Classical and novel analysis of positive invariance and strong stability preserving time integrators

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The state space of time dependent differential equations (ODEs, PDEs, DAEs) of applied mathematics often posses positively invariant subsets (which are subsets of the state space from which trajectories do not escape from the subset forward in time if they initiate from there) and have functionals that are decreasing along the solutions. For examples of positively invariant sets we refer to the positive orthant in many applications (for concentration like variables at modelling chemical or biological diffusion-advection-reaction systems) and regions of feasibility for the Euler equations of gas dynamics. Total variation, entropy, Lyapunov functions form some examples for functionals that are decreasing along the solutions of some PDEs or related semidiscretized ODEs modelling transport processes. Under the numerical time discretization methods these properties are natural and in many cases necessary to be preserved (e.g. negative concentrations or increasing entropy have no physical meaning, furthermore diminition of total variation is a pillar of ensuring convergence of numerical methods in gas dynamics). However, it is not trivial at all what numerical methods and what time step sizes imply the preservations of invariant sets and decreasing functionals along numerical solutions; accuracy and stability of the numerical method together do not imply this. In the last decades an analysis is developed for general numerical time stepping schemes to determine the suitable time steps that fulfill the requirements of the preservation properties. The suitable step sizes are characterized in terms of the scheme coefficients and some simple characteristics of the problem. These are, respectively, the absolute monotonicity radius of the scheme and the largest step size under which the Explicit Euler method has the preservation property. Thus this classical analysis works under the condition that the Explicit Euler method fulfils the preservation property.
In this lecture we present the classical analysis with references. In addition, we shall give some examples when this can be applied to nontrivial situations (e.g. for semidiscretized hyperbolic conservation laws). However, some simple particular examples show that the numerical preservation property is present but the classical analysis can be applied since the Explicit Euler method does not respect the preservation property for any step sizes. Examples for these include, among others, FEM semidis-

cretizations of the heat equations, spectral difference semidiscretization for the advection equation, high order WENO schemes for advection and DAEs for transport. We shall present another condition on the problem and some different terms of the scheme coefficients that guarantee the numerical preservation properties for implicit methods. We shall apply them in the examples above to show the applicability of the novel analysis.