OPTIMIZING OF SUMS AND PRODUCTS OF LINEAR FRACTIONAL FUNCTIONS UNDER LINEAR CONSTRAINTS

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Abstract. In this paper, we consider nonconvex optimization problems whose objective functions are composed of parts easily to optimize on polyhedral sets. This allows to develop algorithms which take advantage of the special structure of the problems. We present a one–parametric algorithm – based on the approved simplex method – that yields a global solution to the problem of optimizing a sum or a product of two linear fractional functions under linear constraints. Results of computational experiments reported at the end of the paper show that our algorithm is promising in comparison with algorithms recently published.

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1. Introduction

We consider nonconvex optimization problems whose objective functions are composed of explicitly quasimonotonic parts and whose feasible regions are polyhedra. Then it is possible to develop algorithms which take advantage of the special structure of the problems.

Previous papers concerned with objective functions in the form of a sum of a linear and a linear fractional function or of several linear fractional functions appeared about 25 years ago (Almogy and Levin [1], Ritter [10], Teterev [13]). Recently Falk and Palocsay [3] and Konno and Yajima [7] proposed algorithms for optimizing the sum and the product of linear fractional functions, respectively.

A general problem stated in the following Section 2 comprehends the special cases just mentioned. In Section 3 some properties of bicriteria problems are given and in Sections 4 and 5 we discuss approaches to global optimizers of sums and products of two explicitly quasimonotonic, especially linear fractional functions. For optimizing a product of two linear fractional functions we propose a one–parametric algorithm which may be adapted to sums, too. This algorithm was elaborated by Schreiber [11] and also implemented in Borland Pascal 7.0 as the two–parametric algorithm and the branch and bound algorithm by Konno and Yajima [7]. Results of computational experiments are reported in Section 6.

2. Problem Statement

Let $X \subset \mathbb{R}^n$ be a nonempty and compact set, especially a polyhedron, and let all functions considered in the following be continuous.

Definition 1. *A function* $\phi(x)$ *defined on a convex set* X *is said to be explicitly quasimonotonic on* X *iff for any* $x^1, x^2 \in X$ *and* $\tilde{x} \in (x^1, x^2)$ $\phi(x^1) \neq \phi(x^2)$ *implies* $\min\{\phi(x^1), \phi(x^2)\} < \phi(\tilde{x}) < \max\{\phi(x^1), \phi(x^2)\}$.

We assume that the functions $f_i : X \to \mathbb{R}^1$, $i = 1(1)k$, are explicitly quasimonotonic and have the ranges F_i on X and that the function $h : F_1 \times F_2 \times \ldots \times F_k \to \mathbb{R}^1$ is strictly increasing with respect to each f_i. We consider the optimization problem

$$\max\{z(x) = h[f_1(x), \ldots, f_k(x)] \; : \; x \in X\} \tag{1}$$

and get important special cases of this problem if $h(f_1, \ldots, f_k)$ is the sum or the product of linear fractional functions f_i whose denominators are positive throughout on X. Assuming in the latter case the numerators of the f_i's also to be positive on X the special cases just mentioned correspond to the replacement of multiple objective functions $f_i(x)$ by their (weighted) arithmetical and their geometrical mean, respectively.

3. Properties of Bicriteria Problems

In this section we consider problem (1) with $k = 2$. As was shown in [4] for the sum of a linear and a linear fractional function by examples the objective function $z(x)$ is on a given feasible region neither quasiconvex nor quasiconcave in general and may have then local extrema which are not global.

Let x^* be a global maximizer of $z(x)$ on X and let X^* be the set of all x^*. Under the assumptions made in the beginning of Section 2 exist the individual maxima of the functions $f_i(x)$ on X. We denote the optimal value of $f_i(x)$ by $\bar{t}_i$ and the set of its maximizers by X_i^*, i. e.

$$\max\{f_i(x) \; : \; x \in X\} = \bar{t}_i, \;\; i = 1, 2, \tag{2}$$

$$X_i^* = \{x \in X \; : \; f_i(x) = \bar{t}_i\}.$$

Furthermore, for $i = 1$, $j = 2$ and $i = 2$, $j = 1$ let be

$$\max\{f_j(x) \; : \; x \in X_i^*\} = t_j', \tag{3}$$

$$X_{ij}^* = \{x \in X_i^* \; : \; f_j(x) = t_j'\}.$$

It holds [5]

Theorem 1. $X_1^* \cap X_2^* \neq \emptyset \iff t_1' = \bar{t}_1 \iff t_2' = \bar{t}_2$.

This means $t'_1 = \bar{t}_1$ which implies $t'_2 = \bar{t}_2$ and vice versa is a necessary and sufficient condition for the (not necessarily unique) original $\bar{x}$ of the utopia point $\bar{t} = (\bar{t}_1, \bar{t}_2)^T$ to be an element of the feasible region X. If this condition is fulfilled we have $\bar{x} \in X^*$, $X^* = X_1^* \cap X_2^* = X_{12}^* = X_{21}^*$ and $z(x^*) = h(\bar{t}_1, \bar{t}_2)$.

Otherwise the following inequality is valid

$$\max\{h(\bar{t}_1, t'_2), h(t'_1, \bar{t}_2)\} \leq z(x^*) < h(\bar{t}_1, \bar{t}_2). \tag{4}$$

We define

$$X_j(t_j) = \{x \in X \ : \ f_j(x) \geq t_j\} \quad \text{for} \quad t_j \in [t'_j, \bar{t}_j], \ j = 1, 2,$$

and (cf. Figure 1)

$$X' = X_1(t'_1) \cap X_2(t'_2).$$

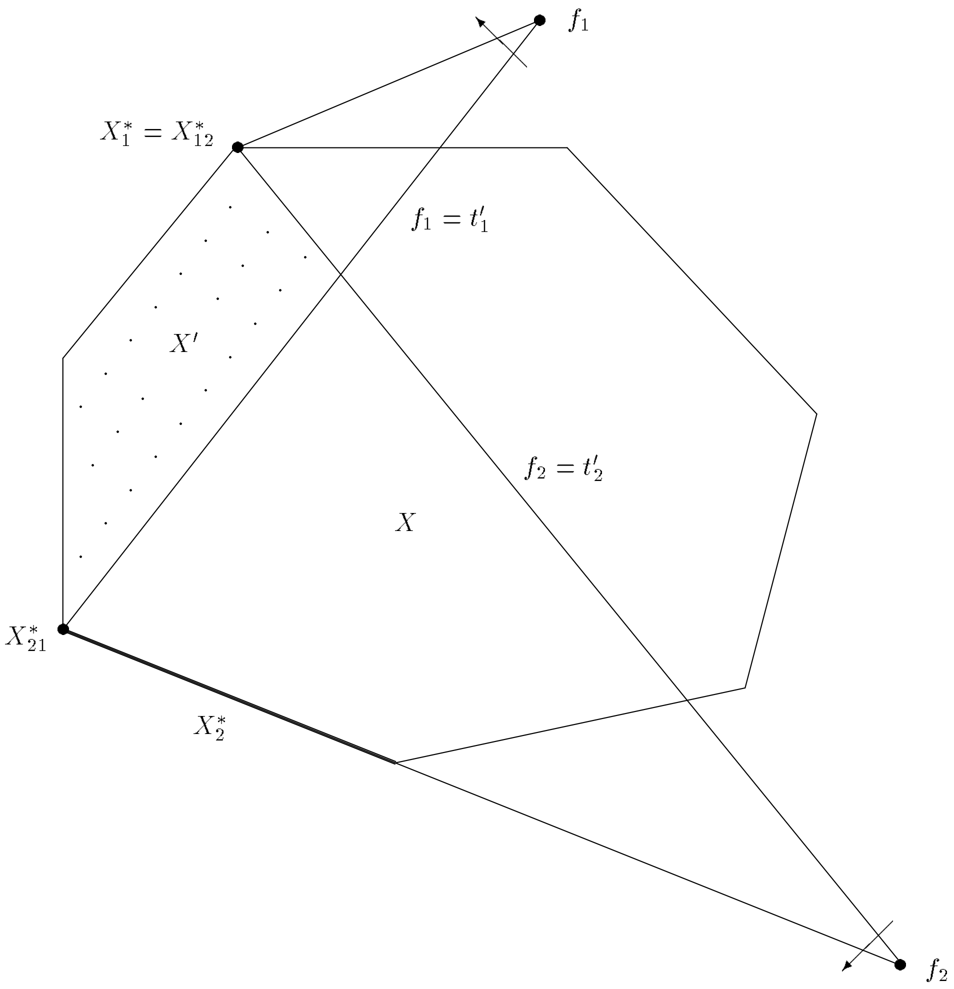

Figure 1. Reduction of the feasible region X to the subset X' by means of the values t'_j as lower bounds for the functions $f_j(x)$, $j = 1, 2$. The arrows near f_1 and f_2 indicate directions of increasing values

Then the following theorems hold [6]

Theorem 2. $x^* \in X' \quad \forall x^*$.

Theorem 3. *For each* $t_j \in [t'_j, \bar{t}_j]$ *the function* $f_i(x)$ *attains its maximum on* $X_j(t_j)$ *in the level surface* $\{x \in X \,:\, f_j(x) = t_j\}$ *and only there.*

Theorem 4. *The function* $z(x)$ *attains its local maxima (and consequently its global maximum) on* X *in edges or pieces of edges of* X *belonging to* X'.

4. Maximizing the Sum of Two Explicitly Quasimonotonic Functions

Now we consider problem (1) with $k = 2$, $h(f_1, f_2) = f_1 + f_2$ and a polyhedral feasible region, that is

$$\max\{z(x) = f_1(x) + f_2(x) \,:\, Ax = b,\ x \geq 0\} \tag{5}$$

Solving the subproblems (2) and (3) we get $\bar{t}_i$ and t'_i. If $\bar{t}_i = t'_i$ then our starting problem (5) is solved in accordance with Theorem 1 and its explanation. Otherwise we distinguish without loss of generality the cases

$$\bar{t}_1 + t'_2 > t'_1 + \bar{t}_2 \tag{6}$$

and

$$\bar{t}_1 + t'_2 = t'_1 + \bar{t}_2. \tag{7}$$

In the first case we restrict the search for the global maximum of $z(x)$ on X to such $x \in X'$ for which

$$t_1 + t_2 \geq \bar{t}_1 + t'_2.$$

This requires at least

$$t_1 \geq \bar{t}_1 - \bar{t}_2 + t'_2 \tag{8}$$

and the term on the right–hand side which we will denote by $\underline{t}_1^1$ improves according to (6) the hitherto known lower bound t'_1 for f_1. By it we exclude from the set X' which is represented as $X'_1 \cup X'_2 \cup X'_3$ in Figure 2 the last two subsets.

Pursuant to Theorem 3 is valid

$$\max\{f_2(x) \,:\, x \in X_1(\underline{t}_1^1) \cap X_2(t'_2)\} = \bar{t}_2^1 < \bar{t}_2$$

and $\bar{t}_2^1$ improves the upper bound for f_2 corresponding to the exclusion of X'_3. Setting $\bar{t}_2^1$ instead of $\bar{t}_2$ in (8) yields a better lower bound for f_1 which acts upon the upper bound for f_2. By iteration we obtain sequences $\{\underline{t}_1^l\}$ and $\{\bar{t}_2^l\}$. If these sequences converge to $\bar{t}_1$ and t'_2, respectively, then $z^* = \bar{t}_1 + t'_2$ is the optimal value and X_{12}^* is the set of the maximizers. If such a convergence does not happen or convergence at all proves to be too slow then it is suitable to bisect an interval $[\underline{t}_1^l, \bar{t}_1]$ and to continue on each of the halves, and so on.

In this way one has also to handle the second case (7) from the outset.

The illustration of the bisection is similar to Figure 2. But one has from $X' = X_1' \cup X_2' \cup X_3'$ (or a set resulted from X' and represented in this form) to exclude the subset X_2' and to continue on each of the remaining subsets in the sense of branch and bound.

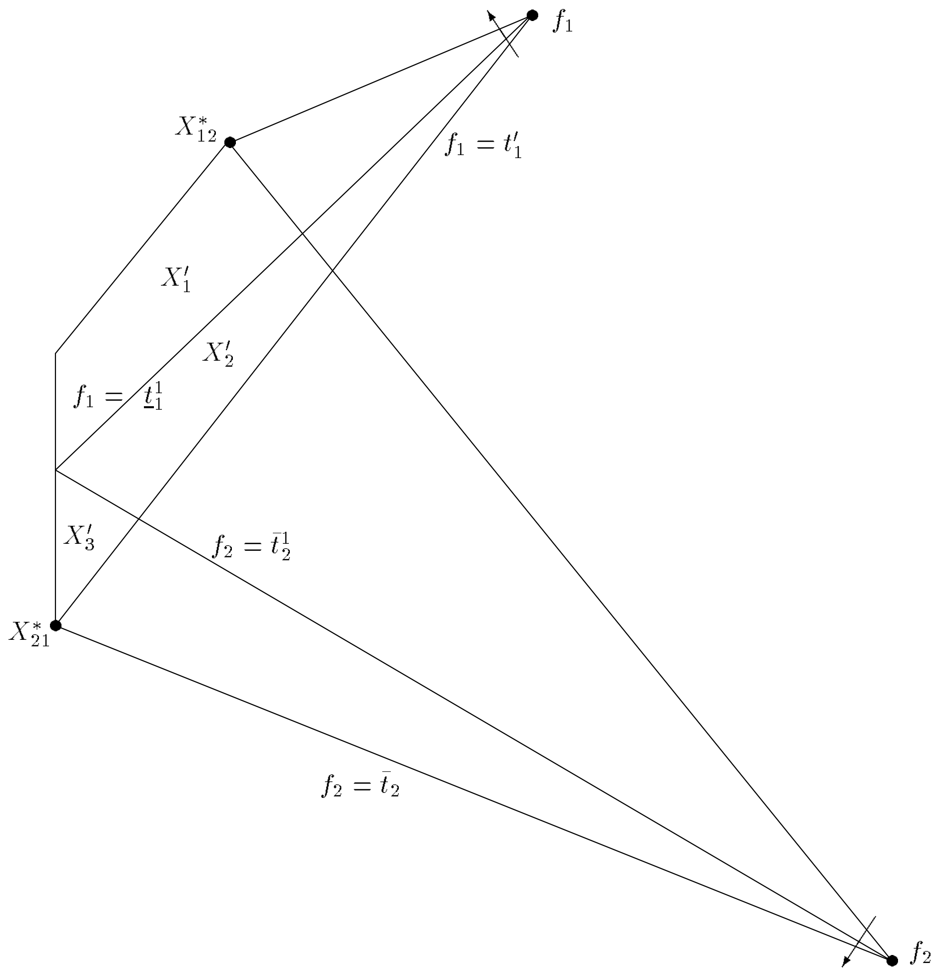

Figure 2. Reduction of $X' = X_1' \cup X_2' \cup X_3'$ to X_1' by the improved lower bound $\underline{t}_1^1$ for $f_1(x)$ which yields the improved upper bound $\bar{t}_2^1$ for $f_2(x)$

Considering the special case with linear fractional functions f_i Falk and Palocsay [3] transformed the starting problem in the criteria space and realized in this way the approach sketched here.

For illustration we refer to Figure 3. There Y denotes the image of X with respect to the mapping $F(x) = (f_1(x), f_2(x))^T$, the sum $h = f_1 + f_2$ is to maximize on Y and $\partial^+ Y$ denotes the upper boundary of Y between the points $(t_1', \bar{t}_2)$ and $(\bar{t}_1, t_2')$.

From Theorem 3 we obtain the following result.

Theorem 5. *The boundary* $\partial^+ Y$ *is strictly decreasing on the interval* $[t_1', \bar{t}_1]$.

Generalizing this approach to sums of three linear fractional functions one cannot expect that the efficient set $\partial^+ Y$ is coordinatewise strictly decreasing.

Falk and Palocsay [3] showed that Dinkelbach's parametric method for linear fractional programming problems [2] in contrast to the claim of Almogy and Levin [1] does not yield correct results if the objective function is a sum of several linear fractional functions. But Falk and Palocsay [3] showed also that the optimal value function of the assigned parametric problem has useful properties which allow to state a sufficient optimality criterion.

For optimizing the sum of two linear fractional functions we propose a modification of the one–parametric algorithm for optimizing a product which we present in the following Section 5.

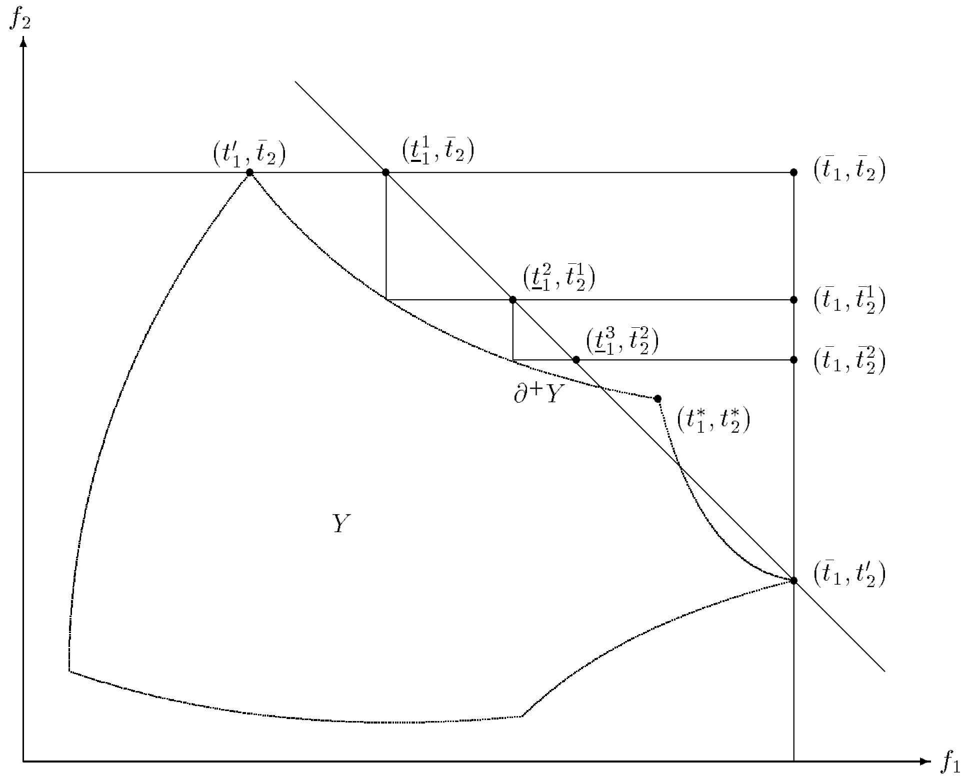

Figure 3. Representation of the problem in the criteria space and of the improvement of the bounds for f_1 and f_2

5. Optimizing the Product of Two Linear Fractional Functions

As another special case of (1) we get the problem

$$\max\left\{z(x)=\frac{c_1^Tx+c_{10}}{d_1^Tx+d_{10}}\cdot\frac{c_2^Tx+c_{20}}{d_2^Tx+d_{20}}\ :\ Ax=b,\ \ x\geq 0\right\} \tag{9}$$

which is equivalent to

$$\min\left\{f(x)=\frac{d_1^Tx+d_{10}}{c_1^Tx+c_{10}}\cdot\frac{d_2^Tx+d_{20}}{c_2^Tx+c_{20}}\ :\ Ax=b,\ \ x\geq 0\right\}. \tag{10}$$

Applying the Charnes–Cooper transformation and setting

$$w=\begin{pmatrix} y\\ y_0\end{pmatrix},\ \tilde{A}=(A,-b),\ \tilde{c}_i^T=(c_i^T,c_{i0}),\ \tilde{d}_i^T=(d_i^T,d_{i0}),\ i=1,2,$$

problem (10) can be rewritten as follows:

$$\min\left\{\tilde{f}(w)=\frac{\tilde{d}_1^Tw}{\tilde{c}_1^Tw}\cdot\frac{\tilde{d}_2^Tw}{\tilde{c}_2^Tw}\ :\ \tilde{A}w=0,\ \ w\geq 0\right\}. \tag{11}$$

For solving (11) and (10), respectively, Konno and Yajima [7] proposed a two–parametric algorithm and a branch and bound algorithm.

We assume without loss of generality that $\tilde{c}_1^Tw=1$ holds in (11) and consider the following one–parametric problem (P_ξ)

$$\text{minimize}\quad \tilde{g}(w,\xi)=\xi\cdot\frac{\tilde{d}_2^Tw}{\tilde{c}_2^Tw} \tag{12}$$

subject to

$$\tilde{A}w=0,\quad \tilde{c}_1^Tw=1,\quad \tilde{d}_1^Tw=\xi,\quad w\geq 0,\quad \xi_{\min}\leq\xi\leq\xi_{\max}, \tag{13}$$

where

$$\begin{aligned}\xi_{\min}&=\min\{\tilde{d}_1^Tw\ :\ \tilde{A}w=0,\ \tilde{c}_1^Tw=1,\ w\geq 0\},\\ \xi_{\max}&=\max\{\tilde{d}_1^Tw\ :\ \tilde{A}w=0,\ \tilde{c}_1^Tw=1,\ w\geq 0\}\end{aligned}$$

exist and are positive under our assumptions.

For each $\xi\in[\xi_{\min},\xi_{\max}]$ we ask for the optimal solution of the following linear fractional programming problem $P(\xi)$

$$\text{minimize}\quad g(w)=\frac{\tilde{d}_2^Tw}{\tilde{c}_2^Tw}$$

subject to

$$\hat{A}w=\begin{pmatrix}0\\1\\ \xi\end{pmatrix},\quad w\geq 0,\quad \text{where}\quad \hat{A}=\begin{pmatrix}\tilde{A}\\ \tilde{c}_1^T\\ \tilde{d}_1^T\end{pmatrix}.$$

Beginning with $\xi=\xi_{\min}$ we apply the modified simplex method developed by Martos ([9], Section 11.4). Let be $\xi_0\in[\xi_{\min},\xi_{\max}]$, B the corresponding optimal basis matrix, N the matrix of the nonbasic columns of $\hat{A}$, v_B (v_N)

the basic (nonbasic) part of the vector v and J the index set of the nonbasic variables. We have

$$w_B = B^{-1}\begin{pmatrix}0\\1\\\xi\end{pmatrix} = q+\xi p, \tag{14}$$

where q and p denote the last but one and the last column of B^{-1}, respectively. From the feasibility condition $w_B \geq 0$ we get an interval $[\alpha_d, \beta_d]$ with

$$\alpha_d = \max\{-q_i/p_i \;:\; p_i > 0\}, \tag{15}$$

$$\beta_d = \min\{-q_i/p_i \;:\; p_i < 0\}. \tag{16}$$

The optimality criterion

$$t_j = z_j - g(w)n_j \geq 0 \quad \forall j \in J$$

reads with respect to (14) and $w_N = 0$ as follows

$$z_j \geq \frac{\tilde{d}_{2B}^T q + \xi \tilde{d}_{2B}^T p}{\tilde{c}_{2B}^T q + \xi \tilde{c}_{2B}^T p} \cdot n_j \quad \forall j \in J.$$

Let $v(j)$ denote the vector $(z_j/n_j)\cdot\tilde{c}_{2B} - \tilde{d}_{2B} \quad \forall j \in J$. The optimality condition yields an interval $[\alpha_p, \beta_p]$ with

$$\alpha_p = \max\{-v(j)^T q/v(j)^T p \;:\; n_j \cdot v(j)^T p > 0\}, \tag{17}$$

$$\beta_p = \max\{-v(j)^T q/v(j)^T p \;:\; n_j \cdot v(j)^T p < 0\}. \tag{18}$$

Correspondingly, B is optimal basis matrix for $P(\xi)$ with $\xi \in [\underline{\xi}, \bar{\xi}]$ where

$$\underline{\xi} = \max\{\alpha_p, \alpha_d\}, \tag{19}$$

$$\bar{\xi} = \min\{\beta_p, \beta_d\}. \tag{20}$$

By the assumptions

$$\underline{\xi} < \bar{\xi}, \quad \alpha_d \neq \alpha_p, \quad \beta_d \neq \beta_p \quad \forall \xi \in [\xi_{\min}, \xi_{\max}]$$

we exclude degeneracy. For the case of degeneracy we refer to Terlaky's pivoting rule of the criss–cross method [12], cf. also [8].

Then we get a partition of the interval $[\xi_{\min}, \xi_{\max}]$ into finitely many subintervals $[\xi_l, \xi_{l+1}]$. Let B_l be the optimal basis matrix belonging to $[\xi_l, \xi_{l+1}] = I_l$.

Setting $w_B = B_l^{-1}\begin{pmatrix}0\\1\\\xi\end{pmatrix}$, $w_N = 0$ in the objective function (12) of (P_ξ) yields

$$\tilde{g}(w(\xi), \xi) = \xi \cdot \frac{\tilde{d}_{2B}^T(q+\xi p)}{\tilde{c}_{2B}^T(q+\xi p)} = \frac{a\xi + b\xi^2}{c + d\xi}. \tag{21}$$

It is easy to determine the minimum of $\tilde{g}$ on I_l and we get the solution of (11) and consequently of (10) by comparison of the minima on all subintervals I_l.

Our approach is summarized in the following algorithm.

One–Parametric Algorithm

Step 1: (Initialization)

$$\xi_{\min} := \min\{\tilde{c}_1^T w \;:\; \tilde{A}w = 0,\; \tilde{d}_1^T w = 1,\; w \geq 0\};$$

$$\xi_{\max} := \max\{\tilde{c}_1^T w \;:\; \tilde{A}w = 0,\; \tilde{d}_1^T w = 1,\; w \geq 0\};$$

Let w^* be optimal basic solution of $P(\xi_{\min})$ and let B_1 be the corresponding basis matrix; set $l := 1$, $rec := \tilde{g}(w^*, \xi_{\min})$, $\underline{\xi} := \xi_{\min}$;

Step 2: Calculate $\bar{\xi}$ according to (16), (18) and (20);

Step 3: Minimize $\tilde{g}(w(\xi), \xi)$ on $[\underline{\xi}, \bar{\xi}]$ und set with the minimizer $\hat{\xi}$

$$cand := \hat{\xi} \cdot \frac{\tilde{d}_{2B}^T(q + \hat{\xi}p)}{\tilde{c}_{2B}^T(q + \hat{\xi}p)};$$

if $cand < rec$ **then**
begin $rec := cand$; $w_B^* := q + \hat{\xi}p$ **end**;
if $\bar{\xi} = \xi_{\max}$ **then terminate**;

Step 4: **if** $\bar{\xi} = \beta_p$ **then**
Calculate a new basis B_{l+1} of $P(\bar{\xi})$ by primal pivoting in the column with $\beta_p = -v(r)^T q / v(r)^T p$;

Step 5: **if** $\bar{\xi} = \beta_d$ **then**
Calculate a new basis B_{l+1} of $P(\bar{\xi})$ by dual pivoting in the row with $\beta_d = -q_s / p_s$;

Step 6: Set $\underline{\xi} := \bar{\xi}$, $l := l + 1$; **goto Step 2**; ■

Remark 1. In Step 1 the value $\xi_{\max}$ is determinated for controlling the progress of the solution process.

Remark 2. With respect to the last condition in Step 3 we note that the algorithm would stop automatically if $\bar{\xi}$ reaches the value $\xi_{\max}$ because then no further iteration is possible in consequence of occuring infeasibility in $P(\xi)$.

For minimizing the sum of two linear fractional functions we propose a modification of this algorithm where the difference consists only in the replacement of the product for $\tilde{g}(w, \xi)$ in (12) and consequently in Step 1 and Step 3 of the algorithm by the corresponding sum.

6. Computational Experiments

The branch and bound algorithm (BAB) and the two–parametric algorithm (TPA) by Konno and Yajima [7] and our one–parametric algorithm (OPA) for optimizing the product of two linear fractional functions has been tested in the form of programs in Borland Pascal 7.0. The test was performed on groups of randomly generated problems of various size, containing ten problems each. The relative expense was measured by the number of simplex iterations. The arithmetical means of such numbers for the algorithms considered here and for solving of linear optimization problems (LOP) of corresponding size are given in Table I.

Table I. Results of computational experiments

Size of problems	Number of simplex iterations			
$m \times n$	BAB	TPA	OPA	LOP
10 × 20	363.5	108.7	53.6	20.0
15 × 20	268.6	63.1	49.3	22.4
15 × 30	614.0	345.6	85.2	27.2
20 × 30	650.3	264.7	91.5	35.5
25 × 50	1154.3	1170.3	168.0	49.1
35 × 50	1023.4	478.6	159.0	57.8
45 × 50	864.8	128.0	119.0	55.5
33 × 65	1603.3	2487.0	242.2	70.1
50 × 65	1341.8	697.0	176.0	74.6
60 × 65	922.1	160.2	144.6	70.5
39 × 78	1985.0	4140.5	318.1	93.1

We note that the CPU time on a PC for solving five problems of size 20×60 by the programs in the given order varied between 58 and 89, 92 and 204, and 18 and 19 seconds, respectively.

With the inclusion of slack variables holds $n \approx 2m$ for the test examples of Konno and Yajima [7]. Then our experiments verify the statement of Konno and Yajima that BAB from about $n = 50$ becomes increasingly superior to TPA, cf. the corresponding numbers in the first row for each n in Table I. But OPA seems to be more promising. Table II shows ratios (rounded off to two decimals) of numbers of simplex iterations from Table I.

Table II. Ratios of numbers of simplex iterations for $n \approx 2m$

Comparison	Dimension n				
between	20	30	50	65	78
BAB and TPA	3.34	1.78	0.99	0.64	0.48
TPA and OPA	2.03	4.06	6.97	10.27	13.02
BAB and OPA	6.78	7.21	6.87	6.62	6.24

For optimizing the sum of two linear fractional functions the expense of our modified one–parametric algorithm also measured by the number of simplex iterations is identical to that for optimizing products under the same constraints. A comparison between the algorithm by Falk and Palocsay [3] (cf. Section 4) and our modified algorithm is intended yet.

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