p0}-$projection of which is in the finite dimensional space $H_h^{1/2}(\Gamma) \subset H^{1/2}(\Gamma) \subset L^2(\Gamma)$. In other words, the numerical approximation for $\mu \in H^{1/2}(\Gamma) \subset H^{-1/2}(\Gamma)$ is constructed by the set of polynomials P_{Kb}, $K \in \mathcal{T}_h$, while the numerical approximation $u_h \in H_h^1(\Omega) \subset H^1(\Omega)$ is defined in the standard way by means of P_K, $K \in \mathcal{T}_h$. In general, the order of the polynomials in P_K and P_{Kb} may be different.

With the assumptions concerning Ω a triangulation of $\overline{\Omega}$ gives automatically a triangulation $\mathcal{T}_{hb}$ of the boundary Γ. According to assumption (H_2) reference elements $\overline{K}$ and $\overline{K}_b$ exist for every $K \in \mathcal{T}_h$ and $\mathcal{T}_{hb,}$ respectively.

We need some definitions. We denote by s_{h1} the projection $s_{h1} : H^1(\Omega) \to H^1_h(\Omega)$ and by s^0_{h2} the projection $s^0_{h2} : H^{1/2}(\Gamma) \to L^2(\Gamma)$. Further, for an integer $k \geq 0$ P_k is the set of polynomials of order $\leq k$. $\mid v \mid_{m,\Omega}$ is defined by

$$\mid v \mid_{m,\Omega} = \left(\sum_{|\alpha|=m} \int_\Omega \mid \partial^\alpha \mid^2 dx \right)^{1/2}$$

where α is a multi index.
With these definitions we cite for later use Lemma 5 which is a special case of Theorem 3.2.1 in [4].

Lemma 5: Suppose $(H_1), (H_2), (H_3)$ are valid.
(I) (Triangulation of $\overline{\Omega}$) Assume there is an integer $k \geq 1$ such that the following inclusions are satisfied

$$P_k(\overline{K}) \subset \overline{P} \subset H^1(\overline{K}) \quad \text{and} \quad H^{k+1}(\overline{K}) \subset\subset \mathcal{C}^0(\overline{K})$$

($\subset\subset$ means continuous injection). Then there exists a constant c independent of h such that, for any function $v \in H^{k+1}(\Omega)$

$$\| v - s_{h1} v \|_{H^1(\Omega)} \leq ch^k \mid v \mid_{k+1,\Omega} \tag{23}$$

(II) (Triangulation of Γ) Assume there exists an integer $k^* \geq 1$ such that

$$P_{k^*}(\overline{K}_b) \subset \overline{P}_b \subset H^1(\overline{K}_b) \quad \text{and} \quad H^{k^*+1}(\overline{K}_b) \subset\subset \mathcal{C}^0(\overline{K}_b).$$

Then there exists a constant c^* independent of h such that for $\mu \in H^{k^*+1}(\Gamma)$

$$\| \mu - s^0_{h2}\mu \|_{L^2(\Gamma)} \leq c^* h^{k*+1} \mid \mu \mid_{k^*+1,\Gamma} . \tag{24}$$

For later use we note that $H^1(\overline{K}_b) \subset H^{1/2}(\overline{K}_b)$.

Theorem 4: Let $u \in H^{k+1}(\Omega)$ and $\lambda \in H^{k^*+1}(\Gamma) \cap L^2(\Gamma)$. Suppose the assumptions of Lemmata 4 and 5 are valid. Then there are constants $c > 0$ and $\overline{c} > 0$ such that

$$\| (u,\lambda) - (u,\lambda)_h \|_V \leq (1 + c_1/c_3) \left(ch^k \| u \|_{H^{k+1}(\Omega)} + c^*\overline{c} h^{k^*+3/2} \| \lambda \|_{H^{k^*+1}(\Gamma)} \right)$$

where c_1, c_3 are defined in Theorem 1. c and c^* are given in eqs. (23) and (24), respectively.

Proof: The starting point is ineq. (22). Using $s_h(u,\lambda) = (s_{h1}u, s^0_{h2}\lambda)$ and Lemma 5, (I), ineq. (22) can be written

$$\| (u,\lambda) - (u,\lambda)_h \|_V \leq (1 + c_1/c_3) \left(ch^k \| u \|_{H^{k+1}(\Omega)} + \| \lambda - s^0_{h2}\lambda \|_{H^{-1/2}(\Gamma)} \right).$$

An estimate of the second term on the right hand side is based on the norm definition and the $L^2(\Gamma)-$orthogonality of $\lambda - s^0_{h2}\lambda$ on $s^0_{h2}w$:

$$\| \lambda - s^0_{h2}\lambda \|_{H^{-1/2}(\Gamma)} = \sup_{w \in H^{1/2}(\Gamma)} \frac{\mid (\lambda - s^0_{h2}\lambda, w) \mid_{L^2(\Gamma)}}{\| w \|_{H^{1/2}(\Gamma)}}$$

$$= \sup_{w \in H^{1/2}(\Gamma)} \frac{| (\lambda - s_{h2}^0\lambda, w - s_{h2}^0 w)_{L^2(\Gamma)}}{\| w \|_{H^{1/2}(\Gamma)}}$$

$$\leq \| \lambda - s_{h2}^0\lambda \|_{L^2(\Gamma)} \sup_{w \in H^{1/2}(\Gamma)} \frac{\| w - s_{h2}^0 w \|_{L^2(\Gamma)}}{\| w \|_{H^{1/2}(\Gamma)}}.$$

Now we use the interpolation inequality in the spaces $H^s(\Gamma)$:

$$\| w - s_{h2}^0 w \|_{L^2(\Gamma)} \leq \overline{c}h^s \| w \|_{H^s(\Gamma)}, \quad s \in [0, k^* + 1].$$

Then Lemma 5, (II), can be applied for s=1/2 to give

$$\| \lambda - s_{h2}^0\lambda \|_{H^{-1/2}(\Gamma)} \leq c^* h^{k^*+1} \| \lambda \|_{H^{k^*+1}(\Gamma)} \sup_{w \in H^{1/2}(\Gamma)} \frac{\overline{c}h^{1/2} \| w \|_{H^{1/2}(\Gamma)}}{\| w \|_{H^{1/2}(\Gamma)}}$$

$$= c^*\overline{c}h^{k^*+3/2} \| \lambda \|_{H^{k^*+1}(\Gamma)}$$

which proves Theorem 4.

5. Numerical solution

Now we consider the finite element solution of Problem 4, eq. (20), for a situation in two dimensions ($n = 2$) illustrated in Figur 1. Ω is assumed to be a polygon. Since $V_h = H_h^1(\Omega) \times H_h^{-1/2}(\Gamma)$, eq. (20) splits into

$$a(u_h, v_h) + \int_{\Gamma_0} \lambda_h v_h = F(v_h) \quad \forall v_h \in H_h^1(\Omega) \tag{25}$$

and

$$\int_{\gamma_1} u_h \mu_h = 0 \quad \forall \mu_h \in H_h^{-1/2}(\Gamma) \tag{26}$$

where $F(.)$ has been given in eq. (9).

First we define the spaces $H_h^1(\Omega)$ and $H_h^{-1/2}(\Gamma)$ with dimension M and M_b resp. The triangulation (e.g. by triangles) of $\overline{\Omega}$ is supposed to have M vertices. The corresponding basis functions are $[w_j]_{j=1}^M$. On each of the boundary parts γ_1 and Γ_0 p vertices ($2p < M$) may be situated. The vertices on γ_1 are numbered by $P_1, P_2, ..., P_p$ and on Γ_0 by $P_{p+1}, P_{p+2}, ..., P_{2p}$. We assume that the points P_1, P_p and P_{p+1}, P_{2p} are on the border of γ_1 and Γ_0, resp. Here we already use the fact that the number of vertices on γ_1 and Γ_0 must be the same, see eq. (31). The basis functions corresponding to the boundary points on γ_1 and Γ_0 are $[t_j]_{j=1}^{2p}$.

The space $H_h^1(\Omega)$ can be constructed by standard methods. The setting of $H_h^{-1/2}(\Gamma) \subset H^{-1/2}(\Gamma) \subset L^2(\Gamma)$ shall be considered shortly. Although different finite dimensional subspaces of $H^{-1/2}(\Gamma)$ can be defined we will base the numerical solution on the subspace

$$H_h^{-1/2}(\Gamma) = \left[v_h \in \mathcal{C}^0(\Gamma);\ v_h \,|_K \in P_{k^*}(K),\ K \in \mathcal{T}_{hb} \right] \tag{27}$$

($k^* = 1, 2$) where Γ is written as the union $\Gamma = \cup_{K \in \mathcal{T}_{hb}} K$. To get the final finite dimensional equations the unknown functions u_h in Ω and λ_h on Γ_0 (only this is needed) are represented in the form

$$u_h = \sum_{i=1}^{M} u_i w_i \quad in\ \overline{\Omega} \tag{28}$$

and

$$\lambda_h \;=\; \sum_{i=p+1}^{2p} \lambda_i t_i \quad on\ \Gamma_0 \tag{29}$$

where u_i, λ_i must be determined. We introduce the following

Definition: Let D be a $(n,m)-$matrix and I, J, K, L, $(I < J, K < L)$ be four positive integers. Then $D(I,J;K,L)$, $n = J - I + 1$, $m = L - K + 1$ means such a $(n,m)-$matrix where the row index runs from I to J and the column index runs from K to L.
Then we consider six matrices

$$A_1 = A_1(1,p;p+1,M), \quad A_2 = A_2(p+1,2p;p+1,M),$$

$$A_3 = A_3(2p+1,M;p+1,M), \quad S = S(p+1,2p;p+1,2p),$$
$$O_1 = O_1(1,p;1,p), \quad O_3 = O_3(2p+1,M;1,p).$$

A_1, A_2, A_3 and S have elements $a_{i,j} \;=\; a(w_j, w_i)$ and $s_{i,j} \;=\; \int_{\Gamma_0} t_j w_i$ resp. The elements of O_1 and O_3 are all zero. Furthermore, according to eqs. (9) and (25) we define the $(M,1)-$right hand side vector b with elements

$$b_j \;=\; \int_\Omega f w_j \;-\; \int_{\Gamma_1} g w_j, \quad j \;=\; 1, ..., M. \tag{30}$$

and the $(M,1)-$vector of the unknowns X with elements

$$X_i = \lambda_{p+i} \quad for\ i = 1, ..., p \qquad \text{and} \qquad X_i = u_i \quad for\ i = p+1, ..., M.$$

With these definitions the first finite dimensional system, eq. (25), takes the form

$$\begin{pmatrix} O_1 & A_1 \\ S & A_2 \\ O_3 & A_3 \end{pmatrix} X = b \qquad \text{or for short} \qquad BX \;=\; b. \tag{31}$$

The second finite dimensional equation, eq. (26), is a linear homogeneous system with a quadratic regular matrix which gives $u \;=\; 0$ on γ_1, i.e. $u_i = 0$ for $i = 1, 2, ..., p$. Therefore, this equation need not be considered further. In the following, eq. (31) is investigated.
If the number of unknowns shall be equal to the number of equations determining the unknowns (we claim this) the number of vertices on γ_1 must be equal to the number of vertices on Γ_0. This is a remarkable result. Essentially this means that the accuracy of u (and λ too) on Γ_0 is directly proportional to the number of measurements on γ_1.
As mentioned in section 1 the problem considered in this work is ill-posed. Therefore it is interesting to investigate eq. (31) under this aspect. In fact, the condition number $\kappa :=\parallel B^{-1} \parallel\parallel B \parallel$ of the matrix B, eq. (31), is in most cases large. In typical numerical examples we found κ is of order 10^3 until 10^{17}. In this manner the ill-posedness of the problem appears. We will estimate κ in terms of the usual parameter h ($\simeq$ maximum diameter of the geometrical finite elements).

In order to do this we note that $a_{i,j}$ is determined by $a(w_j, w_i)$ (i.e. it is a volume integral and it only depends on the basis functions $[w_k]$) whereas $s_{i,j}$ is defined by the surface integral

$s_{i,j} = \int_{\Gamma_0} t_j w_i$ (it depends on the basisfunctions $[t_k]$ too). From this we can suppose that the matrices A_i, $i = 1,2,3$, may be of a different order of h than the matrix S. Let therefore A_i be of the asymptotic form $A_i \simeq h^m A_{i0}$, $i = 1,2,3$, where A_{i0} does not depend on h for $h \to 0$. Analogously S may be assumed to be of the form $S \simeq h^q S_0$ for $h \to 0$ (in general $m \neq q$). Thus B may be written

$$B \simeq \begin{pmatrix} O_1 & h^m A_{10} \\ h^q S_0 & h^m A_{20} \\ O_3 & h^m A_{30} \end{pmatrix} \quad \text{for} \quad h \to 0. \tag{32}$$

With these assumptions we have

Theorem 5: The condition number κ of the matrix B is for $h \to 0$ of order

$$\kappa \;=\; O\left(\frac{1}{h^{|m\,-\,q|}}\right). \tag{33}$$

Proof: It is easy to see from eq. (32) that

$$\det B \simeq h^{mM+(q-m)p} \begin{vmatrix} O_1 & A_{10} \\ S_0 & A_{20} \\ O_3 & A_{30} \end{vmatrix}.$$

Let the elements of B be $b_{i,j}$ and the corresponding algebraic complements be $B_{i,j}$. Then one can see that $B_{i,j}$ (which is a determinant of order $M\,-\,1$) is of $h-$order

$$B_{i,j} \simeq h^{q(p-1)+m(M-p)} \qquad \text{for } i = 1,...,M \text{ and } j = 1,...,p,$$

$$B_{i,j} \simeq h^{qp+m(M-p-1)} \qquad \text{for } i = 1,...,M \text{ and } j = p+1,...,M.$$

The inverse matrix B^{-1} can be written as

$$B^{-1} = \frac{1}{\det B} \begin{pmatrix} B_{11} & \dots & B_{M1} \\ \vdots & \vdots & \vdots \\ B_{1M} & \dots & B_{MM} \end{pmatrix} \simeq \frac{1}{detB} \begin{pmatrix} h^{q(p-1)+m(M-p)} B_1^0 \\ h^{qp+m(M-p-1)} B_2^0 \end{pmatrix} \quad \text{for } h \to 0$$

where the matrices B_1^0 and B_2^0 do not depend on h and are of the form $B_1^0 = B_1^0(1,p;1,M)$, $B_2^0 = B_2^0(p+1,M;1,M)$. The norm of B^{-1} can now be estimated:

$$\| B^{-1} \| = \frac{1}{\mid detB \mid} \left(\sum_{i,j=1}^{M} \mid B_{ij} \mid^2 \right)^{1/2} \simeq O\left((h^{-2q} + h^{-2m})^{1/2} \right) \qquad \text{for} \;\; h \to 0.$$

Since according to eq. (32) $\| B \| \simeq O\left((h^{2q} + h^{2m})^{1/2} \right)$ we get
$\kappa = O\left((2 + h^{2(m-q)} + h^{2(q-m)})^{1/2} \right)$ which for $h \to 0$ proves the order relation of Theorem 5.

Theorem 5 shows that calculations of the elements $a_{i,j}$ and $s_{i,j}$ should be made very carefully. Furthermore a preconditioning and/or a regularization of eq. (31) may be necessary.

The matrix B of the system (31) is neither symmetric nor definite. Therefore, in order to solve eq. (31), a stable procedure should be chosen. In [9,10] several such algorithms (conjugate gradient methods and preconditioning techniques) are presented. Especially we found the Orthodir-algorithm discussed in [9] is a suitable one. It is guaranted to converge even if the symmetric part of B is not positive definite.

6. Example

We choose a two dimensional example. Let Ω be a square from which a disc is removed:

$$\Omega = [(x,y) \in R^2;\ 0 < x < 10;\ -5 < y < 5] \setminus [(x,y) \in R^2;\ (x-5)^2 + y^2 \leq 9].$$

Γ_0 and Γ_1 are given by $\Gamma_0 = [(x,y) \in R^2;\ (x-5)^2 + y^2 = 9]$ and
$\Gamma_1 = [x \in [0;10]$ and $-5 \leq y \leq 5;\ 0 \leq x \leq 10$ and $y \in [-5;5]]$ respectively. $\gamma_1 \subset \Gamma_1$ is defined by $\gamma_1 = [x = 0$ and $1 \leq y \leq 5;\ \ x = 10$ and $1 \leq y \leq 5]$.
Ω is triangulated into N_e elements (triangles; for the table below $N_e = 38$). Given such a triangulation we take the corresponding convex hull of the vertices as the new domain Ω. This then is a polygon as assumed in section 5. All numerical calculations are based on this polygon. The space P_k is chosen to be of first order ($k = 1$). For P_{k^*b} we take first and second order elements ($k^* = 1$ or 2). The data in the Problem 2 are $A = 1,\ f = 0,\ g = 0.5, b_0 = 0.01$.

Since B is expected to have a large condition number we used Tychonovs method to regularize eq. (31), i.e. instead of eq. (31) we solved

$$C_\epsilon X_\epsilon = B'b \quad \text{with} \quad C_\epsilon = B'B + \epsilon I \tag{34}$$

where B' is the transposed of B. I is the unit matrix and $\epsilon > 0$ a small parameter. Of course, the case $\epsilon \to 0$ is of special interest and is considered (among others) in the following table. The solution of the linear system (34) was done by means of the Orthodir-algorithm [9]. In the table, the condition number of C_ϵ, $\kappa(C_\epsilon)$, and two of the a priori unknown function values of the electric potential are given in terms of ϵ for $k^* = 1$. u_{10} ist the potential in the point (5;-3) which is situated on Γ_0. u_{20} is u_h in the point (5;-5) which is on Γ_1.

ϵ	$\kappa(C_\epsilon)\ \|_{k^*=1}$	u_{10}	u_{20}
10^{-2}	4.10^3	-0.26367	-1.22531
10^{-3}	4.10^4	-0.74528	-1.86467
10^{-4}	4.10^5	-1.44806	-2.76307
10^{-6}	4.10^7	-1.65667	-3.02920
10^{-8}	4.10^9	-1.65886	-3.03227
10^{-10}	4.10^{11}	-1.65912	-3.03236
10^{-12}	4.10^{13}	-1.65912	-3.03234
0	2.10^{17}	-1.65912	-3.03234

For $k^* = 2$ the condition number C_ϵ is larger by about one order than for $k^* = 1$. The function values u_h are practically not significantly different from the values for $k^* = 1$. The convergence for $\epsilon \to 0$ is of the same manner as in the table.

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