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A note on the uniform perturbation index[1]

ABSTRACT. For a given differential-algebraic equation (DAE) the perturbation index gives a measure for the sensitivity of a solution w. r. t. small perturbations. If we consider, however, *classes* of DAEs (e. g. all DAEs that arise as semi-discretizations of a given partial DAE by the method of lines) then the error bound in the definition of the perturbation index may become arbitrarily large even if the perturbation index does not exceed 1. We illustrate this fact by 2 examples and define as alternative the *uniform* perturbation index that gives simultaneously error bounds for *all* DAEs of a given class. We prove that in one example each individual DAE has perturbation index 1 but the uniform perturbation index is 2. Another example illustrates that the class of all finite difference semi-discretizations may even have *no* uniform perturbation index if the given partial DAE has perturbation index 2.

KEY WORDS: ifferential-algebraic equations, perturbation index, Baumgarte stabilization, partial DAEs, method of lines

1 Introduction

One main difficulty in the numerical integration of initial value problems for higher index differential-algebraic equations (DAEs)

$$F(x'(t), x(t), t) = 0\ ,\quad x(0) = x_0\ ,\quad t \in [0, T] \tag{1}$$

is the fact, that the solution does not depend continuously on small perturbations in the equations. The discrete analogue is the amplification of small errors during the numerical integration. Such errors arise e. g. as round-off errors or because of stopping the iterative solution of nonlinear equations. A quantitative measure of this effect is given by the perturbation index

[1]This paper is an extended version of a talk presented at the conference "DAEs, Related Fields and Applications" Oberwolfach (Germany), November 1995.

Definition 1 ([8, p. 478f]) *The DAE (1) has* perturbation index m *along a solution* $x(t)$ *on* $[0,T]$*, if* m *is the smallest integer such that, for all functions* $\hat{x}(t)$ *having a defect* $F(\hat{x}'(t),\hat{x}(t),t)=\delta(t)$ *there exists on* $[0,T]$ *an estimate*

$$\|\hat{x}(t)-x(t)\| \leq C_0(\|\hat{x}(0)-x(0)\| + \max_{\tau\in[0,t]}\|\delta(\tau)\| + \ldots + \max_{\tau\in[0,t]}\|\delta^{(m-1)}(\tau)\|) \qquad (2)$$

whenever the expression on the right hand side is sufficiently small.

If appropriate discretization methods are used then the error that is caused by the amplification of perturbations in the *numerical* solution is bounded by $C_0^* \cdot \frac{1}{h^{m-1}}\Delta$. Here Δ denotes an upper bound for the errors that arise in one single step of integration, h is the stepsize of integration. This term can be interpreted as discretization of the error bound (2) for the analytical solution (see e. g. [7], [1]).

As long as the constants C_0 and C_0^* are of moderate size the sensitivity of the solution w. r. t. perturbations can be completely characterized by the integer m in (2), i. e. by the perturbation index. These constants C_0 and C_0^* depend in general on bounds for partial derivatives of F in a neighbourhood of the analytical solution $x(t)$, for many applications they are $\mathcal{O}(1)$, [8, p. 480f]. The situation changes if we consider a class of DAEs with a parameter that may be arbitrarily small. Such a class appears e. g. if a partial DAE ([4], [5]) is discretized in space by the method of lines. The resulting semi-discretized DAEs depend on the space discretization. If the space discretization is refined then the constant C_0 in (2) may become arbitrarily large. Thus for practical computations the error bound $C_0^* \cdot \frac{1}{h^{m-1}}\Delta$ does not give any useful information about the amplification of errors during integration.

In Section 2 we study this effect in detail for a Baumgarte-like stabilization of differential-algebraic systems of index 2. For large values of the Baumgarte coefficient α the error bound of Definition 1 is useless since $\lim_{\alpha\to\infty} C_0 = \infty$. That is why we introduce in Section 3 the *uniform perturbation index* for a class of DAEs. In Section 4 this concept is applied to a system of 2 linear partial differential equations [5, Example 1]. This partial DAE has index 2 but the semi-discretization by finite differences on an equidistant grid is a DAE of perturbation index 1. The main result of Section 4 is that for this example — depending on the coefficients of the partial DAE — either

- the uniform perturbation index of the class of all these semi-discretizations is 2 and coincides thus with the index of the underlying partial DAE or
- the class of all these semi-discretizations has no uniform perturbation index at all.

2 A perturbation analysis for stabilized differential-algebraic systems of index 2

In this we consider the differential-algebraic system

$$\left.\begin{array}{rcl} y'(t) & = & f(y(t),z(t)) \\ 0 & = & g(y(t)) \end{array}\right\} \ , \quad t \in [0,T] \ , \quad y(0)=y_0 \ , \quad z(0)=z_0 \tag{3}$$

that is supposed to have a solution $y : [0,T] \to \mathbb{R}^{n_y}$, $z : [0,T] \to \mathbb{R}^{n_z}$. We assume that in a neighbourhood of this solution functions f and g are sufficiently differentiable and satisfy the index-2 condition "$[g_y f_z](\eta,\zeta)$ non-singular".

The differential-algebraic system (3) has (perturbation and differential) index 2 [8, p. 480f]: We have

$$\begin{aligned} \|\hat{y}(t)-y(t)\|+&\|\hat{z}(t)-z(t)\| \leq \\ &\leq C_0(\|\hat{y}(0)-y(0)\| + \max_{\tau\in[0,t]} \|\delta(\tau)\| + \max_{\tau\in[0,t]} \|\theta(\tau)\| + \max_{\tau\in[0,t]} \|\theta'(\tau)\|) \end{aligned} \tag{4}$$

for all $t \in [0,T]$ if $\hat{y}'(t) = f(\hat{y},\hat{z}) + \delta(t)$ and $g(\hat{y}(t)) = \theta(t)$. Here the constant C_0 depends on the length T of the time interval and on upper bounds for $\|[(g_y f_z)^{-1}](\eta,\zeta)\|$ and for partial derivatives of f and g.

Similar to the stabilization of model equations for constrained mechanical systems that was introduced by Baumgarte ([3]) the index of (3) can be reduced to 1 if the algebraic constraints $g(y)=0$ are substituted by

$$0 = \frac{1}{\alpha}\frac{d}{dt}g(y(t)) + g(y(t)) \tag{5}$$

with a constant $\alpha > 0$. With this substitution the analytical solution of (3) remains unchanged since $g(y(t))=0$ implies $\frac{d}{dt}g(y(t))=0$ and thus also (5). On the other hand consistent initial values for (3) satisfy $g(y(0))=0$ such that (5) results in $g(y(t))=0$, $(t\in[0,T])$.

As for (3) we study the sensitivity of the solution of the stabilized system w. r. t. small perturbations comparing $(y(t),z(t))$ with functions $(\hat{y}_\alpha(t),\hat{z}_\alpha(t))$ that satisfy for $t\in[0,T]$

$$\left.\begin{array}{rcl} \hat{y}'_\alpha(t) & = & f(\hat{y}_\alpha(t),\hat{z}_\alpha(t)) + \delta(t) \\ \theta(t) & = & \dfrac{1}{\alpha}\dfrac{d}{dt}g(\hat{y}_\alpha(t)) + g(\hat{y}_\alpha(t)) \end{array}\right\} \tag{6}$$

The key to these error bounds are the following estimates:

Lemma 1 *a) Let functions* $\tilde{\delta} \in \mathcal{C}[0,T]$, $\tilde{\theta} \in \mathcal{C}^1[0,T]$ *and a constant* $\alpha > 0$ *be given. The solutions of the linear differential equation*

$$w'(t) + \alpha w(t) = \tilde{\delta}(t) + \alpha\tilde{\theta}(t) \tag{7}$$

satisfy

$$\begin{aligned} |w(t) - \tilde{\theta}(t)| &\leq |w(0) - \tilde{\theta}(0)|e^{-\alpha t} + \frac{1}{\alpha} \cdot \max_{\tau\in[0,t]}(|\tilde{\delta}(\tau)| + |\tilde{\theta}'(\tau)|) \ , \\ |w'(t)| &\leq |w'(0)|e^{-\alpha t} + \max_{\tau\in[0,t]}(|\tilde{\delta}(\tau)| + |\tilde{\theta}'(\tau)|) \ . \end{aligned}$$

b) If $\tilde{\delta}(t) \equiv 0$, $\tilde{\theta}(t) = \Theta \cos\frac{t}{\varepsilon}$ *and* $w(0) = \varepsilon^2\alpha^2\Theta/(1+\varepsilon^2\alpha^2)$ *with (small) positive parameters* Θ, ε *then the solution* $w(t)$ *of (7) is*

$$w_\alpha(t) = (1 - \frac{1}{1+\varepsilon^2\alpha^2})\,\Theta\cos\frac{t}{\varepsilon} + \frac{\varepsilon\alpha}{1+\varepsilon^2\alpha^2}\,\Theta\sin\frac{t}{\varepsilon} \ . \tag{8}$$

Proof: The solution of (7) is given by

$$w(t) = w(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-\tau)}(\tilde{\delta}(\tau) + \alpha\tilde{\theta}(\tau))\,d\tau \ . \tag{9}$$

Integration by parts results in

$$\alpha\int_0^t e^{-\alpha(t-\tau)}\tilde{\theta}(\tau)\,d\tau = \left[e^{-\alpha(t-\tau)}\tilde{\theta}(\tau)\right]_0^t - \int_0^t e^{-\alpha(t-\tau)}\tilde{\theta}'(\tau)\,d\tau$$

and finally we have

$\int_0^t e^{-\alpha(t-\tau)}\,d\tau = \frac{1}{\alpha}(1-e^{-\alpha t}) < \frac{1}{\alpha}$,

i. e.

$$|\int_0^t e^{-\alpha(t-\tau)}(\tilde{\delta}(\tau) - \tilde{\theta}'(\tau))\,d\tau\,| < \frac{1}{\alpha} \cdot \max_{\tau\in[0,t]}(|\tilde{\delta}(\tau)| + |\tilde{\theta}'(\tau)|) \ .$$

The estimate for $|w'(t)|$ is obtained from $w'(t) = \tilde{\delta}(t) - \alpha(w(t) - \tilde{\theta}(t))$. To prove part b) of the lemma the given functions $\tilde{\delta}$, $\tilde{\theta}$ are inserted into (9). ■

We now return to Eqs. (6). Because of

$$\frac{d}{dt}g(y(t)) = g_y(y(t))y'(t) = [g_y f](y(t), z(t))$$

Eqs. (5) can be solved w. r. t. the algebraic components z and the stabilized system is of (differential and perturbation) index 1 ([8, p. 480]), we get

$$\begin{aligned} &\|\hat{y}_\alpha(t) - y(t)\| + \|\hat{z}_\alpha(t) - z(t)\| \leq \\ &\qquad \leq C_{0,\alpha}(\|\hat{y}_\alpha(0) - y(0)\| + \|\hat{z}_\alpha(0) - z(0)\| + \max_{\tau\in[0,t]} \|\delta(\tau)\| + \max_{\tau\in[0,t]} \|\theta(\tau)\|) \ . \end{aligned} \tag{10}$$

In general, however, this estimate can be satisfied for large values of α only, if $C_{0,\alpha} \to \infty$, ($\alpha \to \infty$). This is not surprising since (5) approximates for large values of α the algebraic constraint $g(y) = 0$ of the index-2 system (3) and there is (per definitionem) no estimate like (10) for systems of perturbation index 2.

Example 1 The system $y_1' = y_2' = z$, $0 = y_1 + y_2$ is of index 2. The solution of the initial value problem $y_1(0) = y_{1,0}$ is constant:

$$0 = y_1 + y_2 \ \Rightarrow \ 0 = y_1' + y_2' = 2z \ \Rightarrow \ z(t) \equiv 0 \ , \ y_1(t) \equiv y_{1,0} \ , \ y_2(t) \equiv -y_{1,0} \ .$$

Consider now functions $\hat{y}_\alpha$, $\hat{z}_\alpha$ that are defined by

$$\hat{y}_{\alpha,1}(t) = y_{1,0} + \frac{1}{2}(w_\alpha(t) - w_\alpha(0)) \ , \ \hat{y}_{\alpha,2}(t) = w_\alpha(t) - \hat{y}_{\alpha,1}(t) \ , \ \hat{z}_\alpha(t) = \frac{1}{2}w_\alpha'(t)$$

with $w_\alpha(t)$ from (8). These functions satisfy $g(\hat{y}_\alpha(t)) = \hat{y}_{\alpha,1}(t) + \hat{y}_{\alpha,2}(t) = w_\alpha(t)$ and we get in (6) $\delta(t) \equiv 0$, $\theta(t) = \Theta \cos \frac{t}{\varepsilon}$ (see Lemma 1), i. e. $\|\delta(t)\| = 0$, $\|\theta(t)\| = \mathcal{O}(\Theta)$, $\|\theta'(t)\| = \mathcal{O}(\frac{1}{\varepsilon}\Theta)$. Straightforward computations give

$$|\hat{z}_\alpha(t) - z(t)| = \frac{1}{2}|w_\alpha'(t)| = \frac{1}{2}| - \frac{\varepsilon\alpha^2}{1+\varepsilon^2\alpha^2} \sin\frac{t}{\varepsilon} + \frac{\alpha}{1+\varepsilon^2\alpha^2}\cos\frac{t}{\varepsilon}| \cdot \Theta$$

and for $\alpha \geq 1$ the constant $C_{0,\alpha}$ in (10) has to satisfy $C_{0,\alpha} \geq \frac{1}{12}\sqrt{\alpha}$ since the special choice $\varepsilon = \frac{1}{\sqrt{\alpha}}$ results in $\hat{y}_{\alpha,1}(0) = y_1(0)$, $|\hat{y}_{\alpha,2}(0) - y_2(0)| = |w_\alpha(0)| = \frac{\alpha}{1+\alpha}\Theta \leq \Theta$, $|\hat{z}_\alpha(0) - z(0)| = \frac{1}{2}\frac{\alpha}{1+\alpha}\Theta \leq \Theta$, $|\theta(t)| \leq \Theta$ and $|\hat{z}_\alpha(\frac{\pi}{2}\varepsilon) - z(\frac{\pi}{2}\varepsilon)| = \frac{1}{2}\frac{\alpha^{3/2}}{1+\alpha}\Theta \geq \frac{1}{4}\sqrt{\alpha}\Theta$.

I. e. standard perturbation index theory gives with (10) an error estimate for the stabilized system that grows rapidly for $\alpha \to \infty$. If $\max_{\tau\in[0,t]} \|\theta'(\tau)\|$ is of moderate size and $\alpha \gg 1$ then (10) overestimates the influence of small perturbations on $(y(t), z(t))$ substantially, (see Example 2).

It is known from the literature (e. g. [2]) that neither differential nor standard perturbation index is an appropriate measure for the difficulties that one has to expect in the numerical solution of Baumgarte-like stabilized differential-algebraic systems with large Baumgarte coefficients. Baumgarte stabilization reduces the index (in our example from 2 to 1) but because of boundary layers (see the terms $\ldots e^{-\alpha t}$ in Lemma 1) the numerical solution of the index-reduced system might be even more complicated than that of the original higher index system if the Baumgarte coefficients are large. Furthermore for large Baumgarte coefficients the index-reduced system is less robust against perturbations than the (low) perturbation index suggests.

3 The uniform perturbation index

The results of Section 2 motivate the extension of the perturbation analysis to classes of DAEs

$$F_\alpha(x'(t), x(t), t) = 0\ ,\quad x(0) = x_0\ ,\quad t \in [0, T] \tag{11}$$

where $\alpha \in M_\alpha$ denotes some free parameter. In this note we restrict ourselves to scalar parameters α, the dimension of x (and thus also the norm $\|\cdot\|$ in (12)) may vary with α (see Section 4).

Definition 2 *The class of DAEs (11) has* uniform perturbation index m *along solutions $x_\alpha(t)$ on $[0, T]$, if $m \geq 1$ is the smallest integer such that, for all $\alpha \in M_\alpha$ and for all functions $\hat{x}_\alpha(t)$ having a defect $F_\alpha(\hat{x}'_\alpha(t), \hat{x}_\alpha(t), t) = \delta(t)$ there exists on $[0, T]$ an estimate*

$$\|\hat{x}_\alpha(t) - x_\alpha(t)\| \leq C_0(\|\hat{x}_\alpha(0) - x_\alpha(0)\| + \max_{\tau\in[0,t]} \|\delta(\tau)\| + \ldots + \max_{\tau\in[0,t]} \|\delta^{(m-1)}(\tau)\|) \tag{12}$$

whenever the expression on the right hand side is sufficiently small. Here C_0 denotes a constant that is independent of α and $\delta(\tau)$.

Remarks 1 a) The condition $m \geq 1$ in Definition 2 can be relaxed to $m \geq 0$ if for $m = 0$ the term $\delta^{(m-1)}(\tau)$ is interpreted as $\int_0^\tau \delta(w)\, dw$ (cf. [8, p. 479]). I. e. the class of DAEs (11) has the uniform perturbation index $m = 0$ if instead of (12) the (stronger) estimate

$$\|\hat{x}_\alpha(t) - x_\alpha(t)\| \leq C_0(\|\hat{x}_\alpha(0) - x_\alpha(0)\| + \max_{\tau\in[0,t]} \|\int_0^\tau \delta(w)\, dw\|\ )$$

is satisfied.

b) The idea of error bounds that are independent of a (small) parameter is extensively used in the analysis of singular perturbation problems. As an example we refer to the work of Hairer et al. [6] and Lubich [9] who investigate in close connection to DAE-theory singularly perturbed ordinary differential equations (ODEs) that all have perturbation index 0 ([8, p. 479]). In terms of Definition 2 the class of singularly perturbed ODEs in [6] has uniform perturbation index 1. The class of singular singularly perturbed ODEs that is considered in [9] has even uniform perturbation index 3, i. e. the uniform perturbation index exceeds the classical one by 3.

c) Mattheij [10] and Wijckmans [12] analyse linear DAEs that are “close to a higher-index DAE” ([12, pp. 53ff, 73ff]) and study the sensitivity of the solution w. r. t. small perturbations. The present paper is closely related to their approach and uses with Lemma 1 the same basic tool in the proof of uniform error estimates. The extension of the well established concept of perturbation index from individual DAEs to classes of DAEs gives a unified framework for various case studies from the literature.

Example 2 Consider the class of all Baumgarte-like stabilized differential-algebraic systems of index 2 with parameter $\alpha \geq \alpha_0 > 0$ that was introduced in Section 2. If the parameter α is fixed then these stabilized systems have the classical perturbation index 1. If we consider, however, the class of all these systems then estimate (12) can not be satisfied with $m = 1$ and a constant C_0 that is independent of α (see Example 1).
Applying componentwise Lemma 1 to $\frac{d}{dt} g(\hat{y}_\alpha(t)) + \alpha g(\hat{y}_\alpha(t)) = \alpha\theta(t)$

$$\frac{d}{dt} g(\hat{y}_\alpha(t)) + \alpha g(\hat{y}_\alpha(t)) = \alpha\theta(t)$$

we get $g(\hat{y}_\alpha(t)) = \hat{\theta}(t)$ with

$$\begin{aligned} \|\hat{\theta}(t) - \theta(t)\| &\leq \|g(\hat{y}_\alpha(0)) - \theta(0)\| \cdot e^{-\alpha t} + \frac{1}{\alpha} \cdot \max_{\tau \in [0,t]} \|\theta'(\tau)\| \ , \\ \|\hat{\theta}'(t)\| &\leq \|\frac{d}{dt} g(\hat{y}_\alpha(t))\Big|_{t=0}\| \cdot e^{-\alpha t} + \max_{\tau \in [0,t]} \|\theta'(\tau)\| \ , \end{aligned}$$

Because of $\frac{d}{dt} g(\hat{y}_\alpha(t)) = g_y(\hat{y}_\alpha(t))\hat{y}'_\alpha(t) = [g_y f](\hat{y}_\alpha(t), \hat{z}_\alpha(t)) + g_y(\hat{y}_\alpha(t))\delta(t)$ and $g(y(0)) = [g_y f](y(0), z(0)) = 0$ we have

$$\begin{aligned} \hat{y}'_\alpha(t) &= f(\hat{y}_\alpha, \hat{z}_\alpha) + \delta(t) \\ \hat{\theta}(t) &= g(\hat{y}_\alpha(t)) \end{aligned}$$

with

$$\begin{aligned} \|\hat{\theta}(t)\| &\leq \|\theta(t)\| + \frac{1}{\alpha_0} \max_{\tau \in [0,t]} \|\theta'(\tau)\| + \mathcal{O}(1)(\|\hat{y}_\alpha(0) - y(0)\| + \|\theta(0)\|) \\ \|\hat{\theta}'(t)\| &\leq \max_{\tau \in [0,t]} \|\theta'(\tau)\| + \mathcal{O}(1)(\|\hat{y}_\alpha(0) - y(0)\| + \|\hat{z}_\alpha(0) - z(0)\| + \|\theta'(0)\| + \|\delta(0)\|) \ , \end{aligned}$$

(the constants in the $\mathcal{O}(.)$–terms are independent of α). Following the lines of standard perturbation index theory (see (4)) estimate (12) with $m = 2$ is proved. I. e., the class of all Baumgarte-like stabilized differential-algebraic systems of index 2 with parameter $\alpha \geq \alpha_0 > 0$ has uniform perturbation index 2.

Remarks 2 a) The uniform perturbation index remains unchanged if the class of DAEs that is considered in Example 2 is extended by the index-2 system (3), i. e. by the limit case $\alpha \to \infty$.

b) The discrete analogue of the uniform error bound in Example 2 is an error bound $C^*_{0,\infty} \cdot \frac{1}{h}\Delta$ with a constant $C^*_{0,\infty}$ that is independent of α. I. e., if appropriate discretization methods are used then the amplification of small errors Δ during integration is bounded by $\min(C^*_{0,\alpha}\Delta, C^*_{0,\infty}\cdot\frac{1}{h}\Delta)$ with $\lim_{\alpha\to\infty} C^*_{0,\alpha}=\infty$ and $C^*_{0,\infty}=\mathcal{O}(1)$. For large values of α and stepsizes h of moderate size the uniform error bound $C^*_{0,\infty}\cdot\frac{1}{h}\Delta$ is substantially smaller than the error bound $C^*_{0,\alpha}\Delta$ from standard perturbation index theory.

c) If the class of DAEs in Example 2 is restricted to systems with $\alpha<\overline{\alpha}$ and a fixed $\overline{\alpha}$ then the uniform perturbation index of the class is 1 since (12) with $m=1$ can be proved with $C_0=C_{0,\overline{\alpha}}$.

d) The analysis for the index-2 case is straightforwardly extended to prove that the classical Baumgarte stabilization for constrained mechanical systems ([2]) results in a class of index-1 DAEs that has uniform perturbation index 3.

4 Semidiscretizations of partial DAEs – a case study

Uniform error bounds found our special interest since recently partial DAEs and its semi-discretizations have been considered (e. g. [4]). With one example we illustrate in this that uniform error estimates in the sense of Definition 2 usually describe correctly the sensitivity of semi-discretized DAEs w. r. t. small perturbations.

Example 3 [5, Example 1]] Consider the system of 2 linear partial differential equations

$$\begin{aligned} u_t-\frac{1}{4}v_{xx}+\varrho v &= f^u(x,t)\\ -\frac{1}{4}u_{xx}+\frac{1}{4}v_{xx}+v &= f^v(x,t) \end{aligned} \tag{13}$$

for $0\le x\le L$, $0\le t\le T$ with initial conditions

$$u(x,0)=g^u(x)\ ,\ \ v(x,0)=g^v(x)\ ,\ \ (0\le x\le L)$$

and homogenous Dirichlet boundary conditions, $f:=(f^u,f^v)^T$, $g:=(g^u,g^v)^T$. $\varrho\in\mathbb{R}$ denotes some (fixed) parameter, we will consider the cases $\varrho=0$ and $\varrho=-2$ in detail. We suppose that functions f and g are sufficiently differentiable and that g satisfies the boundary conditions. Furthermore we suppose that u, v, f and g have series expansions

$$u(x,t)=\sum_{n=1}^{\infty}\phi_n(x)u_n(t)\ ,\ \ v(x,t)=\sum_{n=1}^{\infty}\phi_n(x)v_n(t)\ ,\ \ \ldots$$

with $\phi_n(x):=\sin(\frac{n\pi x}{L})$. Under suitable smoothness assumptions the coefficients $u_n(t)$, $v_n(t)$, $(n\ge 1)$ are the solutions of the initial value problems

$$u_n(0)=g^u_n\ ,\ \ v_n(0)=g^v_n$$

for the linear constant-coefficient DAE

$$\begin{aligned} u_n'(t) + (\varrho + \frac{1}{4}\lambda_n^2)v_n(t) &= f_n^u(t) \\ \frac{1}{4}\lambda_n^2 u_n(t) + (1 - \frac{1}{4}\lambda_n^2)v_n(t) &= f_n^v(t) \end{aligned} \tag{14}$$

with $\lambda_n := \frac{n\pi}{L}$.

Consider now the discretization by finite differences on an equidistant grid $0 = x_0 < x_1 < < \ldots < x_N < x_{N+1} = L$, $x_i := ih$, $h = L/(N+1)$. Let $U(t) = (u_1^N(t), \ldots, u_N^N(t))^T$, $V(t) = (v_1^N(t), \ldots, v_N^N(t))^T$

$$U(t) = (u_1^N(t), \ldots, u_N^N(t))^T \ , \ \ V(t) = (v_1^N(t), \ldots, v_N^N(t))^T$$

with $u_i^N(t) \approx u(x_i, t)$, $v_i^N(t) \approx v(x_i, t)$, $(i = 1, \ldots, N)$ and

$$F^u(t) = (f^u(x_1,t), \ldots, f^u(x_N,t))^T \ , \ \ F^v(t) = (f^v(x_1,t), \ldots, f^v(x_N,t))^T \ ,$$

$$G^u = (g^u(x_1), \ldots, g^u(x_N))^T \ , \ \ G^v = (g^v(x_1), \ldots, g^v(x_N))^T \ .$$

The finite difference approximation satisfies $U(0) = G^u$, $V(0) = G^v$,

$$\begin{aligned} U'(t) - \frac{1}{4}A_h \cdot V(t) + \varrho \cdot V(t) &= F^u(t) \\ -\frac{1}{4}A_h \cdot U(t) + \frac{1}{4}A_h \cdot V(t) + V(t) &= F^v(t) \end{aligned} \tag{15}$$

with the symmetric tridiagonal matrix

$$A_h = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & & & 0 \\ 1 & -2 & 1 & & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ 0 & & & & 1 & -2 \end{pmatrix} \in \mathbb{R}^{N\times N}$$

that has eigenvalues $\mu_i = -\frac{4}{h^2}\sin^2(\frac{i\pi}{2(N+1)})$ and eigenvectors

$$\Phi_i = (\,\sin(\frac{i\pi}{N+1})\,, \sin(\frac{2i\pi}{N+1})\,, \ldots\,, \sin(\frac{Ni\pi}{N+1})\,)^T \ , \ \ (i = 1, \ldots, N) \ .$$

We rewrite vectors $U(t)$, $V(t)$, $F^u(t)$, $F^v(t)$, G^u, G^v as linear-combinations of eigenvectors of A_h:

$$U(t) = \sum_{i=1}^{N} \frac{1}{\|\Phi_i\|_2}\Phi_i \cdot U_i(t) \quad , \quad V(t) = \sum_{i=1}^{N} \frac{1}{\|\Phi_i\|_2}\Phi_i \cdot V_i(t) \quad , \quad \ldots$$

Multiplying the equations (15) subsequently by $\Phi_1^T,\ \Phi_2^T,\ \dots,\ \Phi_N^T$ we get the equivalent system of equations

$$\begin{aligned} U_i'(t) + (\varrho + \frac{1}{4}\Lambda_i^2)V_i(t) &= F_i^u(t) \\ \frac{1}{4}\Lambda_i^2 U_i(t) + (1 - \frac{1}{4}\Lambda_i^2)V_i(t) &= F_i^v(t) \end{aligned} \tag{16}$$

with $\Lambda_i := \sqrt{-\mu_i} = \frac{2}{h}\sin(\frac{i\pi}{2(N+1)})$, $(i = 1, \dots , N)$ since $A_h\Phi_i = \mu_i\Phi_i = -\Lambda_i^2\Phi_i$ and

$$\Phi_j^T\Phi_i = \begin{cases} \|\Phi_i\|_2^2 & \text{if } i = j \ , \\ 0 & \text{if } i \neq j \ . \end{cases}$$

This change of coordinates has no influence on the sensitivity of the solution w. r. t. small perturbations. For the perturbation analysis we prefer Eqs. (16) since they are quite similar to the corresponding equations (14) for the coefficients $u_n(t)$ of the solution of the partial DAE (13). If $i \le N$ is fixed then we get

$$\lim_{N\to\infty} \Lambda_i = \lambda_i = \frac{i\pi}{L}$$

and in the limit case $N \to \infty$ Eqs. (16) are transferred to (14).
For $\varrho = 0$ this analysis is carried out in [5]. They observe that (14) has (differential and perturbation) index 1 if $\lambda_n^2 \neq 4$ and index 2 if $\lambda_n^2 = 4$. Up to now there is no widely accepted index concept for partial DAEs (see [4] for a comprehensive study of this subject). But for the special example (13) it seems to be natural to call (13) a partial DAE of index 1 if the DAEs (14) have index 1 for *all* $n \in \mathbb{N}$ and a partial DAE of index 2 if there is one $n \in \mathbb{N}$ such that (14) has index 2 ([4, Example 2]). Since λ_n depends on L the index of the partial DAE varies with the length L of the domain.

If $\varrho \in \{0, -2\}$ and L is fixed then the finite difference approximation (16) has always index 1 if the discretization is sufficiently fine (i. e. h is sufficiently small): this follows in the case $\lambda_i^2 \neq 4$ from $\lim_{N\to\infty} \Lambda_i = \lambda_i$ and in the case $\lambda_i^2 = 4$ from $\Lambda_i \neq \lambda_i$. In this sense “the method of lines approximation ... acts like a regularization” ([5]) if the partial DAE (13) has index 2. However, in view of the results of Section 2 we do not expect that the class of *all* semidiscretizations (15) has uniform perturbation index 1 if the partial DAE (13) has index 2, i. e. if there is an $n_0 \in \mathbb{N}$ with $\lambda_{n_0}^2 = 4$.

Therefore the most interesting case is given by DAEs (14) with $\lambda_n^2 \approx 4$ and $\lambda_n^2 \neq 4$. These problems can be interpreted as perturbations of an index-2 DAE (Eqs. (14) with $\lambda_n^2 = 4$), they were studied in great detail by Söderlind ([11]). He proved that the stability of the lower index system (i. e. (14) with $0 < |\lambda_n^2 - 4| \ll 1$) depends strongly on the sign of the perturbation. To analyse this phenomenon we solve the second equation in (14) w. r. t. $v_n(t)$, insert this expression into the first one and get (if $\varrho \in \{0, -2\}$)

$$\begin{aligned} u_n'(t) + \alpha u_n(t) &= f_n^u(t) + \alpha \cdot \frac{1}{\frac{1}{4}\lambda_n^2} f_n^v(t) \\ v_n(t) &= \frac{1}{\varrho + \frac{1}{4}\lambda_n^2}(f_n^u(t) - u_n'(t)) \end{aligned} \tag{17}$$

with

$$\alpha := -\frac{1}{4}\lambda_n^2 \frac{\varrho + \frac{1}{4}\lambda_n^2}{1 - \frac{1}{4}\lambda_n^2} \quad .$$

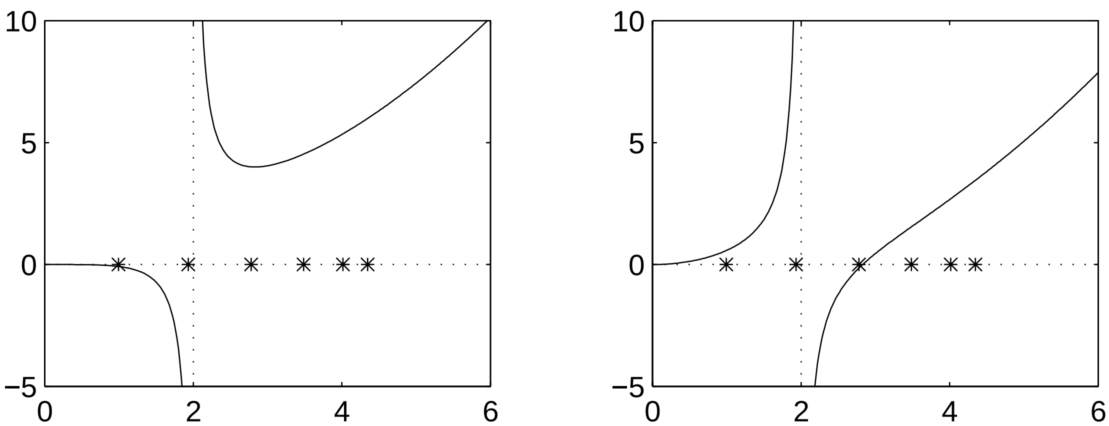

Figure 1: Coefficient α in (17) vs. λ_n for two values of ϱ. The asterisks at the abscissa mark the eigenvalues $\Lambda_1, \Lambda_2, \ldots, \Lambda_6$ of the semi-discretized problem (15) with $N = 6$ and $L = \pi$.

The way in that errors are propagated in (17) is determined by α. If $\alpha > 0$ then these equations have the same basic structure as the Baumgarte-like stabilized systems of Section 2 ($u_n(t) \to g(y(t))$, $v_n(t) \to z(t)$), the perturbation analysis of Sections 2 and 3 can be carried over straightforwardly. If, however, $\alpha < 0$ then errors may grow like $e^{-\alpha t}$. This is still acceptable if $0 \geq \alpha \geq -\alpha_0$ with a positive constant α_0 of moderate size (i. e. $e^{-\alpha t} \leq e^{\alpha_0 t}$), but if $\lambda_n^2 \to 4$ then α may become arbitrarily small. Fig. 1 shows α vs. λ_n in the two cases $\varrho = 0$ (left) and $\varrho = -2$ (right). Depending on ϱ we get the following results:

Lemma 2 *Let a positive constant* $\Delta_\lambda \in (0, 1]$ *be given.*

a) If $\varrho \in \{0, -2\}$ *then the class of all DAEs* (14) *with* $|\lambda_n^2 - 4| \geq \Delta_\lambda > 0$ *has uniform perturbation index 1.*

b) If $\varrho = 0$ *then the class of all DAEs* (14) *with* $\lambda_n^2 \in (0, 4 - \Delta_\lambda] \cup [4, \infty)$ *has uniform perturbation index 2.*

c) If $\varrho = -2$ *then the class of all DAEs* (14) *with* $\lambda_n^2 \in (0, 4] \cup [4 + \Delta_\lambda, \infty)$ *has uniform* *perturbation index 2.*

d) Neither for $\varrho = 0$ *nor for* $\varrho = -2$ *the class of all DAEs* (14) *(with arbitrary* λ_n^2*) has a uniform perturbation index.*

Proof: If $|\lambda_n^2 - 4| \geq \Delta_\lambda$ then $\alpha \geq -\alpha_0$ with a constant α_0 that is independent of λ_n but depends on Δ_λ. Following standard perturbation index theory estimate (2) with $m = 1$ is proved ($C_0 = \mathcal{O}(e^{\alpha_0 t})$ and $\lim\limits_{\Delta_\lambda \to 0} C_0 = \infty$). In the stripe $\{\, \lambda_n : |\lambda_n^2 - 4| \leq \Delta_\lambda \,\}$ the way in that errors are propagated depends on the sign of α (and thus on ϱ, see Fig. 1): if $\alpha > 0$ the results of Sections 2 and 3 can be applied to prove the uniform error estimate (12) with $m = 2$, in the case $\alpha < 0$ there is no estimate (12) at all since $C_0 = \mathcal{O}(e^{-\alpha t})$ and $\alpha \to -\infty$. ■

Söderlind [11] points out that DAEs of the form (14) with $\lambda_n^2 = 4$ are not isolated higher-index problems that are difficult to solve numerically but these problems separate a class of index-1 DAEs that are in the limit case $\lambda_n^2 \to 4$ very similar to index-2 DAEs from another class of index-1 DAEs that exhibit an essential instability. This statement is carried over straightforwardly to partial DAEs (13) if the length L of the domain is such that the index of the partial DAE is 2.

Remarks 3 a) For the partial DAE (13) we consider the set of *all* DAEs (14) and for the semi-discretized DAEs the set of *all* DAEs (16), the dimension of $U(t)$, $V(t)$ varies with N. Uniform error estimates make sense only, if the norms in (12) are compatible for varying N. Throughout this we use the $\mathcal{L}^2$–norm on $[0, L]$ in the partial DAE case and the discrete analogue $\|U\|_{2,N} := \left(\frac{1}{N}\sum_{i=1}^N (U_i^N)^2\right)^{1/2}$ for the semi-discretized DAEs, i. e.

$$\|U(t)\| = \Big(\frac{1}{N}\sum_{i=1}^{N} U_i^2(t)\Big)^{1/2} \quad , \quad \|V(t)\| = \Big(\frac{1}{N}\sum_{i=1}^{N} V_i^2(t)\Big)^{1/2} \quad .$$

b) Because of $\lim_{N\to\infty} \Lambda_i = \lambda_i$ Lemma 2a proves that for a given partial DAE (13) of index 1 with $\varrho \in \{0, -2\}$ the class of all (sufficiently fine) finite difference approximations (15) has uniform perturbation index 1.

c) If the partial DAE (13) has index 2 (i. e. $\frac{2L}{\pi} \in \mathbb{N}$) then the class of all (sufficiently fine) finite difference approximations (15) has either uniform perturbation index 2 (if $\varrho = -2$, see Lemma 2c) or no uniform perturbation index at all (if $\varrho = 0$, see Lemma 2d). This follows from $\lim_{N\to\infty} \Lambda_i = \lambda_i$ and $\Lambda_i < \lambda_i$, $(i = 1, \ldots, N)$, see also the asterisks for $N = 6$ in Fig. 1. If $\varrho = 0$ and $\lambda_n^2 = 4$ then errors in (16) may be amplified by $\exp(3(\frac{N+1}{L})^2 t)$ since $\Lambda_n^2 = (\frac{2}{h}\sin h)^2 = 4(1 - \frac{1}{3}h^2) + \mathcal{O}(h^4)$.

5 Summary

Classes of DAEs may consist of DAEs that all have perturbation index 1 but a (in some sense well-defined) limit is of higher index. We illustrated with 2 examples that the numerical solution of such *index-1* DAEs may cause problems that are typical of *higher index* DAEs. Furthermore, the error bounds from standard perturbation index theory do not give useful information about the sensitivity of the solution w. r. t. perturbations. The uniform perturbation index describes for DAEs close to the higher index DAE the influence of perturbations on the solution correctly.

In the case of partial DAEs with an index that depends on the domain the existence of a uniform perturbation index can not be guaranteed. In one example the analysis of an index-2 partial DAE results in an (ordinary) index-2 DAE that separates a class of DAEs with uniform perturbation index 2 from a class of DAEs that has no uniform perturbation index at all. The same phenomenon is found analysing the sensitivity of the solutions of the semi-discretized systems w. r. t. small perturbations.

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